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ON A CONJECTURE OF ANDREWS

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ABSTRACT. In this paper, we prove a particular case of a conjecture of Andrews on two partition functions $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$.

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1. INTRODUCTION.

For an even integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n into parts such that no part $\not\equiv 0\ (mod\ \lambda+1)$ may be repeated and no part is $\equiv 0,\ \pm(a-\frac{\lambda}{2})(\lambda+1)\ mod\ [(2k-\lambda+1)(\lambda+1)]$.
For an odd integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n into parts such that no part $\not\equiv 0\ (mod\ \frac{\lambda+1}{2})$ may be repeated, no part is $\equiv \lambda+1\ (mod\ 2\lambda+2)$ and no part is $\equiv 0,\ \pm(2a-\lambda)\ \frac{\lambda+1}{2}$ $[mod\ (2k-\lambda+1)(\lambda+1)]$.

Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $b_1+\cdots+b_s$ with $b_i \geq b_{i+1}$, no part $\not\equiv 0\ (mod\ \lambda+1)$ is repeated, $b_i - b_{i+k-1} \geq \lambda+1$ with strict inequality if $\lambda+1/b_i$ and $\sum_{i=j}^{\lambda-j+1} f_i \leq a-j$ for $1 \leq j \leq \frac{\lambda+1}{2}$ and $f_1+\cdots+f_{\lambda+1} \leq a-1$ where f_i is the number of appearances of j in the partition.

Andrews [1] conjectured the following identities for $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$.

CONJECTURE. For $\frac{\lambda}{2} < a \leq k < \lambda$,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$$

for $0 \leq n < \binom{k+\lambda-a+1}{2} + (k-\lambda+1)(\lambda+1)$, while

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + 1$$

when $n = \binom{k+\lambda-a+1}{2} + (k-\lambda+1)(\lambda+1)$.

This conjecture has been verified [1] for $3 \leq \lambda \leq 7$, $\frac{\lambda}{2} < k \leq min(\lambda-1,5)$, $\frac{\lambda}{2} < a \leq k$.

In this paper we prove the case $k = a$ of the above conjecture.

2. PROOF.

We prove the conjecture for $k = a$ by establishing the following identities.

CASE 1. Let λ be even. Then

(1) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n < (a-\frac{\lambda}{2})(\lambda+1)$

(2) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ when $n = (a-\frac{\lambda}{2})(\lambda+1)$

(3) $B_{\lambda,k,a}[(a-\frac{\lambda}{2})(\lambda+1)+\Theta] = A_{\lambda,k,a}[(a-\frac{\lambda}{2})(\lambda+1)+\Theta], \quad 1 \le \Theta < \lambda+1$

(4) $$B_{\lambda,k,a}[(a-\frac{\lambda}{2}+1)(\lambda+1)] = \begin{cases} A_{\lambda,k,a}[(a-\frac{\lambda}{2}+1)(\lambda+1)] & \text{when } k > a. \\ A_{\lambda,k,a}[(a-\frac{\lambda}{2}+1)(\lambda+1)]+1 & \text{when } k = a. \end{cases}$$

CASE 2. Let λ be odd.

(5) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) \quad$ for $n \le \lambda$.

(6) $B_{\lambda,k,a}(\lambda+1) = A_{\lambda,k,a}(\lambda+1)$

(7) $B_{\lambda,k,a}(\lambda+1+\Theta) = A_{\lambda,k,a}(\lambda+1+\Theta), \quad \Theta < \frac{\lambda+1}{2}$

(8) $$B_{\lambda,k,a}[\tfrac{3}{2}(\lambda+1)] = \begin{cases} A_{\lambda,k,a}[\frac{3}{2}(\lambda+1)], & a > \frac{\lambda+1}{2} \text{ and for any } k \\ & a = \frac{\lambda+1}{2} \text{ and } k > a \\ A_{\lambda,k,a}[\frac{3}{2}(\lambda+1)]+1 & \text{when } k = a = \frac{\lambda+1}{2} \end{cases}$$

(9) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n), \quad n = (2a-\lambda+1)(\frac{\lambda+1}{2})+\Theta, \quad \Theta < \frac{\lambda+1}{2}$

(10) For $n = (2a-\lambda+2)(\frac{\lambda+1}{2})$

$$B_{\lambda,k,a}(n) = \begin{cases} A_{\lambda,k,a}(n) & \text{when } k > a \\ A_{\lambda,k,a}(n)+1 & \text{when } k = a \end{cases}$$

CASE 1. Let λ be even.

PROOF OF (1). Let $P_{B_{\lambda,k,a}}(n)$ and $P_{A_{\lambda,k,a}}(n)$ denote the set of partitions enumerated by $B_{\lambda,k,a}(n)$ and $A_{\lambda,k,a}(n)$ respectively. To prove (1) we prove the following stronger result.

(11) $$P_{B_{\lambda,k,a}}(n) = P_{A_{\lambda,k,a}}(n) \qquad \text{for } n < (a-\tfrac{\lambda}{2})(\lambda+1)$$

In fact we show that both are equal to

(12) $$P_D(n) \cup P_E(n)$$

where $P_D(n)$ is the set of partitions of n into distinct parts and $P_E(n)$ is the set of partitions of n in which only $(\lambda+1)$ can be repeated.

From the definition of $A_{\lambda,k,a}(n)$ it is clear that $P_A(n)$ is equal to (12). Also $\pi \in P_B(n)$ implies that $\pi \in P_D(n)$ if $\lambda+1$ is not repeated and $\pi \in P_E(n)$ otherwise. Hence $P_B(n) \subset P_D(n) \cup P_E(n)$.

On the other hand, let $\pi \in P_D(n)$. If $n = b_1 + \cdots + b_k + \cdots + b_s$ has more than k parts, then

$$\begin{aligned} n &\ge 1+2+\cdots+k = 1+2+\cdots+(\tfrac{\lambda}{2}+\alpha), \qquad \text{where } k = \tfrac{\lambda}{2}+\alpha,\ \alpha < \tfrac{\lambda}{2} \\ &= (\tfrac{\lambda}{2}-\alpha+1+\tfrac{\lambda}{2}+\alpha)+(\tfrac{\lambda}{2}-\alpha+2+\tfrac{\lambda}{2}+\alpha-1)+\cdots+(\tfrac{\lambda}{2}+\tfrac{\lambda}{2}+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\alpha) \\ &= (\lambda+1)+\cdots+(\lambda+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\alpha) \\ &= \alpha(\lambda+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\alpha) > (a-\tfrac{\lambda}{2})(\lambda+1). \end{aligned}$$

Thus for $n < (a-\frac{\lambda}{2})(\lambda+1)$ and for $\pi \in P_D(n)$, no partition of n contains more than k parts and hence the condition on b's is satisfied.

Let us now verify the condition on f's for $\pi \in P_D(n)$. Let $a = \frac{\lambda}{2}+\Theta$, $\Theta < \frac{\lambda}{2}$. If

$$\sum_{i=1}^{\lambda+1} f_i > a-1 \qquad \text{or} \qquad \sum_{i=1}^{\lambda} f_i > a-1$$

then the number being partitioned is

$$\begin{aligned}&\geq 1+2+\cdots+a=1+2+\cdots+(\tfrac{\lambda}{2}+\Theta)\\&=(\tfrac{\lambda}{2}-\Theta+1+\tfrac{\lambda}{2}+\Theta)+(\tfrac{\lambda}{2}-\Theta+2+\tfrac{\lambda}{2}+\Theta-1)+\cdots+(\tfrac{\lambda}{2}+\tfrac{\lambda}{2}+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\Theta)\\&=\Theta(\lambda+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\Theta)>(a-\tfrac{\lambda}{2})(\lambda+1).\end{aligned}$$

Thus for $n<(a-\frac{\lambda}{2})(\lambda+1)$ and for $\pi\in P_D(n)$, we have $\sum_{i=1}^{\lambda+1}f_i\leq a-1$ and $\sum_{i=1}^{\lambda}f_i\leq a-1$.

Similarly if $\sum_{i=2}^{\lambda-1}f_i>a-2$, then the number being partitioned is

$$\begin{aligned}&\geq 2+3+\cdots+(\tfrac{\lambda}{2}+\Theta)\\&=\Theta(\lambda+1)+2+3+\cdots+(\tfrac{\lambda}{2}-\Theta)\\&>(a-\tfrac{\lambda}{2})(\lambda+1)\qquad\text{if }\tfrac{\lambda}{2}-\Theta\geq 2.\end{aligned}$$

Hence $\sum_{i=2}^{\lambda-1}f_i\leq a-2$ for $\frac{\lambda}{2}-\Theta\geq 2$ and $n<(a-\frac{\lambda}{2})(\lambda+1)$. Let $\frac{\lambda}{2}-\Theta=1$. Then $a=\lambda-1$ and for $\pi\in P_D(n)$. $f_i\leq 1$ for all $i=1,2,\cdots,\lambda-1$ and hence

$$\sum_{i=2}^{\lambda-1}f_i\leq\lambda-2=a-1$$

If $\sum_{i=2}^{\lambda-1}f_i=\lambda-2$, then the number being partitioned is

$$\begin{aligned}&\geq 2+3+\cdots+(\lambda-1)\\&=(\lambda-1+2)+(\lambda-2+3)+\cdots+(\tfrac{\lambda}{2}+1+\tfrac{\lambda}{2})\\&=(\tfrac{\lambda}{2}-1)(\lambda+1)=\Theta(\lambda+1)=(a-\tfrac{\lambda}{2})(\lambda+1).\end{aligned}$$

Thus for $n<(a-\frac{\lambda}{2})(\lambda+1)$, $\sum_{i=2}^{\lambda-1}f_i\leq\lambda-3=a-2$.

Proceeding on the same lines we can show that the other conditions on f's are satisfied for partitions in $P_D(n)$. This proves that $P_D(n)\subset P_B(n)$. Similarly, $P_E(n)\subset P_B(n)$. Hence $P_B(n)=P_D(n)\cup P_E(n)$.

PROOF OF (2). Let $P'_A(n)$ [resp. $P'_B(n)$] denote the set of partitions enumerated by $A_{\lambda,k,a}(n)$ [resp. $B_{\lambda,k,a}(n)$] but not by $B_{\lambda,k,a}(n)$ [resp. $A_{\lambda,k,a}(n)$]. Then we claim

$$P'_A(n)=[a+(a-1)+\cdots+(\lambda-a+2)+(\lambda-a+1)]\text{ and }P'_B(n)=[a-\tfrac{\lambda}{2})(\lambda+1)]\text{ for }n=(a-\tfrac{\lambda}{2})(\lambda+1)$$

Clearly $\pi=a+(a-1)+\cdots+(\lambda-a+1)\in P_A(n)$ but $\pi\notin P_B(n)$ as it violates the condition on f's when $j=\lambda-a+1$. In fact $f_{\lambda-a+1}+\cdots+f_a=a-(\lambda-a)=2a-\lambda\nleq a-(\lambda-a+1)=2a-\lambda-1$. On the other hand, $(a-\frac{\lambda}{2})(\lambda+1)\in P_B(n)$ but it does not belong to $P_A(n)$ since for partitions enumerated by $A_{\lambda,k,a}(n)$ no part is $\equiv(a-\frac{\lambda}{2})(\lambda+1)\ mod[(2k-\lambda+1)(\lambda+1)]$.

As in the proof of (1), we can show that partitions $\pi\neq a+(a-1)+\cdots+(\lambda-a+1)\in P_A(n)$ are the same as the partitions $\pi\neq(a-\frac{\lambda}{2})(\lambda+1)\in P_B(n)$. This proves (2).

PROOF OF (3). To prove (3) we establish a bijection of $P'_A(n)$ onto $P'_B(n)$ where $n=(a-\frac{\lambda}{2})(\lambda+1)+\Theta$, $\Theta<\lambda+1$. Now $\pi\in P'_A(n)$ implies that it violates one of the conditions on f's or b's. Let $S_j(j=1,2,\cdots,\frac{\lambda}{2})$ denote the condition

$$\sum_{i=j}^{\lambda-j+1}f_i\leq a-j$$

and let S denote the condition

$$\sum_{i=j}^{\lambda+1}f_i\leq a-1$$

and let S^* be the condition on b's. In the following steps 1 to $\frac{\lambda}{2}+2$ we enumerate the partitions in P_A violating $S_{\frac{\lambda}{2}}, \cdots, S_1, S$ and S^* and also give the necessary bijection of $P'_A(n)$ onto $P'_B(n)$.

STEP 1. Consider $S_{\frac{\lambda}{2}}$: $f_{\frac{\lambda}{2}} + f_{\frac{\lambda}{2}+1} \leq 2 \leq a - \frac{\lambda}{2}$. For $a - \frac{\lambda}{2} \geq 2$ there are no partitions in P_A violating $S_{\frac{\lambda}{2}}$. If $a - \frac{\lambda}{2} = 1$ then the set of partitions violating $S_{\frac{\lambda}{2}}$ is $\left\{(\frac{\lambda}{2}+1) + \frac{\lambda}{2} + \pi : \pi \in P_D(\Theta) \text{ with parts } < \frac{\lambda}{2}\right\} \cup \left\{(\frac{\lambda}{2}+\Theta') + (\frac{\lambda}{2}+1) + \frac{\lambda}{2} + \pi : \pi \in P_D(\Theta - \frac{\lambda}{2} - \Theta') \text{ with parts } < \frac{\lambda}{2},\ 2 \leq \Theta' \leq \frac{\lambda}{2}\right\}$. For an element in the first set we associate $(\lambda+1)+\pi$ in P'_B while for an element in the second set we associate $(\lambda+1)+(\frac{\lambda}{2}+\Theta')+\pi$ in P'_B.

STEP 2. Consider $S_{\frac{\lambda}{2}-1}$: $f_{\frac{\lambda}{2}-1} + f_{\frac{\lambda}{2}} + f_{\frac{\lambda}{2}+1} + f_{\frac{\lambda}{2}+2} \leq 4 \leq a - \frac{\lambda}{2} + 1$. For $a - \frac{\lambda}{2} \geq 3$ there are no partitions in P_A violating $S_{\frac{\lambda}{2}-1}$. Let $a - \frac{\lambda}{2} = 1$. Then the set of partitions violating $S_{\frac{\lambda}{2}-1}$ is

$$\left\{(\tfrac{\lambda}{2}+1) + \tfrac{\lambda}{2} + (\tfrac{\lambda}{2}-1) + \pi : \pi \in P_D(\Theta - \tfrac{\lambda}{2} + 1) \quad \text{with parts } < \tfrac{\lambda}{2} - 1\right\}$$
$$\cup \left\{(\tfrac{\lambda}{2}+2) + (\tfrac{\lambda}{2}+1) + \tfrac{\lambda}{2} + \pi : \pi \in P_D(\Theta - \tfrac{\lambda}{2} - 2) \quad \text{with parts } < \tfrac{\lambda}{2} - 1\right\}$$
$$\cup \left\{(\tfrac{\lambda}{2}+2) + \tfrac{\lambda}{2} + (\tfrac{\lambda}{2}-1) + \pi : \pi \in P_D(\Theta - \tfrac{\lambda}{2}) \quad \text{with parts } < \tfrac{\lambda}{2} - 1\right\}$$
$$\cup \left\{(\tfrac{\lambda}{2}+2) + (\tfrac{\lambda}{2}+1) + (\tfrac{\lambda}{2}-1) + \pi : \pi \in P_D(\Theta - \tfrac{\lambda}{2} - 1) \quad \text{with parts } < \tfrac{\lambda}{2} - 1\right\}$$

We note that the partitions in the first two sets violate $S_{\frac{\lambda}{2}}$. For a partition in the third set we associate $(\lambda+1)+\frac{\lambda}{2}+\pi$ in P'_B while we associate $(\lambda+1)+(\frac{\lambda}{2}+1)+\pi$ in P'_B for a partition in the last set.

Let $a - \frac{\lambda}{2} = 2$. The set of partitions of $2(\lambda+1)+\Theta$ in P'_A violating $S_{\frac{\lambda}{2}-1}$ is

$$\left\{(\tfrac{\lambda}{2}+2) + (\tfrac{\lambda}{2}+1) + \tfrac{\lambda}{2} + (\tfrac{\lambda}{2}-1) + \pi : \pi \in P_D(\Theta) \quad \text{with parts } < \tfrac{\lambda}{2} - 1\right\}$$
$$\cup \left\{(\tfrac{\lambda}{2}+\Theta') + (\tfrac{\lambda}{2}+2) + (\tfrac{\lambda}{2}+1) + \tfrac{\lambda}{2} + (\tfrac{\lambda}{2}-1) + \pi : \pi \in P_D(\Theta - \tfrac{\lambda}{2} - \Theta'),\ \text{parts } < \tfrac{\lambda}{2} - 1,\ 3 \leq \Theta' \leq \tfrac{\lambda}{2}\right\}$$

For an element in the first set we associate $2(\lambda+1)+\pi$ in P'_B while for an element in the second set we associate $2(\lambda+1)+(\frac{\lambda}{2}+\Theta')+\pi$ in P'_B. Proceeding like this we arrive at the following step.

STEP $\frac{\lambda}{2}$. Consider $S_1 : f_1 + \cdots + f_\lambda \leq a - 1$. Since $f_i \leq 1$ for all $i = 1, 2, \cdots, \lambda$ we have $f_1 + f_2 + \cdots + f_\lambda \leq \lambda$. Let $f_1 + f_2 + \cdots + f_\lambda = \lambda$. Then $1 + 2 + \cdots + \lambda = \frac{\lambda}{2}(\lambda+1) > n$. Thus there are no partitions of n in P_A in which all parts $1, 2, \cdots, \lambda$ appear. Let $f_1 + \cdots + f_\lambda = \lambda - 1$. Let the deleted part among $1, 2, \cdots, \lambda$ be x. Consider

(13) $1 + 2 + \cdots + (x-1) + (x+1) + \cdots + (\lambda-1) + \lambda = (\frac{\lambda}{2}-1)(\lambda+1) + (\lambda+1-x)$ with $1 \leq \lambda+1-x \leq \lambda$.

If $a - \frac{\lambda}{2} = \frac{\lambda}{2} - 1$, then the only partition of n violating S_1 is

$$\lambda + (\lambda-1) + \cdots + (x+1) + (x-1) + \cdots + 2 + 1$$

with $\lambda + 1 - x = \Theta$ for which we associate $(\frac{\lambda}{2}-1)(\lambda+1)+\Theta$ in P'_B.

When $a - \frac{\lambda}{2} < \frac{\lambda}{2} - 1$, there are no partitions of n violating S_1 since $(13) > n$. More generally, if $f_1 + \cdots + f_\lambda = \lambda - y, 2 \leq y \leq \lambda - a$, and if $x_1, \cdots, x_y$ are the parts which are left out with $1 \leq x_1 < x_2 < \cdots < x_y \leq \lambda$, then

$$\begin{aligned}(14) \qquad & \lambda + (\lambda-1) + \cdots + (x_y+1) + (x_y-1) + \cdots + (x_1+1) + (x_1-1) + \cdots + 2 + 1 \\ & = (\tfrac{\lambda}{2} - y)(\lambda+1) + (\lambda+1-x_1) + \cdots + (\lambda+1-x_y)\end{aligned}$$

If $a-\frac{\lambda}{2}<\frac{\lambda}{2}-y$, then there are no partitions of n violating S_1 since (14) $> n$. If $a-\frac{\lambda}{2}=\frac{\lambda}{2}-y$, then

$$n=(a-\tfrac{\lambda}{2})(\lambda+1)+(\lambda+1-x_1)+\cdots+(\lambda+1-x_y).$$

There are no partitions of n violating S_1 if $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)>\Theta$. The partition (14) violates S_1 when $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)=\Theta$ and for this partition we associate

$$(\tfrac{\lambda}{2}-y)(\lambda+1)+(\lambda+1-x_1)+\cdots+(\lambda+1-x_y) \text{ in } P'_B.$$

If $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)<\Theta$, then there are no partitions of n violating S_1 since parts have to be repeated.

Let $a-\frac{\lambda}{2}>\frac{\lambda}{2}-y$. Then $\frac{\lambda}{2}-y+1\le a-\frac{\lambda}{2}\le\frac{\lambda}{2}-1$ and there are no partitions of n violating S_1 since $f_1+\cdots+f_\lambda=\lambda-y\le a-1$.

STEP $\frac{\lambda}{2}+1$. Consider $S: f_1+\cdots+f_{\lambda+1}\le a-1$. Clearly $f_i\le 1$ for $i=1,2,\cdots,\lambda$ and $f_{\lambda+1}\le a-\frac{\lambda}{2}$. Let $f_1+\cdots+f_{\lambda+1}=\lambda+\alpha$, where $f_{\lambda+1}=\alpha$ with $1\le\alpha\le a-\frac{\lambda}{2}$. Since $1+2+\cdots+(\lambda+1)=(\frac{\lambda}{2}+1)(\lambda+1)>n$, it follows that there are no partitions of n violating S if $f_1+\cdots+f_{\lambda+1}\ge\lambda+1$. Thus let us consider the case when $f_1+\cdots+f_\lambda+f_{\lambda+1}=\lambda$ with $f_{\lambda+1}=\alpha$. Then the number being partitioned is

$$\begin{aligned}&\ge 1+2+\cdots+(\lambda-\alpha)+\alpha(\lambda+1)\\ &=1+2+\cdots+\alpha+(\tfrac{\lambda}{2}-\alpha)(\lambda+1)+\alpha(\lambda+1)\\ &=\tfrac{\lambda}{2}(\lambda+1)+1+2+\cdots+\alpha>n.\end{aligned}$$

Thus there are no partitions of n violating S in this case also.

More generally, let $f_1+\cdots+f_{\lambda+1}=\lambda-y, f_{\lambda+1}=\alpha$ with $1\le y\le\lambda-a$. Let $x_1,\cdots,x_{y+\alpha}$ be the parts deleted among $1,2,\cdots,\lambda$ with $1\le x_1<x_2<\cdots<x_{y+\alpha}\le\lambda$. Consider

$$\begin{aligned}(15)\qquad &\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{\alpha\ times}+\lambda+(\lambda-1)+\cdots+(x_{y+\alpha}+1)+(x_{y+\alpha}-1)+\cdots\\ &\qquad+(x_1+1)+(x_1-1)+\cdots+2+1\\ &=\alpha(\lambda+1)+(\tfrac{\lambda}{2}-\alpha-y)(\lambda+1)+(\lambda+1-x_1)+\cdots+(\lambda+1-x_{y+\alpha})\\ &=(\tfrac{\lambda}{2}-y)(\lambda+1)+(\lambda+1-x_1)+\cdots+(\lambda+1-x_{y+\alpha}).\end{aligned}$$

As in the case of S_1 we can show that there are no partitions of n violating S when $a-\frac{\lambda}{2}$ is less or greater than $\frac{\lambda}{2}-y$ and even when $a-\frac{\lambda}{2}=\frac{\lambda}{2}-y$ and $(\lambda+1-x_1)+\cdots+(\lambda+1-x_{y+\alpha})$ is less or greater then Θ. If $(\lambda+1-x_1)+\cdots+(\lambda+1-x_{y+\alpha})=\Theta$ then the partition on the extreme left hand side of (15) violates S for which we associate the last partition of (15) which belongs to P'_B.

STEP $\frac{\lambda}{2}+2$. We now prove that if a partition violates the condition S^* on b's then it violates one of the conditions on f's. Before proving this we first note that when $k>a$ for a partition of $n=(a-\frac{\lambda}{2})(\lambda+1)+\Theta$, $\Theta<\lambda+1$ having $\ge k$ parts

$$\begin{aligned}&1+2+\cdots+k\\ &=1+2+\cdots+(\tfrac{\lambda}{2}+\alpha)\qquad\text{where } k=\tfrac{\lambda}{2}+\alpha, 1\le\alpha<\tfrac{\lambda}{2}.\\ &=(\tfrac{\lambda}{2}+\alpha)+(\tfrac{\lambda}{2}-\alpha+1)+\cdots+(\tfrac{\lambda}{2}+1)+\tfrac{\lambda}{2}+1+2+\cdots+(\tfrac{\lambda}{2}-\alpha)\\ &=(k-\tfrac{\lambda}{2})(\lambda+1)+1+2+\cdots+(\tfrac{\lambda}{2}-\alpha)\\ &>(a-\tfrac{\lambda}{2})(\lambda+1)+\lambda+1>n,\end{aligned}$$

And hence there are no partitions of n violating S^* in this case.

Thus it suffices to consider the case when $k=a$. If a partition violates S^* then there exists a partition

(16) $$n = b_1 + \cdots + b_i + \cdots + b_{i+k-1} + \cdots + b_k + \cdots + b_s$$

and an integer i with $b_i - b_{i+k-1} < \lambda + 1$. If $b_{i+k-1} \geq \lambda + 1$, then the number being partitioned is

$$\geq (\lambda+1) + \cdots + (\lambda+1) + \cdots$$

$$\geq k(\lambda+1) \geq (a - \tfrac{\lambda}{2} + 1)(\lambda+1) > n.$$

Thus let $b_{i+k-1} < \lambda + 1$. If $b_i < \lambda + 1$ then (16) contains at least k parts $\leq \lambda$ and hence $\sum_{i=1}^{\lambda} f_i \geq k$ which implies that such a partition violates S_1.

Let $b_{i+k-1} < \lambda + 1$ and $b_i \geq \lambda + 1$. Since $n = (a - \frac{\lambda}{2})(\lambda+1) + \Theta, \Theta < \lambda + 1$, the number of parts $\geq \lambda + 1$ among $b_i, \cdots, b_{i+k-1}$ is $\leq a - \frac{\lambda}{2}$. If $a - \frac{\lambda}{2}$ parts are equal to $\lambda + 1$, then $f_{\lambda+1} = a - \frac{\lambda}{2}$ and the remaining $k - a + \frac{\lambda}{2}$ parts are $\leq \lambda$ and hence

$$f_1 + \cdots + f_\lambda + f_{\lambda+1} \geq k - a + \tfrac{\lambda}{2} + a - \tfrac{\lambda}{2} = k$$

and such a partition violates S.

If a partition of a number violates S^* and if there are parts $> \lambda + 1$ then the number being partitioned is

(17) $$(\lambda + x_\alpha) + (\lambda + x_{\alpha-1}) + \cdots + (\lambda + x_1) + y_1 + \cdots + y_{k-\alpha}$$

where $\alpha < a - \frac{\lambda}{2}, 1 \leq x_1 < x_2 < \cdots < x_\alpha$ and $y_1, \cdots, y_{k-\alpha}$ are among $1, 2, \cdots, \lambda$. Since $b_i - b_{i+k-1} < \lambda + 1$ we have $\lambda + x_\alpha - y_{k-\alpha} < \lambda + 1$ which implies $x_\alpha - y_{k-\alpha} < 1$ and hence $x_\alpha = y_{k-\alpha}$. If $y_{k-\alpha} = x_\alpha > 1$ then (17) is

$$\geq \alpha(\lambda+1) + (k - \alpha + 1) + \cdots + 3 + 2 + 1$$

$$= \alpha(\lambda+1) + (\tfrac{\lambda}{2} + \beta - \alpha + 1) + \cdots + 2 + 1 \qquad \text{where } k = \tfrac{\lambda}{2} + \beta, 1 \leq \beta < \tfrac{\lambda}{2}.$$

$$= \alpha(\lambda+1) + (\beta - \alpha + 1)(\lambda+1) + 1 + 2 + \cdots + (\tfrac{\lambda}{2} - \beta + \alpha - 1)$$

$$= (\beta+1)(\lambda+1) + 1 + 2 + \cdots + (\tfrac{\lambda}{2} - \beta + \alpha - 1)$$

$$= (k - \tfrac{\lambda}{2} + 1)(\lambda+1) + 1 + 2 + \cdots + (\tfrac{\lambda}{2} - \beta + \alpha - 1) > n.$$

From this it is clear that if a partition of $(a - \frac{\lambda}{2})(\lambda+1) + \Theta$, $\Theta < \lambda + 1$, violates S^* then it does not contain a part $> \lambda + 1$ and hence all the parts will be among $1, 2, \cdots, \lambda + 1$. This implies that

$$f_1 + \cdots + f_{\lambda+1} \geq k = a \nleq a - 1$$

and hence such a partition violates S. This completes the proof of (3).

PROOF OF (4). First part of (4) can be proved on the same lines of (3). The second part of (4) is the case $k = a$ of the Conjecture.

As in the proof of (3) we can show that every partition in P'_B has an associate in P'_A except

$$(a - \tfrac{\lambda}{2} + 1)(\lambda+1)$$

and this proves (4).

CASE 2. Let λ be odd.

PROOF OF (5). We prove (5) by establishing the following stronger result

(18) $$P_{B_{\lambda,k,a}}(n) = P_D(n) = P_{A_{\lambda,k,a}}(n) \text{ for } n \leq \lambda.$$

From the definitions of $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ it is clear that $P_{A_{\lambda,k,a}}(n) = P_D(n)$ and that $P_B(n) \subset P_D(n)$. On the other hand, if $\pi \in P_D(n)$ then $f_i \leq 1$ for $i = 1, 2, \cdots, \lambda$ and $f_{\lambda+1} = 0$ as $n \leq \lambda$. Also

$$f_{\frac{\lambda+1}{2}}+\cdots+f_\lambda \le 1$$

and

$$f_1+\cdots+f_\lambda = f_1+\cdots+f_{\frac{\lambda-1}{2}}+f_{\frac{\lambda+1}{2}}+\cdots+f_\lambda \le \frac{\lambda-1}{2}+1=\frac{\lambda+1}{2}$$

But $f_1+\cdots+f_\lambda=\frac{\lambda+1}{2}$ implies that the number being partitioned is $\ge 1+2+\cdots+\frac{\lambda-1}{2}+\frac{\lambda+1}{2}>\lambda$ Thus $f_1+\cdots+f_\lambda \le \frac{\lambda-1}{2} \le a-1$ since $\frac{\lambda-1}{2}<a$. Consider

$$f_2+\cdots+f_{\lambda-1} \le f_2+\cdots+f_{\frac{\lambda-1}{2}}+1 \le (\frac{\lambda-1}{2}-1)+1=\frac{\lambda-1}{2}.$$

As before if $f_2+\cdots+f_{\frac{\lambda-1}{2}}=\frac{\lambda-1}{2}$ then the number being partitioned $\ge 2+3+\cdots+\frac{\lambda-1}{2}>\lambda$ and hence $f_2+\cdots+f_{\lambda-1} \le \frac{\lambda-1}{2}-1 \le a-2$ since $\frac{\lambda-1}{2}<a$. Proceeding like this we arrive at $f_{\frac{\lambda+1}{2}} \le 1$ as $n \le \lambda$ from which we obtain $f_{\frac{\lambda+1}{2}} \le a-\frac{\lambda+1}{2}$.

For $\pi \in P_D(n)$ and $n \le \lambda$ the condition on b's is satisfied since no partition of n has more than $\frac{\lambda+1}{2}$ parts. This proves that $P_D(n) \subset P_B(n)$ and hence (5) is established.

PROOF OF (6). From the definitions of $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ it is clear that

$$P'_A(\lambda+1)=\begin{cases} \frac{\lambda+3}{2}+\frac{\lambda-1}{2} \text{ when } a=\frac{\lambda+1}{2} \\ \frac{\lambda+1}{2}+\frac{\lambda+1}{2} \text{ when } a>\frac{\lambda+1}{2} \end{cases}$$

and $P'_B(\lambda+1)=\{(\lambda+1)\}$

PROOF OF (7). For $n=(\lambda+1+\Theta)$, $\quad \Theta<\frac{\lambda+1}{2}$

$$P'_A(n)=\begin{cases} \frac{\lambda+3}{2}+\frac{\lambda-1}{2}+\pi:\pi\in P_D(\Theta) & \text{with parts } <\frac{\lambda-1}{2},\ \Theta<\frac{\lambda-1}{2},\ a=\frac{\lambda+1}{2} \\ \frac{\lambda+5}{2}+\frac{\lambda-1}{2}+\frac{\lambda-3}{2},\ \frac{\lambda+3}{2}+\frac{\lambda-1}{2}+\pi:\pi\in P_D(\Theta) & \text{with parts } <\frac{\lambda-1}{2},\ \Theta=\frac{\lambda-1}{2}, a=\frac{\lambda+1}{2} \\ \frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta) & \text{and } a>\frac{\lambda+1}{2} \end{cases}$$

$$P'_B(n)=\{(\lambda+1)+\pi:\pi\in P_D(\Theta)\}$$

PROOF of (8). Clearly

$$P'_A(n)=\begin{cases} \frac{\lambda+5}{2}+\frac{\lambda-1}{2}+\pi:\pi\in P_D(\frac{\lambda-1}{2}) & \text{with parts } <\frac{\lambda-1}{2} \text{ and } a=\frac{\lambda+1}{2} \\ \frac{\lambda+3}{2}+\frac{\lambda+1}{2}+\frac{\lambda-1}{2}, \frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\frac{\lambda+1}{2}) \text{ and } a=\frac{\lambda+3}{2} \\ \frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\frac{\lambda+1}{2}) \text{ and } a>\frac{\lambda+3}{2} \end{cases}$$

$$P'_B(n)=\begin{cases} \frac{3}{2}(\lambda+1), (\lambda+1)+\pi:\pi\in P_D(\frac{\lambda+1}{2}) & \text{with parts } <\frac{\lambda+1}{2} \text{ and } a=\frac{\lambda+1}{2} \\ \frac{3}{2}(\lambda+1), (\lambda+1)+\pi:\pi\in P_D(\frac{\lambda+1}{2}) \text{ and } a=\frac{\lambda+3}{2} \\ (\lambda+1)+\pi:\pi\in P_D(\frac{\lambda+1}{2}) \text{ and } a>\frac{\lambda+3}{2} \end{cases}$$

When $a = \frac{\lambda+1}{2} = k$, the n in the conjecture becomes $\frac{3}{2}(\lambda+1)$ and $\frac{3}{2}(\lambda+1) \in P'_B$ has no associate in P'_A and this establishes the conjecture when $k = a = \frac{\lambda+1}{2}$.

PROOF OF (9). Let $n = (2a-\lambda+1)(\frac{\lambda+1}{2}) + \Theta, \Theta < \frac{\lambda+1}{2}$. Now $\pi \in P'_A(n)$ implies π violates one of the conditions $S_1, \cdots, S_{\frac{\lambda+1}{2}}, S, S^*, S^{**}$ where S^{**} is the condition "no parts $\not\equiv 0 \pmod{\lambda+1}$ are repeated". A proof similar to that of Step $\frac{\lambda}{2}+2$ of even λ will show that partitions violating S^* will also violate S_1. Since no part is $\equiv \lambda+1 \pmod{2\lambda+2}$ for partitions enumerated by $A_{\lambda,k,a}(n)$ we have $f_{\lambda+1} = 0$ and hence S reduces to S_1. In the following steps 1 to $\frac{\lambda+3}{2}$, we enumerate the partitions in P_A violating $S_{\frac{\lambda+1}{2}}, \cdots, S_1, S^{**}$ and also give the bijection of $P'_A(n)$ onto $P'_B(n)$.

STEP 1. Consider $S_{\frac{\lambda+1}{2}}$: $f_{\frac{\lambda+1}{2}} \le 1 \le (a - \frac{\lambda+1}{2})$. Clearly there are no partitions in P_A violating $S_{\frac{\lambda+1}{2}}$ for $a - \frac{\lambda+1}{2} \ge 1$. Since $\frac{\lambda+1}{2}$ is not a part of partitions enumerated by both $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ when $a = \frac{\lambda+1}{2}$ it follows that there are no partitions violating $S_{\frac{\lambda+1}{2}}$ when $a = \frac{\lambda+1}{2}$ also.

STEP 2. Consider $S_{\frac{\lambda-1}{2}}$: $f_{\frac{\lambda-1}{2}} + f_{\frac{\lambda+1}{2}} + f_{\frac{\lambda+3}{2}} \le 3 \le a - \frac{\lambda-1}{2}$

For $a \ge \frac{\lambda+5}{2}$ there are no partitions in P_A violating $S_{\frac{\lambda-1}{2}}$. If $a = \frac{\lambda+1}{2}$, then $n = (\lambda+1)+\Theta$, $\Theta < \frac{\lambda+1}{2}$ and the set of partitions violating $S_{\frac{\lambda-1}{2}}$ is $\left\{\frac{\lambda+3}{2} + \frac{\lambda-1}{2} + \pi : \pi \in P_D(\Theta)\right\}$
For each partition in the above set we associate $(\lambda+1)+\pi$ in P'_B. Let $a = \frac{\lambda+3}{2}$. Then $n = 2(\lambda+1)+\Theta$, $\Theta < \frac{\lambda+1}{2}$ and the set of partitions violating $S_{\frac{\lambda-1}{2}}$ is

$$\left\{\frac{\lambda+3}{2} + \frac{\lambda+1}{2} + \frac{\lambda-1}{2} + \pi : \pi \in P_D(\frac{\lambda+1}{2}+\Theta) \quad \text{with parts } < \frac{\lambda-1}{2}\right\}$$

$$\cup \left\{(\frac{\lambda+1}{2}+\Theta') + \frac{\lambda+3}{2} + \frac{\lambda+1}{2} + \frac{\lambda-1}{2} + \pi : \pi \in P_D(\Theta-\Theta'),\ 2 \le \Theta' \le \frac{\lambda-1}{2}\right\}$$

We associate $\frac{3}{2}(\lambda+1)+\pi \in P'_B$ for every partition in the first set while for a partition in the second set we associate $\frac{3}{2}(\lambda+1) + (\frac{\lambda+1}{2}+\Theta') + \pi$ in P'_B.

Proceeding like this we arrive at the following step.

STEP $\frac{\lambda+1}{2}$. Consider $S_1: f_1 + \cdots + f_\lambda \le a-1$. By the definition of $A_{\lambda,k,a}(n)$, $f_i \le 1$ for all $i = 1, \cdots, \lambda$ except for $i = \frac{\lambda+1}{2}$. But $1 \le f_{\frac{\lambda+1}{2}} \le 2a-\lambda+1$. The case $f_{\frac{\lambda+1}{2}} > 1$ will be considered in step $\frac{\lambda+3}{2}$. Hence let us now assume $f_{\frac{\lambda+1}{2}} \le 1$.
In this case $f_1 + \cdots + f_\lambda \le \lambda$. If $f_1 + \cdots + f_\lambda = \lambda$, then $1+2+\cdots+\lambda = \frac{\lambda}{2}(\lambda+1) = \frac{\lambda-1}{2}(\lambda+1) + \frac{\lambda+1}{2} \ge (a - \frac{\lambda-1}{2})(\lambda+1) + \frac{\lambda+1}{2} > n$. Thus there are no partitions violating S_1 in P'_A. Let $f_1 + \cdots + f_\lambda = \lambda - 1$ and let the deleted part be x. Consider

$$1+2+\cdots+(x-1)+(x+1)+\cdots+(\lambda-1)+\lambda \tag{19}$$
$$= (\lambda-2)(\frac{\lambda+1}{2}) + (\lambda+1-x) \text{ where } 1 \le (\lambda+1-x) \le \lambda.$$

If $2a-\lambda+1 < \lambda-2$ then (19) is $> n$ and hence there will be no partitions of n violating S_1. Clearly $2a-\lambda+1 \ne \lambda-2$. When $2a-\lambda+1 > \lambda-2$ the only partition of n violating S_1 is

$$\lambda + (\lambda-1) + \cdots + (x+1) + (x-1) + \cdots + 2 + 1 \qquad \text{with } \frac{\lambda+1}{2} - x = \Theta$$

for which we associate the following partition in P'_B

$$\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{\frac{\lambda-3}{2}\ times}+(\frac{\lambda+1}{2}+\Theta)+\frac{\lambda+1}{2}$$

More generally, let $f_1+\cdots+f_\lambda=\lambda-y$ $(1\le y\le\lambda-a)$ and let $x_1,\cdots,x_y$ with $1\le x_1<x_2<\cdots<x_y\le\lambda$ be the parts deleted among $1,2,\cdots,\lambda$. Then

$$\begin{aligned}(20)\qquad &\lambda+(\lambda-1)+\cdots+(x_y+1)+(x_y-1)+\cdots+(x_1+1)+(x_1-1)+\cdots+2+1\\ &=(\lambda-2y)(\frac{\lambda+1}{2})+(\lambda+1-x_1)+\cdots+(\lambda+1-x_y).\end{aligned}$$

If $2a-\lambda+1<\lambda-2y$ then (20) is $>n$ and hence there are no partitions of n violating S_1. Also $2a-\lambda+1\ne\lambda-2y$. Let $2a-\lambda+1>\lambda-2y$. Then $\lambda-2y+1\le 2a-\lambda+1\le\lambda-1$. If $2a-\lambda+1>\lambda-2y+1$ then $f_1+\cdots+f_\lambda=\lambda-y\le a-1$ and hence there will be no partitions of n violating S_1. If $2a-\lambda+1=\lambda-2y+1$ and if $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)>\frac{\lambda+1}{2}+\Theta$ then (20) is $>n$. On the other hand, if $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)<\frac{\lambda+1}{2}+\Theta$ then also there are no partitions of n violating S_1 since in this case parts have to be repeated. Since $\frac{\lambda+1}{2}+\Theta<\lambda+1$ we note that $(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)=\frac{\lambda+1}{2}+\Theta$ is possible only if

(a) $x_1<\frac{\lambda+1}{2}$, $x_2=\frac{\lambda+1}{2}$ and $x_i>\frac{\lambda+1}{2}$ for $i=3,\cdots y$

(b) $x_1<\frac{\lambda+1}{2}$ and $x_i>\frac{\lambda+1}{2}$ for $i=2,\cdots,y$

(c) $x_1=\frac{\lambda+1}{2}$ and $x_i>\frac{\lambda+1}{2}$ for $i=2,\cdots,y$

(d) $x_i>\frac{\lambda+1}{2}$ for $i=1,\cdots,y$

In each of the cases (a)-(d) the partition on the left hand side of (20) violates S_1 for which we respectively associate the following partitions in P'_B.

$$\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{(\frac{\lambda-2y+1}{2})\ times}+(\lambda+1-x_1)+(\lambda+1-x_3)+\cdots+(\lambda+1-x_y)$$

$$\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{(\frac{\lambda-2y-1}{2})\ times}+(\lambda+1-x_1)+\frac{\lambda+1}{2}+(\lambda+1-x_2)+\cdots+(\lambda+1-x_y)$$

$$\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{(\frac{\lambda-2y+1}{2})\ times}+(\lambda+1-x_2)+\cdots+(\lambda+1-x_y)$$

$$\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{(\frac{\lambda-2y-1}{2})\ times}+\frac{\lambda+1}{2}+(\lambda+1-x_1)+\cdots+(\lambda+1-x_y)$$

STEP $\frac{\lambda+3}{2}$. Consider S^{**}: 'no parts $\not\equiv 0$ $(mod\ \lambda+1)$ are repeated'. This implies that $f_{\frac{\lambda+1}{2}}\ge 2$.

When $a=\frac{\lambda+1}{2}$ there are no partitions violating S^{**} since $\frac{\lambda+1}{2}$ is not a part for partitions enumerated by both $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$.

Let $a=\frac{\lambda+3}{2}$. Then $n=2(\lambda+1)+\Theta,\Theta<\frac{\lambda+1}{2}$. The set of partitions in P'_A violating S^{**} is

$$\left\{\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta)\right\}$$

$$\cup\left\{\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\frac{\lambda+1}{2}+\Theta)\quad \text{with parts } <\frac{\lambda+1}{2}\right\}$$

$$\cup\left\{(\frac{\lambda+1}{2}+\Theta')+\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta-\Theta'),1\le\Theta'\le\frac{\lambda-1}{2}\right\}$$

$$\cup\left\{\frac{\lambda+1}{2}+\frac{\lambda+1}{2}+\pi:\pi\in P_D(\lambda+1+\Theta)\quad \text{with parts } <\frac{\lambda+1}{2}\right\}$$

$$\cup\left\{(\tfrac{\lambda+1}{2}+\Theta')+\tfrac{\lambda+1}{2}+\tfrac{\lambda+1}{2}+\pi:\pi\in P_D(\tfrac{\lambda+1}{2}+\Theta-\Theta')\text{ parts }<\tfrac{\lambda+1}{2},\ 1\le\Theta'\le\tfrac{\lambda-1}{2}\right\}$$

$$\cup\left\{(\tfrac{\lambda+1}{2}+\Theta'')+(\tfrac{\lambda+1}{2}+\Theta')+\tfrac{\lambda+1}{2}+\tfrac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta-\Theta'-\Theta''),1\le\Theta'<\Theta''\le\tfrac{\lambda-1}{2}\right\}$$

For each of the above sets of partitions in P'_A we respectively associate the following sets of partitions in P'_B.

$$\left\{\tfrac{3}{2}(\lambda+1)+\tfrac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta)\right\}$$

$$\cup\left\{(\lambda+1)+(\tfrac{\lambda+1}{2})+\pi:\pi\in P_D(\tfrac{\lambda+1}{2}+\Theta)\text{ parts }<\tfrac{\lambda+1}{2}\right\}$$

$$\cup\left\{(\lambda+1)+(\tfrac{\lambda+1}{2}+\Theta')+\tfrac{\lambda+1}{2}+\pi:\pi\in P_D(\Theta-\Theta'),\ 1\le\Theta'\le\tfrac{\lambda-1}{2}\right\}$$

$$\cup\left\{(\lambda+1)+\pi:\pi\in P_D(\lambda+1+\Theta)\text{ parts }<\tfrac{\lambda+1}{2}\right\}$$

$$\cup\left\{(\lambda+1)+(\tfrac{\lambda+1}{2}+\Theta')+\pi:\pi\in P_D(\tfrac{\lambda+1}{2}+\Theta-\Theta')\text{ parts }<\tfrac{\lambda+1}{2},\ 1\le\Theta'\le\tfrac{\lambda-1}{2}\right\}$$

$$\cup\left\{(\lambda+1)+(\tfrac{\lambda+1}{2}+\Theta'')+(\tfrac{\lambda+1}{2}+\Theta')+\pi:\pi\in P_D(\Theta-\Theta'-\Theta''),\ 1\le\Theta'<\Theta''\le\tfrac{\lambda-1}{2}\right\}$$

For any given 'a' we can similarly enumerate the partitions in P'_A violating S^{**} and also can obtain the bijection of P'_A onto P'_B. The proof of (9) now follows from Steps 1 to $\frac{\lambda+3}{2}$.

PROOF OF (10). The first part of (10) follows on a line similar to the proof of (9). The second part of (10) is the case $k=a$ of the conjecture. As in the proof of (9) we can show that every partition in P'_B has an associate in P'_A except $(2a-\lambda+2)(\frac{\lambda+1}{2})$ and this proves (10).

We now consider some numerical examples.

EXAMPLE 1. Let $\lambda=4, k=3=a, n=\binom{k+\lambda-a+1}{2}+(k-\lambda+1)(\lambda+1)=10$.

TABLE 1

n	$P_{A_{4,3,3}}(n)$	$P_{B_{4,3,3}}(n)$
1	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$
3	$\{3,2+1\}$	$\{3,2+1\}$
4	$\{4,3+1\}$	$\{4,3+1\}$
5	$\{4+1\}\cup\{3+2\}$	$\{4+1\}\cup\{5\}$
6	$\{6,4+2\}\cup\{3+2+1\}$	$\{6,4+2\}\cup\{5+1\}$
7	$\{7,6+1,4+3\}\cup\{4+2+1\}$	$\{7,6+1,4+3\}\cup\{5+2\}$
8	$\{8,7+1,6+2\}\cup\{4+3+1\}$	$\{8,7+1,6+2\}\cup\{5+3\}$
9	$\{9,8+1,7+2,6+3,6+2+1\}\cup\{4+3+2\}$	$\{9,8+1,7+2,6+3,6+2+1\}\cup\{5+4\}$
10	$\{9+1,8+2,7+3,7+2+1,6+4,6+3+1\}\cup\{4+3+2+1\}$	$\{9+1,8+2,7+3,7+2+1,6+4,6+3+1\}\cup\{10,5+5\}$

According to the proofs of (1)-(4), we have

(a) $P_{A_{4,3,3}}(n)=P_{B_{4,3,3}}(n)$ for $n\le 4$

(b) $P'_{A_{4,3,3}}(5)=\{3+2\}$, $P'_{B_{4,3,3}}(5)=\{5\}$

(c) The partitions enumerated by $A_{4,3,3}(n)$ for $n = 6,7,8,9$ violating S_2 according to Step 1 in the proof of (3) are

$$\{3+2+1\} \cup \{4+3+2\}$$

for which their associates in P'_B are

$$\{5+1\} \cup \{5+4\}$$

(d) The partitions enumerated by $A_{4,3,3}(n)$ for $n = 6,7,8,9$ violating S_1 as proved in Step 2 are

$$\{4+2+1\} \cup \{4+3+1\}$$

for which the corresponding partitions in P'_B are

$$\{5+2\} \cup \{5+3\}$$

(e) The partitions enumerated by $A_{4,3,3}(n)$ for $n = 6,7,8,9$ violating S also violate S_1 or S_2.

(f) The partition $10 = 2\times(4+1) \in P'_{B_{4,3,3}}(10)$ has no associate in P'_A while all other partitions have.

From Table 1 it is clear that (a)-(f) are indeed true.

EXAMPLE 2. Let $\lambda = 5, k = a = 3, n = \binom{k+\lambda-a+1}{2} + (k-\lambda+1)(\lambda+1) = 9$.

TABLE 2

n	$P_{A_{5,3,3}}(n)$	$P_{B_{5,3,3}}(n)$
1	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$
3	$\{2+1\}$	$\{2+1\}$
4	$\{4\}$	$\{4\}$
5	$\{5, 4+1\}$	$\{5, 4+1\}$
6	$\{5+1\} \cup \{4+2\}$	$\{5+1\} \cup \{6\}$
7	$\{7, 5+2\} \cup \{4+2+1\}$	$\{7, 5+2\} \cup \{6+1\}$
8	$\{8, 7+1\} \cup \{5+2+1\}$	$\{8, 7+1\} \cup \{6+2\}$
9	$\{8+1, 7+2, 5+4\}$	$\{8+1, 7+2, 5+4\} \cup \{9\}$

From the proofs of (5)-(8) we have the following:

(g) $P_{A_{5,3,3}}(n) = P_{B_{5,3,3}}(n)$ for $n \le 5$

(h) $P'_{A_{5,3,3}}(6) = \{4+2\}$ $P'_{B_{5,3,3}}(6) = \{6\}$

(i) $P'_{A_{5,3,3}}(7) = \{4+2+1\}$ $P'_{B_{5,3,3}}(7) = \{6+1\}$

(j) $P'_{A_{5,3,3}}(8) = \{5+2+1\}$ $P'_{B_{5,3,3}}(8) = \{6+2\}$

(k) The partition $(2\times 3 - 5 + 2)(\frac{5+1}{2}) = 9$ in $P'_{B_{5,3,3}}(9)$ has no associate in $P'_{A_{5,3,3}}(9)$ while all others have.

From Table 2 it is evident that the results (g)-(k) are true.

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REFERENCES

1. ANDREWS, G.E., On the general Rogers-Ramanujan theorem, Mem. Amer. Math. Soc. No. 152 (1974), 1-86.