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ON STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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ABSTRACT. Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable Banach space of type r, $1 \le r \le 2$. Complete convergence of $n^{1/p} \sum_{k=1}^{n} X_{nk}$ to 0 , $0 < p < r \le 2$ is obtained when $\sup_{1 \le k \le n} E\|X_{nk}\|^{\nu} = O(n^{\alpha})$, $\alpha \ge 0$ with $\nu\left(\frac{1}{p} - \frac{1}{r}\right) > \alpha + 1$. An application to density estimation is also given.

KEY WORDS AND PHRASES. *Random elements, strong laws of large numbers, complete convergence, Rademacher type r spaces.*
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1. INTRODUCTION AND PRELIMINARIES.

Let $(\mathcal{E}, \|.\|)$ be a real separable Banach space. Let $(\Omega, \mathcal{A}, P)$ denote a probability space. A random element X in $\mathcal{E}$ is a function from Ω into $\mathcal{E}$ which is $\mathcal{A}$ - measurable with respect of the Borel subsets $B(\mathcal{E})$. The p^{th} absolute moment of a random element X is $E\|X\|^p$ where E is the expected value of the random variable $\|X\|^p$. The expected value of a random element X is defined to be the Bochner integral (when $E\|X\| < \infty$) and is denoted by EX. The concepts of independence and identical distributions for real-valued random variables extend directly to $\mathcal{E}$. A separable Banach space is said to be of (Rademacher) type r, $1 \le r \le 2$, if there exist a constant C such that

$$E\left\|\sum_{k=1}^{n} X_k\right\|^r \leq C\sum_{k=1}^{n} E\|X_k\|^r$$

for all independent random elements $X_1, ..., X_n$ with zero means and finite r^{th} moments. Every separable Hilbert space and finite dimensional Banach space is of type 2. Every separable Banach space is at least type 1 while l^r and L^r spaces are of type min(2, r) for $r \geq 1$.

Throughout this paper $\{X_{nk} : 1 \leq k \leq n, n \geq 1\}$ will denote rowwise independent random elements in $\mathcal{E}$ such that

$$EX_{nk} = 0 \qquad \text{for all n and k.} \tag{1.1}$$

The major results of this paper show that

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} \to 0 \qquad \text{completely} \tag{1.2}$$

where complete convergence is defined (as in Hsu and Robbins [1]) by

$$\sum_{n=1}^{\infty} P\left[\left\|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk}\right\| > \epsilon\right] < \infty \tag{1.3}$$

for each $\epsilon > 0$.

Erdös [2] showed that for an array of i.i.d. random variables $\{X_{nk}\}$, (1.3) holds if and only if $E|X_{11}|^{2p} < \infty$. Jain [3] obtained a uniform strong law of large numbers for sequences of i.i.d. random elements in separable Banach spaces of type 2 which would yield (1.2) with p = 1 for an array of i.i.d. random elements $\{X_{nk}\}$ in a type 2 space. Woyczynski [4] showed that

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_k \to 0 \qquad \text{completely} \tag{1.4}$$

for any sequence $\{X_n\}$ of independent random elements in a Banach space of type r, $1 \leq p < r \leq 2$ with $EX_n = 0$ for all n which is uniformly bounded by a random variable X satisfying $E|X|^p < \infty$. Recall that an array $\{X_{nk}\}$ of random elements is said to be uniformly bounded by a random variable X if for all n and k and for every real number $t > 0$

$$P\left[\|X_{nk}\| > t\right] \leq P\left[|X| > t\right]. \tag{1.5}$$

Note that i.i.d. random elements are uniformly bounded by $\|X_{11}\|$. Moricz, Hu, and Taylor [5] showed that Erdös' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.5). Taylor and Hu [6] obtained complete convergence in type r spaces, $1 < r \leq 2$ for uniformly bounded; rowwise independent random elements. The results of this paper relaxes the assumption of uniformly bounded random elements in Taylor and Hu [6]. Moreover, a major application of the main result of this paper is indicated for kernel density estimators where uniformly bounded random variables can not be assumed.

2. MAJOR RESULTS.

The following lemma from Woyczynski [4] will be used in obtaining the major result, Theorem 2.

LEMMA 1. *Let $1 \leq r \leq 2$ and $q \geq 1$. The following properties are equivalent:*

(i) *$\mathcal{E}$ is of type r*

(ii) *There exists a C such that for all independent random elements $X_1, \ldots, X_n$ in $\mathcal{E}$ with $EX_k = 0$, and $E\|X_k\|^q < \infty$, $k = 1, 2, \ldots, n$*

$$E\|\sum_{k=1}^{n} X_k\|^q \leq CE\left(\sum_{k=1}^{n}\|X_k\|^r\right)^{q/r}$$

THEOREM 2. *Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable Banach space of type r. If $EX_{nk} = 0$ and*

$$\sup_{1\leq k\leq n} E\|X_{nk}\|^\nu = O(n^\alpha), \qquad \alpha \geq 0 \tag{2.1}$$

where $\nu\left(\frac{1}{p} - \frac{1}{r}\right) > \alpha + 1$, $0 < p < r \leq 2$. Then

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} \to 0 \qquad \textit{completely.}$$

PROOF: Let $\epsilon > 0$ be given. By Markov's inequality

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk}\| > \epsilon\right) &\leq \sum_{n=1}^{\infty}\frac{1}{\epsilon^\nu}E\left\|\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk}\right\|^\nu \\ &\leq C_1\sum_{n=1}^{\infty}\frac{1}{n^{\nu/p}}E\left\|\sum_{k=1}^{n} X_{nk}\right\|^\nu . \end{aligned} \tag{2.2}$$

By Lemma 1 and Hölder's inequality,

$$\begin{aligned} C_1\sum_{n=1}^{\infty}\frac{1}{n^{\nu/p}}E\left\|\sum_{k=1}^{n} X_{nk}\right\|^\nu &\leq C_1\sum_{n=1}^{\infty}\frac{1}{n^{\nu/p}}E\left(\sum_{k=1}^{n}\|X_{nk}\|^\nu\right)\left(\left(\sum_{k=1}^{n}1\right)^{1-\frac{r}{\nu}}\right)^{\frac{\nu}{r}} \\ &\leq C_1\sum_{n=1}^{\infty}\frac{n^{\nu/r-1}}{n^{\nu/p}}\sum_{k=1}^{n}E\|X_{nk}\|^\nu \\ &\leq C_1\sum_{n=1}^{\infty}\frac{n^{\nu/r-1}}{n^{\nu/p}}\cdot n\sup_{1\leq k\leq n}E\|X_{nk}\|^\nu \\ &\leq C_2\sum_{n=1}^{\infty}\frac{n^{\nu/r-1}\cdot n\cdot n^\alpha}{n^{\nu/p}} \\ &= C_2\sum_{n=1}^{\infty}\frac{1}{n^{\nu(\frac{1}{p}-\frac{1}{r})-\alpha}} < \infty \end{aligned}$$

since $\nu\left(\frac{1}{p}-\frac{1}{r}\right) > 1+\alpha$. Therefore,

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} \to 0 \qquad \text{completely.}$$

REMARK 1.

For values of p and r, $1 \le p \le r \le 2$, it follows that $\nu > 2$. Moreover, as p and r move close to each other ν increases without bound. However, for certain values of p strictly less than one, a value of $\nu = 1$ is possible to obtain complete convergence. To see this let $p = \frac{1}{3}$, $r = 1$ and $\alpha = 0$. It follows that $\nu\left(\frac{1}{p}-\frac{1}{r}\right) = \nu(3-1) = 2\nu > 1$ implies that $\nu > \frac{1}{2}$. However, the proof of Theorem 2 requires that $\nu \ge 1$. Thus, $\nu = 1$ is the smallest moment necessary (given suitable conditions on p, r and α) to obtain complete convergence, via Theorem 2

REMARK 2.

The condition $\sup_{1\le k\le n} E\|X_{nk}\|^{\nu} = O(n^{\alpha})$ is somewhat stronger than (1.5) used by Taylor and Hu [6]. However, the bound in each row increases as $n \to \infty$ which is a substantial improvement in Theorem 4 of Taylor, Moricz and Hu [5]. This substantial improvement will be illustrated in Example 1.

An immediate corollary to Theorem 2 is obtained for i.i.d. random elements.

COROLLARY 3. *Let $\{X_{nk}\}$ be an array of i.i.d. random elements in a Banach space $\mathcal{E}$ of type r such that $EX_{11} = 0$. Let $E\|X_{11}\|^{\nu} < \infty$ where $\nu\left(\frac{1}{p}-\frac{1}{r}\right) > 1$, $0 < p < r \le 2$. Then,*

$$\frac{1}{n^{1/p}}\sum_{k=1}^{n} X_{nk} \to 0 \qquad \textit{completely.}$$

REMARK 3.

The moment condition in Corollary 3 can be considerably smaller than the moment condition in Theorem 6 of Taylor and Hu [6], (see Remark 1) but in general will be much larger.

3. EXAMPLE 1.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with common density function f. The kernel estimator for f with constant bandwidths h_n is given by

$$f_n(t) = \frac{1}{nh_n}\sum_{k=1}^{n} K\left(\frac{t-X_k}{h_n}\right) \tag{3.1}$$

where K is a bounded (integrable) kernel with compact support [a, b] and the sequence $\{h_n\}$ is bounded and monotonically decreasing to 0 as $n \to \infty$. Let X_{nk} be defined as follows:

$$X_{nk} = \frac{1}{h_n}\left[K\left(\frac{t-X_k}{h_n}\right) - E\left(K\left(\frac{t-X_1}{h_n}\right)\right)\right]. \tag{3.2}$$

Since the sequence $\{X_n\}$ is i.i.d., it follows that $\{X_{nk} : k = 1, 2, \dots\}$ is i.i.d. for each n. Verification of Condition (2.1) depends on the choice of K, the bandwidth sequence $\{h_n\}$ and the particular Banach space. Typically, $h_n = O(n^{-d})$ where $0 < d < \frac{1}{2}$. To illustrate the applicability of Condition (2.1), consider the Banach space L^r, $1 < r \leq 2$. Then for each k and n

$$\begin{aligned} E\|X_{nk}\|^\nu &\leq 2^\nu \left(\int_\infty^\infty \left| \frac{1}{h_n} K\left(\frac{t - X_1}{h_n}\right) \right|^r dt \right)^{\nu/r} \\ &\leq C_1 {h_n}^{\nu(1-r)/r} \\ &\leq C_2 n^{d\nu(r-1)/r}. \end{aligned}$$

Since $d < \frac{1}{2}$ and $r > 1$, ν can be chosen so that

$$\sup_{1 \leq k \leq n} E\|X_{nk}\|^\nu = O(n^\alpha)$$

and

$$\nu \left(\frac{1}{p} - \frac{1}{r} \right) > \alpha + 1 \quad \text{by letting p} = 1 \text{ and } \alpha = d\nu \left(\frac{r-1}{r} \right).$$

Verification of (2.1) follows easily for L^q, $q \geq 2$, since they are of type 2. Thus, $n^{-1} \sum_{k=1}^n X_{nk} \to 0$ completely or $(nh_n)^{-1} \sum_{k=1}^n \left(K\left(\frac{t - X_k}{h_n}\right) - E\left(K\left(\frac{t - X_1}{h_n}\right)\right)\right) \to 0$ completely. Hence, consistency for (3.1) follows since $(h_n)^{-1} E\left(K\left(\frac{t - X_1}{h_n}\right)\right) \to f(t)$ by traditional techniques.

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REFERENCES

1. Hsu, P. L., and Robbins, H. (1947). Complete Convergence and the Law of Large Numbers, Proc.Nat.Sci. U.S.A., 33, p. 25-31.
2. Erdős, P. (1949). On a Theorem of Hsu and Robbins, Ann. Math. Statistics, 20, p. 286-291.
3. Jain, N. C. (1975). Tail Probabilities for Sums of Independent Banach Space Random Variables, Z. Wahr.V.Geb, 33, p. 155-166.
4. Woyczynski, W. A. (1980). On Marcinkiewicz-Zygmund Laws of Large Numbers, Prob. and Math.Statist., 1, p. 117-131.
5. Hu, T. C., Moricz, F., and Taylor, R. L. (1986). Strong Laws of Large Numbers for Arrays of Rowwise Independent Random Variables, Acta Math. Hung., 54, p. 153-162.
6. Taylor, R. L. and Hu, T. C. (1987). Strong Laws of Large Numbers for Arrays of Rowwise Independent Random Variables, Internat.J. Math.and Math Sci., 10, p. 805-814.