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A NOTE ON DILATIONS AND MARTINGALES

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ABSTRACT. The purpose of this note is to investigate the effect of dilations on martingales and to give conditions under which a dilated martingale will retain the martingale property. The technique of dilating the joint distribution of a sequence of random variables has applications in optimal stopping theory and the study of "prophet inequalities" where martingales play a significant role.

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1. INTRODUCTION.

Martingales play a significant role in many areas of probability and one such area is in the establishment of "prophet inequalities" in optimal stopping theory. Prophet inequalities are comparisons between the expected maximum of a reward sequence and the maximum expected reward of the same sequence when stopped by non-anticipating stopping times. The name "prophet inequalities" stems from the interpretation of the expected maximum of the sequence as the expected reward of a prophet, or observer with complete foresight, while the ordinary observer must use non-anticipating stopping times to decide when to stop observing the sequence. In establishing these inequalities it is often useful to ascertain the "extremal distribution" which attains or nearly attains the inequality. Often this distribution is that of a martingale, but to discover this, given distributions must be manipulated in such a way that the gain of the prophet is increased while that of the ordinary observer is held constant or decreased. The dilation of a joint distribution of a random vector is a technique which is used to create a new distribution for which the resulting random variables have the same expectation, but a larger variance than the original ones. The purpose of this short note is to consider the effect of dilations on martingales and to decide, under what conditions on the original martingale, will the resulting "dilated" distribution remain that of a martingale.

2. MAIN RESULTS.

For ease of reference the definitions of the two main objects of interest, martingales and dilations, will be given first.

DEFINITION 1. Let $X_1, X_2, \ldots$ be integrable random variables defined on a probability space (Ω, F, P). For each $k = 1, 2, \ldots$ let $F_k = F(X_1, X_2, \ldots, X_k)$, the sub-borel field of F generated by the first k random variables. The sequence $X_1, X_2, \ldots$ is a <u>martingale</u> provided for each $n = 2, 3, \ldots$ that $E(X_n \mid F_{n-1}) = X_{n-1}$ almost surely.

In this article attention will be focused initially on two-term martingales so the definition of the dilation of a joint distribution will be given for a pair of random variables X and Y. The definition is easily extended to include the joint distribution of random vectors of length $n > 2$.

DEFINITION 2. Let X and Y be integrable random variables defined on a probability space (Ω, F, P) and let a, b, c, and d be real numbers such that $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.

Define X_a^b and Y_c^d as follows:

$$(X_a^b, Y_c^d) = (X,Y) \text{ on the set } A^c = \{(X,Y) \notin [a, b] \times [c, d]\}$$

$$= (a,c) \text{ with probability } \int_A \frac{b-X}{b-a}\,\frac{d-Y}{d-c}\, dP$$

$$= (a,d) \text{ with probability } \int_A \frac{b-X}{b-a}\,\frac{Y-c}{d-c}\, dP$$

$$= (b,c) \text{ with probability } \int_A \frac{X-a}{b-a}\,\frac{d-Y}{d-c}\, dP$$

$$= (b,d) \text{ with probability } \int_A \frac{X-a}{b-a}\,\frac{Y-c}{d-c}\, dP.$$

The distribution of (X_a^b, Y_c^d) will be referred to as a the joint dilation of the distribution of (X,Y) over the square $[a, b] \times [c, d]$.

Notice that the joint dilation of X and Y over $[a, b] \times [c, d]$ is the joint distribution with the largest variance that agrees with the joint distribution of X and Y on the complement of the square $[a, b] \times [c, d]$.

A property of dilations which make them useful for establishing certain inequalities is that the expected maximum of the dilation of a distribution is at least as large as the expected maximum of the original distribution as is shown in the following proposition.

PROPOSITION 1. $E[\max\{X,Y\}] \le E[\max\{X_a^b, Y_c^d\}]$.

PROOF. Let $\psi(x,y) = \max\{x,y\}$ and note that ψ is convex in the variables x and y. Since $(X_a^b, Y_c^d) = (X,Y)$ on A^c it will suffice to show that the inequality holds when the integration takes place over the set A as follows:

$$E[\max\{X,Y\}; A] = \int\int_A \psi(x,y)\, \mu^2(dx,dy)$$

$$= \int\int_A \psi(b\,\tfrac{x-a}{b-a} + a\tfrac{b-x}{b-a}, y)\, \mu^2(dx,dy)$$

$$\le \int\int_A \tfrac{x-a}{b-a}\psi(b,y) + \tfrac{b-x}{b-a}\psi(a,y)\, \mu^2(dx,dy)$$

$$= \int\int_A \tfrac{x-a}{b-a}\psi(b,d\tfrac{y-c}{d-c} + c\tfrac{d-y}{d-c}) + \tfrac{b-x}{b-a}\psi(a,d\tfrac{y-c}{d-c} + c\tfrac{d-y}{d-c})\, \mu^2(dx,dy)$$

$$\le \int\int_A \tfrac{x-a}{b-a}\tfrac{y-c}{d-c}\,\psi(b,d) + \tfrac{x-a}{b-a}\tfrac{d-y}{d-c}\,\psi(b,c) + \tfrac{b-x}{b-a}\tfrac{y-c}{d-c}\,\psi(a,d) + \tfrac{b-x}{b-a}\tfrac{d-y}{d-c}\,\psi(a,c)\, \mu^2(dx,dy)$$

$$= E[\max\{X_a^b, Y_c^d\}; A],$$

thus concluding the proof.

If X, Y forms a two-term martingale it is not always true that X_a^b, Y_c^d will also be a martingale as the following example indicates.

EXAMPLE 1. Let $X \equiv 1/2$, and Y be Bernoulli with $p = 1/2$. Easily X, Y is seen to be martingale. The distribution of X_0^1, Y_0^1 is given as follows:

$(X_0^1, Y_0^1) = (1,1), (1,0), (0,1)$, and $(0,0)$ each with probability 1/4.

Note that $E(Y_0^1 \mid X_0^1 = 0) = P(Y_0^1 = 1 \mid X_0^1 = 0) = 1/2$ which shows that X_0^1, Y_0^1 is not a martingale.

A natural question is the following: under what restrictions will the dilation of a two-term martingale X, Y remain a martingale? A partial answer to this question is afforded by the following.

THEOREM 1. Let X, Y be a two-term martingale taking values in $[a, b] \times [c, d]$ where $-\infty < c \leq a < b \leq d < +\infty$. Then X_a^b, Y_c^d is martingale if and only if X only takes values in $\{a, b\}$ with positive probability.

PROOF. Note that $X_a^b, Y_c^d \in \{(a,c), (a,d), (b,c), (b,d)\}$ almost surely. Easy calculations show that X_a^b, Y_c^d is martingale if and only if

$$P(Y_c^d = c \mid X_a^b = a) = \frac{d-a}{d-c}, \qquad P(Y_c^d = d \mid X_a^b = a) = \frac{a-c}{d-c}$$

$$P(Y_c^d = c \mid X_a^b = b) = \frac{d-b}{d-c}, \qquad P(Y_c^d = d \mid X_a^b = b) = \frac{b-c}{d-c}. \tag{2.1}$$

Since $\Omega = \{(X,Y) \in [a, b] \times [c, d]\}$ then from the definition of X_a^b, Y_c^d and conditional probability it follows that (2.1) holds if and only if $E((X - a)(X - b)) = 0$, where $E(XY) = E[E(XY \mid X)] = E[X\,E(Y \mid X)] = E(X^2)$ is used to obtain the latter expression. Since X takes values in [a, b] it follows that the integrand, $(X - a)(X - b)$, must be zero almost surely, therefore $X \in \{a, b\}$ almost surely.

Theorem 1 shows that for a two-term martingale taking values in the square, the first random variable must be already "dilated". The theorem extends easily to the more general situation where (X, Y) has a range extending beyond the square $[a, b] \times [c, d]$ and is recorded as Corollary 1 below.

COROLLARY 1. Let X, Y be a two-term martingale and let $-\infty < c \leq a < b \leq d < +\infty$, be real numbers. Then X_a^b, Y_c^d is martingale if and only if on the set $\{(X, Y) \in [a, b] \times [c, d]\}$, X only takes values in $\{a, b\}$ with positive probability.

3. REMARKS. There is a natural extension of the definition of a dilation to finite sequences of integrable random variables $X_1, X_2, \ldots, X_n$ where $n > 2$. Similarly, there are extensions of Proposition 1 and Theorem 1 in such cases. In this setting, if the martingale $X_1, X_2, \ldots, X_n$ is to be dilated over the set $A = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, then on A, $X_i \in \{a_i, b_i\}$ almost surely for all $i = 1, 2, \ldots, n-1$.

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