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GENERALIZATIONS OF INEQUALITIES OF LITTLEWOOD AND PALEY

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ABSTRACT. For a function f, holomorphic in the open unit ball B_n in C^n, with $f(0) = 0$, we prove

(I) If $0 < s < 2$ and $s < p < \infty$. Then

$$\|f\|_p^p < C\int_0^1\int_{\partial B_n}|f(\rho\zeta)|^{p-s}|Rf(\rho\zeta)|^s(\log 1/\rho)^{s-1}\rho^{-1}d\sigma(\zeta)d\rho$$

(II) If $2 < s < p < \infty$. Then

$$\int_0^1\int_{\partial B_n}|f(\rho\zeta)|^{p-s}|Rf(\rho\zeta)|^s(\log 1/\rho)^{s-1}\rho^{-1}d\sigma(\zeta)d\rho < C\|f\|_p^p$$

where Rf is the radial derivative of f, generalizing the known caeses p = s ([1]) and p = s, n = 1 ([2]).

KEY WORDS AND PHRASES. Radial derivative, slice function.
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1. INTRODUCTION

Let C^n denote the n-dimensional vector space over C. let B_n denote the open unit ball in C^n with boundary ∂B^n and let σ denote the rotation-invariant positive measure on ∂B_n for which $\sigma(\partial B_n) = 1$.

Throughout this paper, we assume that f is holomorphic in B_n with $f(0) = 0$, and $Rf(z) = \sum_{\alpha>0}|\alpha|a_\alpha z^\alpha$ is the radial derivative of $f(z) = \sum_{\alpha>0}a_\alpha z^\alpha$.

For $0 < p < \infty$ and $0 < s < \infty$, we set

$$M_p^p(r, f) = \int_{\partial B_n}|f(r\zeta)|^p d\sigma(\zeta)$$

$$\|f\|_p = \sup_{0<r<1} M_p(r, f) \text{ and}$$

$$G_{p,s}[f] = \int_0^1\int_{\partial B_n}|f(\rho\zeta)|^{p-s}|Rf(\rho\zeta)|^s(\log 1/\rho)^{s-1}\rho^{-1}d\sigma(\zeta)d\rho$$

In [1, Theorem 4 and Theorem 7] J. H Shi generalizes the inequalities of Littlewood and Paley of one complex variable ([2]) to the unit ball B_n. That is

THEOREM A (1) Let $0 < p \leq 2$. Then

$$\| f \|_p^p \leq C\, G_{p,p}[f] \tag{1}$$

(2) Let $2 \leq p < \infty$. Then

$$G_{p,p}[f] \leq C \| f \|_p^p \tag{2}$$

In this notes, we generalize these results, namely, we prove the following

THEOREM (I) Let $0 < s \leq 2$ and $s \leq p < \infty$. Then

$$\| f \|_p^p \leq C\, G_{p,s}[f] \tag{3}$$

(II) Let $2 \leq s \leq p < \infty$. Then

$$G_{p,s}[f] \leq C \| f \|_p^p \tag{4}$$

Throughout this paper C denotes a positive constant depending only on p and s. The magnitude of C may vary from occurrence to occurrence even in the proof of the same theorem.

2. PROOF OF THE THEOREM

For the proof of the Theorem, we need the following

LEMMA. For $0 < p < \infty$. Then

$$\| f \|_p^p = p^2\, G_{p,2}[f] \tag{5}$$

PROOF. For $\zeta \in \partial B_n$ the slice functions are defined by $f_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in B_1$. Then $Rf(\lambda\zeta) = \lambda f'_\zeta(\lambda)$.

By the Hardy_Stein identity for one complex variable ([3]) we have

$$M_p^p(r, f_\zeta) = (p^2/2\pi) \int_0^r \int_0^{2\pi} | f_\zeta(\rho e^{i\theta}) |^{p-2} | f'_\zeta(\rho e^{i\theta}) |^2 \log(r/\rho)\, \rho\, d\rho\, d\theta$$

$$= (p^2/2\pi) \int_0^r \int_0^{2\pi} | f(\rho\zeta e^{i\theta}) |^{p-2} | Rf(\rho\zeta e^{i\theta}) |^2 \rho^{-1} \log(r/\rho)\, d\theta\, d\rho$$

Integrating with respect to $d\sigma(\zeta)$, using the Fubini theorem and the formular

$$\int_{\partial B_n} g(\zeta)\, d\sigma(\zeta) = (1/2\pi) \int_{\partial B_n} d\sigma(\zeta) \int_0^{2\pi} g(e^{i\theta}\zeta)\, d\theta, \qquad g \in L^1(\sigma).$$

(see [4. P. 16]), we have

$$M_p^p(r, f) = p^2 \int_0^r \int_{\partial B_n} | f(\rho\zeta) |^{p-2} | Rf(\rho\zeta) |^2 \rho^{-1} \log(r/\rho)\, d\sigma(\zeta)\, d\rho \tag{6}$$

By letting $r \to 1$ in (6), we obtain (5).

We also need the following fact whose easy proof (by Holder's inequality) we omit.

For a fixed p, $\log G_{p,s}[f]$ is a convex function of s ($0 < s < \infty$), That is, if $0 < s_1 < s < s_2 < \infty$ then

$$G_{p,s}[f] \le G_{p,s_1}[f]^{t}\, G_{p,s_2}[f]^{1-t} \tag{7}$$

Where $t = (s_2 - s)/(s_2 - s_1)$.

We now turn to the proof of the Theorem

(I) Case 1. $s < p < 2$. Set $t = (2-p)/(2-s)$

$$\|f\|_p^p \le C\, G_{p,p}[f] \qquad (\text{by } (1))$$

$$\le C\, G_{p,s}[f]^{t}\, G_{p,2}[f]^{1-t} \qquad (\text{by } (7))$$

$$\le C\, G_{p,s}[f]^{t}\, \|f\|_p^{p(1-t)} \qquad (\text{by } (6))$$

so that

$$\|f\|_p^p \le C\, G_{p,s}[f]$$

Case 2. $s < 2 < p$, Set $t = (p-2)/(p-s)$

$$\|f\|_p^p = C\, G_{p,2}[f] \qquad (\text{by } (6))$$

$$\le C\, G_{p,s}[f]^{t}\, G_{p,p}[f]^{1-t} \qquad (\text{by } (7))$$

$$\le C\, G_{p,s}[f]^{t}\, \|f\|_p^{p(1-t)} \qquad (\text{by } (2))$$

so that

$$\|f\|_p^p \le C\, G_{p,s}[f]$$

This gives (3).

(II) Set $t = (p-s)/(p-2)$

$$G_{p,s}[f] \le G_{p,2}[f]^{t}\, G_{p,p}[f]^{1-t} \qquad (\text{by } (7))$$

$$\le C\|f\|_p^{pt}\, G_{p,p}[f]^{1-t} \qquad (\text{by } (6))$$

$$\le C\|f\|_p^{pt}\, \|f\|_p^{p(1-t)} \qquad (\text{by } (2))$$

$$= C\|f\|_p^p$$

This gives (4).

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