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ON THE SPEED OF CONVERGENCE OF ITERATION OF A FUNCTION

VLADIMIR DROBOT

Department of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192

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ABSTRACT. Let $f_n(x)$ be the n^{th} iterate of a function in some interval [0, c]. It is known that if $f(x) \sim x - x^{\alpha}$, $\alpha > 1$, then $f_n(x) \sim An^a$ for some A and a. In this paper we prove a converse of this theorem: The rate of convergence of the iterates determines the form of a function.

KEY WORDS AND PHRASES: Iterations of a function, slow convergence.
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Let $f(x)$ be a real valued function; denote the n^{th} iterate of $f(x)$ by $f_n(x)$, i.e., $f_0(x) = x$, $f_{n+1}(x) = f(f_n(x))$. If on some interval $[0,c]$ the function f is continuous and satisfies the inequality $0 < f(x) < x$ $(x \neq 0)$, then $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in [0,c]$. Indeed, for every such x, $f_n(x)$ is monotonically decreasing and it is easy to see that the limit must be 0. The rates of convergence of the sequence $f_n(x)$ have been studied extensively, see Ostrowski [1] or Seneta [3]. If $f'(0) < 1$, the sequence converges at least geometrically fast: There is a constant $0 < \gamma < 1$ such that $f_n(x) < \gamma^n$ for large n. The situation is more delicate when $f'(0) = 1$. This is known as "slow convergence problem". A. M. Ostrowski [1] has proved the following result:

THEOREM 1. Suppose $f(x)$ is a continuous increasing function on some interval $[0,c]$ such that $0 < f(x) < x$ for $0 < x \leq c$. If $f(x) = x - Kx^p + o(x^p)$ as $x \to 0$, where $K > 0, p > 1$, then for all $x \in [0,c]$

$$\lim_{n\to\infty} n^a f_n(x) = A$$

where $a - ap + 1 = 0$ and $a - KA^{p-1} = 0$.

These sufficient conditions for $f_n(x)$ to behave like An^{-a} are also, in some sense, necessary, as the next theorem shows. We recall that a function $f(x)$ is said to be concave if

$$f(ux + (1-u)y) \geq uf(x) + (1-u)f(y)$$

for any x, y in the domain and $0 \leq u \leq 1$. If the function $f(x)$ is concave, then "the slopes decrease": For $x_1 < x_2 < x_3$ we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \geq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

For the proof see, for instance, Rudin [2].

THEOREM 2. Suppose $f(x)$ is an increasing, continuous, and concave function on $[0,c]$ satisfying $0 < f(x) < x$ $(x \neq 0)$. Suppose also that for some fixed $a > 0$, $\lim_{n\to\infty} n^a f_n(x)$ exists and is different from 0 for every $x \in [0,c]$. Then for every $\epsilon > 0$ there is $c_\epsilon > 0$ such that

$$x - x^{p-\epsilon} \leq f(x) \leq x - x^{p+\epsilon}, \text{ for } 0 \leq x \leq c_\epsilon,$$

where $a - ap + 1 = 0$, i.e., $p = 1 + 1/a$.

We need the following lemma.

LEMMA. Let $0 < V < v$ and let $g(x) = sx - L$ be a linear function with slope $0 < s < 1$ such that $g(v) = V$. Let w be the number such that $g(w) = w$ and let $w < z < v$. Put

$$N = N(v,V,s,z) = \frac{\log\left(z + \frac{v-V}{1-s} - v\right) - \log\left(\frac{v-V}{1-s}\right)}{\log(s)}.$$

If $k > N$ then $g_k(v) < z$, and if $k > N$ then $g_k(v) > z$.

PROOF. Put $t_1 = v - V = v - g(V)$ and $t_{k+1} = g_k(v) - g_{k+1}(v)$, $k = 1,2,\ldots$. Then

$$t_{k+1} = s\left[g_{k-1}(v) - g_k(V)\right] = st_k$$

and so

$$t_1 + t_2 + \ldots t_k = t_1\,(1 + s + s^2 + \ldots + s^{k-1}) = t_1\,\frac{1-s^k}{1-s} = \frac{v-V}{1-s}(1-s^k).$$

But then

$$g_k(v) = v - (t_1 + t_2 + \ldots + t_k) = v - \frac{v-V}{1-s}(1-s^k) = \frac{v-V}{1-s}\,s^k + v - \frac{v-V}{1-s}.$$

This is a decreasing sequence in k, hence $g_k(v) < z$ is equivalent to $k > N$, and $g_k(v) > z$ is equivalent to $k < N$.

PROOF of Theorem 2. It is enough to show that, under the hypothesis of the theorem,

$$\lim_{x\to 0+} \frac{\log(x - f(x))}{\log(x)} = p = 1 + 1/a \tag{1}$$

We break the proof into two parts: I. lim inf $\geq p$, and II. lim sup $\leq p$.

Proof of I. Let $t = \liminf$ and assume that $t < p$. We notice, by the way, that $t \geq 1$ because $|\log(x - f(x)| > |\log(x)|$ for $0 < x < 1$, and $p > 1$. Thus there exists a sequence $c > x_1 \geq x_2 \geq \cdots \to 0$ such that

$$\frac{\log(x_k - f(x_k))}{\log(x_k)} = t_k \to t < p. \tag{2}$$

Since the function $f(x)$ is concave and $f(0) = 0$, the ratio $f(x)/x$ is a decreasing function of x. From (2) it follows that $f(x_k) = x_k - x_k^{t_k}$, and thus

$$s_k = \frac{f(x_k)}{x_k} = 1 - x_k^{t_k - 1}$$

increases as $k \to \infty$, hence $|\log s_k|$ decreases as $k \to \infty$. We may thus require that the sequence $\{x_k\}$ satisfies

$$\frac{\log x_{k+1}}{\log s_{k+1}} \geq 2\frac{\log x_k}{\log s_k} \quad \text{and} \quad \frac{\log x_k}{\log s_k} \geq k \tag{3}$$

Let k be fixed. The slope of the line joining $(0, f(0))$ and $(x_k, f(x_k))$ is equal to $s_k = 1 - x_k^{t_k - 1}$. If $x > x_k$, then the slope of the line joining $(x_k, f(x_k))$ and $(x, f(x))$ is less than s_k (the function $f(x)$ is concave), hence $f(x) \leq s_k x$ for $x \geq x_k$. Define function $g(x)$ by

$$g(x) = s_k x \quad \text{if} \quad x_k \leq x \leq x_{k-1},\ k = 2, 3, \ldots .$$

We have just proved that $f(x) \leq g(x)$, so $f_m(x) \leq g_m(x)$ for all integers m (f is monotone, i.e., $f_2(x) \leq f(g(x)), \ldots, f_m(x) = f(f_{m-1}(x)) \leq f(g_{m-1}(x)) \leq g_m(x)$). Let n_k be the smallest integer such that $g_{n_k}(x_{k-1}) \leq x_k$. We apply the Lemma with $v = x_{k-1}$, $V = f(x_k) = x_{k-1} - x_{k-1}^{t_{k-1}}$, $z = x_k$, $s = V/v$. A simple calculation leads to

$$n_k \leq \frac{\log x_k - \log x_{k-1}}{\log s_k} + 1. \tag{4}$$

We remark that if $y < x_{k-1}$ then $g_{n_k}(y) \leq x_k$. Indeed, if $y < x_k$, the result is immediate since $g(x) < x$; if $x_k \leq y < x_{k-1}$, then $g_n(y) < x_k$ for some $n \leq n_k$, so $g_{n_k - n}(g_n(y)) \leq x_k$. Thus

$$\begin{aligned} f_{n_2}(x_1) &\leq g_{n_2}(x) \leq x_2 \\ f_{n_2+n_3}(x_1) &\leq g_{n_3}(f_{n_2}(x_1)) \leq x_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{n_2+n_3+\ldots+n_k}(x_1) &\leq g_{n_k}(f_{n_2+\ldots+n_{k-1}}(x_1)) \leq x_k. \end{aligned} \tag{5}$$

Setting $N_k = n_2 + n_3 + \ldots + n_k$, the last inequality in (5) becomes $f_{N_k}(x_1) \leq x_k$, which implies that for any $b > 0$

$$N_k^b f_{N_k}(x_1) \leq N_k^b x_k. \tag{6}$$

By hypothesis of the theorem, if $b > a$ then the left side of (6) goes to ∞ as $k \to \infty$. To obtain the desired contradiction we will show that the right hand side of (6) goes to 0 as $k \to \infty$ for some $b > a$. We now estimate N_k. From (4) we obtain

$$N_k = \sum_{m=2}^{k} n_m = k + \sum_{m=2}^{k} \frac{\log x_m - \log x_{m-1}}{\log s_m} \leq k + \sum_{m=2}^{k} \frac{\log x_m}{\log s_m} \tag{7}$$

However, the requirement (3) gives

$$\frac{\log x_{k-1}}{\log s_{k-1}} \le \frac{1}{2}\frac{\log x_k}{\log s_k}$$

$$\frac{\log x_{k-2}}{\log s_{k-2}} \le \frac{1}{2}\frac{\log x_{k-1}}{\log s_{k-1}} \le \left(\frac{1}{2}\right)^2 \frac{\log x_k}{\log s_k}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\log x_2}{\log s_2} \le \frac{1}{2}\frac{\log x_3}{\log s_3} \le \cdots\cdot \le \left(\frac{1}{2}\right)^{k-2} \frac{\log x_k}{\log s_k}.$$

Substituting these in (7) we obtain

$$N_k \le k + \sum_{m=2}^{k} \left(\frac{1}{2}\right)^{m-1} \frac{\log x_k}{\log s_k} \le \frac{\log x_k}{\log s_k} + 2\frac{\log x_k}{\log s_k} = 3\frac{\log x_k}{\log s_k},$$

the last inequality being justified bu (3). It is thus sufficient to show that for some $b > a$

$$\lim_{k\to\infty} \left(\frac{\log x_k}{\log s_k}\right)^b x_k = 0,$$

or, what comes to the same thing

$$\lim_{k\to\infty} \frac{x_k^{1/b} \log x_{k-1}}{\log s_{k-1}} = 0. \tag{8}$$

Now, $t_k - 1$ is monotonically decreasing to $t-1$ (see (2)), hence $t_k - 1 < t - 1 + \epsilon$ for arbitrary ϵ and k sufficiently large. Thus

$$1 - x_k^{t_k - 1} \le 1 - x_k^{t_k - 1 + \epsilon}$$

or

$$|\log s_k| = |\log(1 - x_k^{t_k - 1})| \ge |\log(1 - x_k^{t_k - 1 + \epsilon})|,$$

so

$$\frac{x_k^{1/b}\log x_k}{\log s_k} \le \frac{x_k^{1/b}\log x_k}{\log(1 - x_k^{t_k - 1 + \epsilon})}$$

for arbitrary ϵ and k sufficiently large. To establish (8) it is sufficient now to show that there exists $\epsilon > 0$ and $b > a$ so that

$$\lim_{x\to 0+} \frac{x^{1/b}\log x}{\log(1 - x^{t-1+\epsilon})} = 0, \tag{9}$$

where $t < 1 + 1/a$. Since $\log(1+u) \sim u$ as $u \to 0$, the expression in (9) is less than

$$-2x^{1/b + 1 - t - \epsilon}\log(x) \tag{10}$$

for sufficiently small x. But $t < 1 + (1/a)$, hence there is $\epsilon > 0$ such that $1 - t + (1/a) - \epsilon > 0$, and so for some $b > a$ the exponent in (10) is strictly positive, i.e., (9) holds ($x^r \log x \to 0$ for any $r > 0$). This proves I.

Proof of II. Again, we argue by contradiction. Assume that there exists a sequence $c > x_1 \geq x_2 \geq \cdots \to 0$ such that

$$\frac{\log(x_k - f(x_k))}{\log(x_k)} = t_k \to t > p. \tag{11}$$

Without loss of generality we may assume that

$$\left|\log\left(x_{k+1}^{t_{k+1}}\right) - \log\left(x_k^{t_k}\right)\right| \geq \frac{1}{2}\left|\log\left(x_{k+1}^{t_{k+1}}\right)\right| \tag{12}$$

$$\left|\log\left(\frac{1}{2} + \frac{1}{2}x_{k+1}^{t_{k+1}} x_k^{-t_k}\right)\right| \geq \left|\log\frac{3}{4}\right| \tag{13}$$

$$(x_k - x_{k+1})\left(x_k^{t_k} - x_{k+1}^{t_{k+1}}\right)^{-1} \geq \frac{1}{2}x_k^{1-t_k} \tag{14}$$

It follows from (11) that $f(x_k) = x_k - x_k^{t_k}$. Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} f(x_k) & \text{if } x = x_k \\ \text{linear} & \text{if } x_{k+1} \leq x \leq x_k \\ 0 & \text{if } x = 0 \end{cases}$$

Since the function $f(x)$ is concave, we see that $f(x) \geq h(x)$ and so, as in the proof of part I, $f_m(x) \geq h_m(x)$ for all integers m. Define two integers n_k and m_k as follows: n_k is the largest integer such that $h_{n_k}(x_k) \geq x_{k+1}$ and m_k is the largest integer such that $h_{m_k}(x_k) \geq \frac{1}{2}\left(x_k + x_{k+1}\right)$. We now obtain estimates on m_k using the Lemma. In this case $v = x_k$, $V = x_k^{t^k}$, $z = \frac{1}{2}\left(x_k + x_{k+1}\right)$, and

$$s = \frac{f(x_k) - f(x_{k+1})}{x_k - x_{k+1}} = 1 - \frac{x_k^{t_k} - x_{k+1}^{t_{k+1}}}{x_k - x_{k+1}} = s_k$$

Applying the Lemma, we obtain

$$m_k \geq \frac{\log\left[\frac{1}{2}(x_k + x_{k+1}) + x_k^{t_k}\dfrac{x_k - x_{k+1}}{x_k^{t_k} - x_{k+1}^{t_{k+1}}} - x_k\right] - \log\left(x_k^{t_k}\dfrac{x_k - x_{k+1}}{x_k^{t_k} - x_{k+1}^{t_{k+1}}}\right)}{\log s_k} - 1$$

After direct simplification this reduces to

$$m_k \geq (\log s_k)^{-1}\log\left(\frac{1}{2} + \frac{1}{2}x_{k+1}^{t_{k+1}} x_k^{-t_k}\right) - 1.$$

We apply (13) and the fact that $s_k \to 1$ to obtain

$$m_k \geq c_1\frac{1}{1 - s_k} = c_1\frac{x_k - x_{k+1}}{x_k^{t_k} - x_{k+1}^{t_{k+1}}}$$

for large k, where c_1 is a constant. Finally, from (14) we obtain

$$m_k \geq c_2 x_k^{1-t_k} \tag{15}$$

for sufficiently large k, where c_2 is some constant. For $k \geq 2$ set $N_k = n_1 + n_2 + \ldots + n_{k-1} + m_k$. It follows from the definition of n's and m's that

$$h(x) \geq \tfrac{1}{2}(x_k + x_{k+1}) \geq \tfrac{1}{2} x_k$$

hence for $b > 0$ we have

$$N_k^b f_{N_k}(x_1) \geq N_k^b h_{N_k}(x_1) \geq \tfrac{1}{2} N_k^b x_k \tag{16}$$

Since $n^a f_n(x_1)$ converges to a limit that is different from 0, it follows that as $k \to \infty$, $N_k^b f_{N_k}(x_1) \to 0$ if $b < a$. But $N_k > m_k$, so (16) implies that $m_k^b x_k \to 0$ as $k \to \infty$ whenever $b < a$. From (15) we see that

$$m_k^b x_k \geq c_2 x_k^{b(1-t_k)+1} \tag{17}$$

Now, $1 + 1/a < t_k$ so $a(1-t_k) + 1 < 0$, and thus $b(1-t_k) + 1 < 0$ for some $b < a$ and k sufficiently large. We see from (17) that for such b, $m_k^b x_k \to \infty$. This contradiction completes the proof.

A word or two regarding the concavity assumption in the Theorem 2. The assumption is certainly needed in the proof. The result is also not true without it. The idea is this: Construct an arbitrary sequence $0 < x_n < 1$, $x_n \downarrow 0$ at an arbitrary rate. It is easy to see that one can construct a function $f(x)$ such that $f(1) = x_1$ and $f(x_n) = x_{n+1}$ (Just draw a picture). The values of $f(x)$ at other point can be taken completely arbitrarily so that the conclusion of Theorem 2 need not hold.

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