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COMPLETELY POSITIVE LINEAR OPERATORS FOR BANACH SPACES

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ABSTRACT. Using ideas of Pisier, the concept of complete positivity is generalized in a different direction in this paper, where the Hilbert space $\mathcal{H}$ is replaced with a Banach space and its conjugate linear dual. The extreme point results of Arveson are reformulated in this more general setting.

KEY WORDS AND PHRASES: Banach spaces, completely positive operators, extreme points, pure elements.

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1. INTRODUCTION.

In [6], Pisier studied completely bounded maps from a C^*-algebra to $B(X, Y)$, the space of bounded operators between two arbitary Banach spaces X and Y. Of course , there is a generalization of ordinary completely bounded maps. In this paper, we first define complete positivity for a map from C^*-algebra to $B(X, \overline{X^*})$, where $\overline{X^*}$ denotes the antilinear dual space of X (the set of all conjugate linear functionals on X). Then we give a representation theorem, and give complete solutions to three extremal problems.

In this paper, the C^*-algebra A always has an identity.

2. COMPLETELY POSITIVE OPERATORS.

DEFINITION 2.1. Let X be a Banach space, and $T \in B(X, \overline{X^*})$. We call T positive if, for all positive integers n and $x_1, \dots, x_n \in X$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} T(x_i)(x_j) \geq 0.$$

REMARK 2.2. We have $\overline{\ell_2^n(X)^*} = \ell_2^*(\overline{X^*})$, and so $M_n(B(X, \overline{X^*})) = B(\ell_2^n(X), \ell_2^n(\overline{X^*})) = B(\ell_2^n(X), \overline{\ell_2^n(X)^*})$. Thus we may define positivity for $M_n(B(X, \overline{X^*}))$.

DEFINITION 2.3. Let A be a C^*-algebra, ϕ a linear map from A to $B(X, \overline{X^*})$ and let $\phi_n(a_{ij}) = (\phi(a_{ij}))$ for $(a_{ij}) \in M_n(A)$. If ϕ_n is positive for all n, then we say ϕ is completely positive.

THEOREM 2.4. Let $\phi : A \longrightarrow B(X, \overline{X^*})$ be a completely positive map. There is a Hilbert space $\mathcal{H}$, a representation π of A on $\mathcal{H}$ and a bounded operator $V \in B(X, \mathcal{H})$ such that, for all $a \in A$,

$$\phi(a) = \overline{V^*}\pi(a)V,$$

and $\mathcal{H} = [\pi(A)VX]$, where $\overline{V^*}(h)(x) = < h, V(x) >$, for all $h \in \mathcal{H}, x \in X$.

PROOF: Consider the vector space tensor product $A \otimes X$ and define a bilinear form as follows: If $u = x_1 \otimes \xi_1 + \ldots + x_m \otimes \xi_m$, $v = y_1 \otimes \eta_1 + \ldots + y_n \otimes \eta_n$,

$$< u, v >= \sum_{i,j} (\phi(y_i^* x_j)(\xi_j))(\eta_i).$$

Because ϕ is completely positive, we have the fact that $<,>$ is positive semi-definite. For each $a \in A$, define a linear transformation $\pi_0(a)$ on $A \otimes X$ by

$$\pi_0(a)(\sum_{j=1}^{n} x_j \otimes \xi_j) = \sum (a x_j) \otimes \xi_j.$$

π_0 is an algebra homomorphism for which

$$< u, \pi_0(a) v >=< \pi_0(a^*) u, v >$$

for all $u, v \in A \otimes X$.

For fixed u, $\rho(a) =< \pi_0(a)u, u >$ defines a positive linear functional on A; i.e, $\rho(a^*a) \geq 0$. Hence, $< \pi_0(a)u, \pi_0(a)u >=< \pi_0(a^*a)u, u >= \rho(a^*a) \leq \|a^*a\| \rho(1) = \|a\|^2 < u, u >$, where 1 is the identity of A.

Now let $R = \{u \in A \otimes X :< u, u >= 0\}$. R is a linear subspace $A \otimes X$, invariant under $\pi_0(a)$, for all $a \in A$. So $<,>$ determines a positive definite inner product on the quotient $(A \otimes X)/R$ in the usual way.

Let $\mathcal{H} = \overline{(A \otimes X)/R}$. There is a unique representation π of A on $\mathcal{H}$ such that

$$\pi(a)(u + R) = \pi_0(a)u + R$$

$a \in A$, $u \in A \otimes X$.

We define a linear map V: $X \longrightarrow \mathcal{H}$ by

$$V(\xi) = 1 \otimes \xi + R$$

for all $\xi \in X$.

We may verify that V is bounded, and $\phi(a) = \overline{V^*}\pi(a)V$ for all $a \in A$.

Let $R_1 = [\pi(A)VX] \subseteq \mathcal{H}$, and $\pi_1(a) = \pi(a)|_{R_1}$ for all $a \in A$. Because $\pi(1) = I$, so $V(X) \subseteq R_1$. We have $\overline{V^*}\pi(a)V(x_1) = \overline{V^*}\pi(a)|_{R_1}V(x_1) = \overline{V^*}\pi_1(a)V(x_1) = \phi(a)(x_1)$, for all $x_1 \in X$, $a \in A$. So we may assume that $\mathcal{H} = [\pi(A)VX]$.

Suppose ϕ: $A \longrightarrow B(X, \overline{X^*})$ is a completely positive map. If there exists Hilbert spaces $\mathcal{H}_i$, representations π_i of A on $\mathcal{H}_i$, and bounded operators $V_i : X \longrightarrow \mathcal{H}_i$ then

$$\phi(a) = \overline{V_i^*}\pi_i(a)V_i,$$

for $i = 1, 2$, where $\mathcal{H}_i = [\pi_i(A)V_iX]$. Define U: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ by

$$U(\sum_{i=1}^{n} \pi_1(a_i)V_1\xi_i) = \sum_{i=1}^{n} \pi_i(a_i)V_2\xi_i,$$

for all $a_1, \ldots, a_n \in A, \xi_1, \ldots, \xi_n \in X$. Then we need to extend to $\mathcal{H}_1$. We may verify that $UV_1 = V_2$ and $U\pi_1(a) = \pi_2(a)U$ for all $a \in A$.

Next we verify that U is an unitary.

$$\begin{aligned}
<\sum_{i=1}^{n}\pi_2(a_i)V_2\xi_i,\sum_{i=1}^{n}\pi_2(a_i)V_2\xi_i> &= \sum_i\sum_j <\pi_2(a_i)V_2\xi_i,\pi_2(a_j)V_2\xi_j> \\
&= \sum_i\sum_j <\pi_2(a_j^*a_i)V_2\xi_i,V_2\xi_j> \\
&= \sum_i\sum_j \phi(a_j^*a_i)(\xi_i)(\xi_j) \\
&= \sum_i\sum_j <\pi_1(a_j^*a_i)V_1\xi_i,V_1\xi_j> \\
&= <\sum_i\pi_1(a_i)V_1\xi_i,\sum_i\pi_1(a_i)V_1\xi_i>.
\end{aligned}$$

So the representation given in Theorem 2.4 is unique up to unitary equivalence.

3. PREPARATIONS.

NOTATION 3.1. Let $CP(A,X)$ denote all completely positive linear maps from A to $B(X,\overline{X^*})$.

LEMMA 3.2. Let ϕ_1 and ϕ_2 belong to $CP(A,X)$, and suppose that $\phi_1 \leq \phi_2$. Let $\phi_i(a) = \overline{V_i^*}\pi_i(a)V_i$ be the canonical expression of ϕ_i, where π_i is a representation of A on R_i such that $[\pi_i(A)V_iX] = R_i$, $i = 1,2$. Then there exsists a contraction $T \in B(R_2,R_1)$ such that

$$TV_2 = V_1,$$

$$T\pi_2(a) = \pi_1(a)T$$

for all $a \in A$.

PROOF: For every $\xi_1\dots,\xi_n \in X, a_1,\dots,a_n \in A$,

$$\begin{aligned}
\|\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j\|^2 &= <\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j,\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j> \\
&= \sum_i\sum_j \pi_1(a_j^*a_i)V_1(\xi_i)(\xi_j) \\
&= \sum_i\sum_j \phi_1(a_j^*a_i)(\xi_i)(\xi_j) \\
&\leq \sum_i\sum_j \phi_2(a_j^*a_i)(\xi_i)(\xi_j) \\
&= \|\sum_{j=1}^{n}\pi_2(a_j)V_2\xi_j\|^2
\end{aligned}$$

Define T: $R_2 \longrightarrow R_1$ by

$$T(\sum_{j=1}^{n}\pi_2(a_j)V_2\xi_j) = \sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j$$

We can verify that above two statements hold.

NOTATION 3.3. For $\phi \in CP(A,X)$, let $[0,\phi] = \{\psi \in CP(A,X);\psi \leq \phi\}$. Let $\phi(a) = \overline{V^*}\pi(a)V$ for all $a \in A$. For each operator $T \in \pi(A)'$, define a map $\phi_T(a) = \overline{V^*}T\pi(a)V$. Then $T \longrightarrow \phi_T$ is linear. If $\phi_T = 0$, we have

$$<T\pi(a)V\xi,\pi(b)V\eta> = <T\pi(b^*a)V\xi,V\eta> = \phi_T(b^*a)(\xi)(\eta) = 0$$

$$<T(\sum_{i=1}^{n}\pi_i(a_i)V\xi_i),\sum_{i=1}^{n}\pi(b_j)V\xi_j> = 0.$$

So $T = 0$. That is, $T \longrightarrow \phi_T$ is injective.

THEOREM 3.4. $T \longrightarrow \phi_T$ is an affine order isomorphism of the partially ordered convex set of $\{T \in \pi(A)' : 0 \leq T \leq I\}$ onto $[0, \phi]$.

The proof of this theorem is exactly the same way as the proof of theorem in Arveson's paper [1].

4. THE THREE EXTREMAL PROBLEMS.

Now we come to discuss three extremal problems.

DEFINITION 4.1. A completely positive map $\phi \in CP(A,X)$ is pure if, for every $\psi \in CP(A,X)$, $\psi \leq \phi$ implies that ψ is a scalar multiple of ϕ.

REMARK 4.2. According to [3], the extreme rays of $CP(A,X)$ can be characterized as the half lines $\{t\phi : t \geq 0\}$, where ϕ is a pure element of $CP(A,X)$.

We state the following theorems without proofs, for the proofs are almost the same as those in Arveson's paper [1].

THEOREM 4.3. All nonzero pure elements of $CP(A,X)$ are precisely those of the form $\phi(a) = \overline{V^*}\pi(a)V$, where π is an irreducible representation of A on some Hilbert space R and $V \in B(X,R)$, such that $R = [\pi(A)VX]$.

THEOREM 4.4. Let $\phi \in CP(A,X)$ and let $\phi(a) = \overline{V^*}\pi(a)V$ be its canonical representation. The extreme points of $[0,\phi]$ are those maps of the form $\overline{V^*}P\pi(a)V$, where P is a projection in $\pi(A)'$.

We consider the extreme points of the set $CP(A,X;K) = \{\phi \in CP(A,X); \phi(1) = K\}$, where K is a fixed positive operator in $B(X, \overline{X^*})$.

THEOREM 4.5. Let $\phi \in CP(A,X;K)$ and let $\pi(a) = \overline{V^*}\pi(a)V$ be its canonical representation with $\overline{V^*}V = K$. Then ϕ is an extreme point of $CP(A,X;K)$ if and only if $[VX]$ is a faithful subspace for the commutant $\pi(A)'$ of $\pi(A)$.

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