Internat. J. Math. & Math. Sci.
VOL. 17 NO. 2 (1994) 293-300

EXPANSION OF A CLASS OF FUNCTIONS INTO AN INTEGRAL INVOLVING ASSOCIATED LEGENDRE FUNCTIONS

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(Received February 4, 1992 and in revised form March 18, 1993)

ABSTRACT. A theorem for expansion of a class of functions into an integral involving associated Legendre functions is obtained in this paper. This is a somewhat general integral expansion formula for a function $f(x)$ defined in (x_1, x_2) where $-1 < x_1 < x_2 < 1$, which is perhaps useful in solving certain boundary value problems of mathematical physics and of elasticity involving conical boundaries.

KEY WORDS AND PHRASES. Integral expansion of a function, associated Legendre function, Mehler-Fok integral transform.
1991 AMS SUBJECT CLASSIFICATION CODE. 44.

1. INTRODUCTION.

Integral transforms are often used to solve the problems of mathematical physics involving linear partial differential equations and also other problems. Integral expansions involving spherical functions of a class of functions are known as Mehler-Fok type transforms. In these transform formulae, the subscript of the Legendre functions appear as the integration variable while its superscript is either zero or a fixed integer (see Sneddon [10]). There is another class of integral transforms involving associated Legendre functions somewhat related to the Mehler-Fok transforms, in which the superscript of the associated Legendre function appears in the integration formula while the subscript (complex) is kept fixed. Felsen [2] first developed this type of transform formulae involving $P^{-\mu}_{-1/2+i\tau}(\cos\theta)$ as kernel where $0<\theta<\pi$ from a unique δ-function representation. Later Mandal ([6], [7]) obtained somewhat similar types of two transform formulae from the solution of two appropriately designed boundary value problems. In the first type, the argument x of $P^{-\mu}_{-1/2+i\tau}(x)$ ranges from -1 to 1 while in the second, the argument z of $P^{\mu}_{-1/2+i\tau}(z)$ ranges from 1 to ∞. Recently Mandal and Guha Roy [8] used a similar technique to establish another Mehler-Fok type integral transform formula involving $P^{-\mu}_{-1/2+i\tau}(\cos\theta)$ as kernel $(0<\theta<\alpha)$.

In the present paper, an integral expansion of a class of functions defined in (x_1, x_2) where $-1<x_1<x_2<1$, involving associated Legendre functions is obtained. Based on direct investigation of the properties of spherical functions, sufficient conditions which would establish the validity of this expansion formula for a wide class of functions are obtained in a manner similar to the ideas used in ([3]-[5]). The main result is given in section 2 in the form of a theorem. Recently, we have used a similar technique to establish another type of integral representation [9] involving $P^{-\mu}_{-1/2+i\tau}(\cosh\alpha)$ as kernel where $0<\alpha<\alpha_0$.

2. INTEGRAL EXPANSION OF A FUNCTION IN (x_1,x_2) WHERE $-1<x_1<x_2<1$.

We present the main result of this paper in the form of the following theorem.

THEOREM. Let $f(x)$ be a given function defined on the interval (x_1,x_2) where $-1<x_1<x_2<1$ and satisfies the following conditions:

(1) The function $f(x)$ is piecewise continuous and has a bounded variation in the open interval (x_1,x_2).

(2) The function $f(x)(1-x^2)^{-1}\ell n(1-x^2)^{-1}\in L(x_1,x_2)$, $-1<x_1<x_2<1$.

Then we have

$$f(x)=\sum_k \sigma_k\,\Gamma\left(\tfrac12+i\tau-i\sigma_k\right)\Gamma\left(\tfrac12-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_{\hat{k}})}F(\sigma_{\hat{k}})$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sigma\,\Gamma\left(\tfrac12+i\tau-i\sigma\right)\Gamma\left(\tfrac12-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}F(\sigma)\,d\sigma \tag{2.1}$$

where

$$F(\sigma)=\int_{x_1}^{x_2}\frac{f(x)}{1-x^2}M(x,x_1;i\sigma)\,d\sigma, \tag{2.2}$$

$$-1<x_1<x_2<1,\ M(x,y;i\sigma)=P^{i\sigma}_{-1/2+i\tau}(x)P^{i\sigma}_{-1/2+i\tau}(-y)-P^{i\sigma}_{-1/2+i\tau}(-x)P^{i\sigma}_{-1/2+i\tau}(y)$$

and $\sigma_k's,\sigma,\tau$ are real. The equation (2.2) may be regarded as an integral transform of the function $f(x)$ defined in (x_1,x_2) and (2.1) is its inverse. (2.1) and (2.2) together give the integral expansion of the function $f(x)$.

PROOF OF THE EXPANSION THEOREM. To prove this expansion theorem, we first note that the representation (cf. Erdélyi [1])

$$P^{i\sigma}_{-1/2+i\tau}(x)=\left(\frac{1+x}{1-x}\right)^{i\sigma/2}F\left(\tfrac12+i\tau,\ \tfrac12-i\tau;\ 1-i\sigma;\ \frac{1-x}{2}\right)/\ \Gamma(1-i\sigma),$$

$-1<x_1<x<x_2<1$, where $F(a,b;c;x)$ denotes the hypergeometric series, implies $P^{i\sigma}_{-1/2+i\tau}(x)$ is continuous in the region defined by $-1<x_1<x<x_2<1$, $-\infty<\sigma<\infty$ and satisfies the inequality

$$\left|P^{i\sigma}_{-1/2+i\tau}(x)\right|\le\sqrt{sh\pi\sigma/\pi\sigma}\ P_{-1/2+i\tau}(x), \tag{2.3}$$

where the Legendre function $P_{-1/2+i\tau}(x)$ is positive.
Using (2.3) it follows from (2.2) that

$$\int_{x_1}^{x_2}\left|\frac{f(x)}{1-x^2}\left[P^{i\sigma}_{-1/2+i\tau}(x)\ P^{i\sigma}_{-1/2+i\tau}(-x_1)-\ P^{i\sigma}_{-1/2+i\tau}(-x)\ P^{i\sigma}_{-1/2+i\tau}(x_1)\right]\right|dx$$

$$\le\sqrt{sh\pi\sigma/\pi\sigma}\int_{x_1}^{x_2}\frac{|f(x)|}{1-x^2}\left\{P_{-1/2+i\tau}(x)\ P_{-1/2+i\tau}(-x_1)-P_{-1/2+i\tau}(-x)\ P_{-1/2+i\tau}(x_1)\right\}dx,$$

and this shows that the conditions imposed on $f(x)$ imply that the integral $F(\sigma)$ is absolutely and uniformly convergent for $\sigma\in[-T,T]$ where T is a positive large number. Hence $F(\sigma)$ is continuous on $[-T,T]$ and the repeated integral

$$J(x,T)=\frac{1}{2\pi i}\int_{-T}^{T}\sigma\Gamma\left(\tfrac12+i\tau-i\sigma\right)\Gamma\left(\tfrac12-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2;x_1;i\sigma)}\,d\sigma\cdot\int_{x_1}^{x_2}\frac{f(y)}{1-y^2}M(y,x_1;i\sigma)\,dy$$

is meaningful. Also, uniform convergence allows us to change the order of integration and write $J(x,T)$ as

$$J(x,T) = \int_{x_1}^{x_2} \frac{f(y)}{1-y^2} K(x,y,T)\, dy, \tag{2.4}$$

where

$$K(x,y,T) = \frac{1}{2\pi i} \int_{-T}^{T} \sigma\, \Gamma\left(\tfrac{1}{2}+i\tau - i\sigma\right) \Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right) \frac{M(x,x_2;i\sigma)M(y,x_1;i\sigma)}{M(x_2,x_1;i\sigma)}\, d\sigma. \tag{2.5}$$

Now we shall show that the kernel $K(x,y,T)$ is symmetric in the variables x and y. By definition, we have

$$K(x,y,T) - K(y,x,T) = \frac{1}{2\pi i} \int_{-T}^{T} \sigma\, \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{1}{M(x_2,x_1;i\sigma)}$$

$$\cdot\left[M(x,x_2;i\sigma)M(y,x_1;i\sigma) - M(y,x_2;i\sigma)M(x,x_1;i\sigma)\right] d\sigma.$$

It follows from the properties of associated Legendre functions (cf. Erdélyi [1]) that the integrand in the above integral is an odd function of σ, hence the integral vanishes. Thus

$$K(y,x,T) = K(x,y,T). \tag{2.6}$$

To investigate the behavior of $K(x,y,T)$ as $T\to\infty$, by writing $\mu = -i\sigma$, we write (2.5) as

$$K(x,y,T) = \frac{1}{2\pi i} \int_{-iT}^{iT} \mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}\, d\mu. \tag{2.7}$$

Expression under the integral sign in (2.7) is analytic function fo the complex variable μ and it has no singularity in the semi-plane $Re\mu \geq 0$, except for simple poles at $\mu = -i\sigma_k$ (k is positive integer) (cf. Felsen [2]), where

$$M(x_2,x_1;i\sigma_k) = 0,\ \sigma_k > 0. \tag{2.8}$$

Completing the contour of integration on (2.7) with the arc Γ_T of radius T situated in the semi-plane $Re\mu \geq 0$ and applying the residue theorem, we obtain

$$K(x,y,T) = K_1(x,y,T) - \sum_k \sigma_k\, \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\cdot\frac{M(x,x_2;i\sigma_k)M(y,x_1;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2;x_1;i\sigma_k)}, \tag{2.9}$$

where

$$K_1(x,y,T) = \frac{1}{2\pi i} \int_{\Gamma_T} \mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}\, d\mu. \tag{2.10}$$

Suppose that $y \leq x$. By virtue of the definition

$$P^{-\mu}_{-1/2+i\tau}(x) = \left(\frac{1+x}{1-x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1 + 0(|\mu|^{-1})\right],$$

$$P^{-\mu}_{-1/2+i\tau}(-x) = \left(\frac{1-x}{1+x}\right)^{-\mu/2}\frac{1}{\Gamma(1+\mu)}\left[1 + 0(|\mu|^{-1})\right] \tag{2.11}$$

Using (2.11) and asymptotic properties of the gamma function for large μ, we conclude that

$$\mu\, \Gamma\left(\tfrac{1}{2}+i\tau+\mu\right)\Gamma\left(\tfrac{1}{2}-i\tau+\mu\right)\frac{M(x,x_2;-\mu)M(y,x_1;-\mu)}{M(x_2,x_1;-\mu)}$$

$$= \frac{\left[\left(\frac{1+x}{1-x}\,\frac{1-x_2}{1+x_2}\right)^{-\mu/2} - \left(\frac{1-x}{1+x}\,\frac{1+x_2}{1-x_2}\right)^{-\mu/2}\right]\left[\left(\frac{1+y}{1-y}\,\frac{1-x_1}{1+x_1}\right)^{-\mu/2} - \left(\frac{1-y}{1+y}\,\frac{1+x_1}{1-x_1}\right)^{-\mu/2}\right]}{\left[\left(\frac{1+x_2}{1-x_2}\,\frac{1-x_1}{1+x_1}\right)^{-\mu/2} - \left(\frac{1-x_2}{1+x_2}\,\frac{1+x_1}{1-x_1}\right)^{-\mu/2}\right]}$$

$$\cdot\left[1 + 0(|\mu|^{-1})\right]. \tag{2.12}$$

Now introduce the new variables

$$\xi = \frac{1}{2}\,\ell n\,\frac{1+x}{1-x},\ \eta = \frac{1}{2}\,\ell n\,\frac{1+y}{1-y},\ \alpha = \frac{1}{2}\,\ell n\,\frac{1+x_1}{1-x_1} \text{ and } \beta = \frac{1}{2}\,\ell n\,\frac{1+x_2}{1-x_2}.$$

Then, for large μ, from (2.10) - (2.12) we obtain for $y \le x$

$$K_1(x,y,T) = \frac{1}{2\pi i} \int_{\Gamma_T} [exp\{-\mu(\xi-\eta)\} + exp\{-\mu(2\beta - 2a - \xi + \eta)\}$$

$$- exp\{-\mu(\xi+\eta-2\alpha)\} - exp\{-\mu(2\beta-\xi-\eta)\}]\,d\mu$$

$$+ O(1) \int_0^{\pi/2} exp\{-\mu(\xi-\eta)\cos\varphi\} + exp\{-\mu(2\beta-2\alpha-\xi+\eta)\cos\varphi\}]$$

$$- exp\{-\mu(\xi+\eta-2\alpha)\cos\varphi\} - exp\{-\mu(2\beta-\xi-\eta)\cos\varphi\}]\,d\varphi,$$

$$\text{for } \alpha < \eta \le \xi < \beta.$$

Using the identity

$$\frac{2}{\pi}\int_0^{\pi/2} exp\{-\lambda T\cos\varphi\}\,d\varphi \le \frac{1 - exp(-\lambda T)}{\lambda T},\ \lambda \ge 0,$$

we obtain for $y \le x$,

$$K_1(x,y,T) = \frac{1}{\pi}\left[\frac{\sin T(\xi-\eta)}{\xi-\eta} + \frac{\sin T(2\beta-2\alpha-\xi+\eta)}{2\beta-2\alpha-\xi+\eta} - \frac{\sin T(\xi+\eta-2a)}{\xi+\eta-2a}\right.$$

$$\left. - \frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}\right] + O(1)\left[\frac{1-exp\{-T(\xi-\eta)\}}{T(\xi-\eta)} + \frac{1-exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}\right.$$

$$\left. - \frac{1-exp\{-T(\xi+\eta-2\alpha)\}}{T(\xi+\eta-2\alpha)} - \frac{1-exp\{-T(2\beta-\xi-\eta)\}}{T(2\beta-\xi-\eta)}\right], \qquad \alpha < \eta \le \xi < \beta, \tag{2.13}$$

where the factor $O(1)$ is independent of y.

Again for $y \ge x$, we use the symmetry property (2.6) and the representation (2.10) of $K_1(x,y,T)$ with the variables x,y replaced by y,x.

Now we write (2.4) as

$$J(x,T) = \int_{x_1}^{x} \frac{f(y)}{1-y^2}\,K_1(x,y,T)\,dy + \int_{x}^{x_2} \frac{f(y)}{1-y^2}\,K_1(x,y,T)\,dy$$

$$- \sum_k \sigma_k\,\Gamma\!\left(\frac{1}{2}+i\tau-i\sigma_k\right)\Gamma\!\left(\frac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}\int_{x_1}^{x_2}\frac{f(y)}{1-y^2}\,M(y,x_1;i\sigma_k)\,dy$$

$$= J_1(x,T) + J_2(x,T) - \sum_k \sigma_k\,\Gamma\!\left(\frac{1}{2}+i\tau-i\sigma_k\right)\Gamma\!\left(\frac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)} \times$$

$$\times \cdot \int_{x_1}^{x_2}\frac{f(y)}{1-y^2}M(y,x_1;i\sigma_k)dy. \tag{2.14}$$

Using (2.13) in J_1, we obtain

$$J_1(x,T) = \frac{1}{\pi}\left[\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi-n)}{\xi-\eta}d\eta + \int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-2\alpha-\xi+\eta)}{2\beta-2\alpha-\xi+\eta}d\eta\right.$$

$$\left. - \int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi+\eta-2\alpha)}{\xi+\eta-2\alpha}d\eta - \int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}d\eta\right]$$

$$+O(1)\left[\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(\xi-\eta)\}}{T(\xi-\eta)}d\eta\right.$$

$$+\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}d\eta$$

$$-\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(\xi+\eta-2\alpha)\}}{T(\xi+\eta-2\alpha)}d\eta$$

$$\left.-\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(2\beta-\xi-\eta)\}}{T(2\beta-\xi-\eta)}d\eta\right]. \tag{2.15}$$

The conditions satisfied by $f(x)$ imply that $f(\tanh\eta)\in L(\alpha,\beta)$; hence, by virtue of Dirichlet's theorem, for $T\to\infty$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi-\eta)}{\xi-\eta}d\eta = \frac{1}{2}f(\tanh\xi-o)+o(1)$$

$$=\frac{1}{2}f(x-o)+o(1),$$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-2a-\xi+\eta)}{2\beta-2\alpha-\xi+\eta}d\eta = o(1),$$

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(\xi+\eta-2\alpha)}{\xi+\eta-2a}d\eta = o(1),$$

and

$$\frac{1}{\pi}\int_\alpha^\xi f(\tanh\eta)\frac{\sin T(2\beta-\xi-\eta)}{2\beta-\xi-\eta}d\eta = o(1).$$

Moreover, if the integral of integration is divided into the subintervals $(\xi-\delta,\xi)$ and $(\alpha,\xi-\delta)$ and if a sufficiently small positive δ (implying a sufficiently large T) is chosen, then we have

$$\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(\xi-\eta)\}}{T(\xi-\eta)}d\eta$$

$$\le\frac{1}{\delta T}\int_\alpha^{\xi-\delta}|f(\tanh\eta)|d\eta+\int_{\xi-\delta}^\xi|f(\tanh\eta)|d\eta$$

$$=O(T^{-1})+o(1)=o(1) \text{ for } T\to\infty,$$

$$\int_\alpha^\xi |f(\tanh\eta)|\frac{1-\exp\{-T(2\beta-2\alpha-\xi+\eta)\}}{T(2\beta-2\alpha-\xi+\eta)}d\eta\le\frac{1}{\xi T}\int_\alpha^\xi|f(\tanh\eta)|d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty,$$

$$\int_\alpha^\xi |f(\tanh \eta)| \frac{1 - exp\{-T(\xi+\eta-2\alpha)\}}{T(\xi+\eta-2\alpha)} d\eta \leq \frac{1}{\xi T} \int_\alpha^\xi |f(\tanh \eta)| d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty,$$

and

$$\int_\alpha^\xi |f(\tanh \eta)| \frac{1 - exp\{-T(2\beta-\xi-\eta)\}}{T(2\beta-\xi-\eta)} d\eta \leq \frac{1}{\xi T} \int_\alpha^\xi |f(\tanh \eta)| d\eta$$

$$= O(T^{-1}) = o(1) \text{ for } T \to \infty. \tag{2.17}$$

Thus (2.15) to (2.17) leads to

$$\lim_{T\to\infty} J_1 (\tanh \xi, T) = \frac{1}{2} f(\tanh \xi - o) = \frac{1}{2} f(x - o). \tag{2.18}$$

Similarly,

$$\lim_{T\to\infty} J_2(\tanh \xi, T) = \frac{1}{2} f(\tanh \xi + o) = \frac{1}{2} f(x + o). \tag{2.19}$$

Hence,

$$\lim_{T\to\infty} J(x,T) = \frac{1}{2}[f(x+o) + f(x-o)] - \sum_k \sigma_k \Gamma\left(\frac{1}{2} + i\tau - i\sigma_k\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma_k\right)$$

$$\cdot \frac{M(x, x_2; i\sigma_k)}{(\partial/\partial\sigma_k) M(x_2, x_1; i\sigma_k)} F(\sigma_k). \tag{2.20}$$

Thus, at the points of continuity of $f(x)$ we obtain (2.1). We note that (2.1) becomes a result in [5] when $x_1 = -1$ and $x_2 = 1$.

It follows from the foregoing theorem that, at points of continuity of $f(x)$, we have

$$f(x) = \sum_k \sigma_k \Gamma\left(\frac{1}{2} + i\tau - i\sigma_k\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma_k\right) \frac{R(x, x_2; i\sigma_k)}{(\partial^2/\partial x_2 \partial\sigma_k) R(x_2, x_1; i\sigma_k)} F(\sigma_k)$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sigma \Gamma\left(\frac{1}{2} + i\tau - i\sigma\right) \Gamma\left(\frac{1}{2} - i\tau - i\sigma\right) \frac{R(x, x_2; i\sigma)}{(\partial/\partial x_2) R(x_2, x_1; i\sigma)} F(\sigma) d\sigma, \tag{2.21}$$

where

$$F(\sigma) = \int_{x_1}^{x_2} \frac{f(x)}{1-x^2} R(x, x_1; i\sigma)\, dx, \; -1 < x_1 < x_2 < 1, \tag{2.22}$$

$$R(x, y; i\sigma) = P^{i\sigma}_{-1/2+i\tau}(x) \frac{\partial}{\partial y} P^{i\sigma}_{-1/2+i\tau}(-y) - P^{i\sigma}_{-1/2+i\tau}(-x) \frac{\partial}{\partial y} P^{i\sigma}_{-1/2+i\tau}(y)$$

and σ_k's, σ, τ are real.

The integrand in (2.21) has singularities at $\sigma = \sigma_k$(k is positive integers) which are simple poles along the positive σ-axis, where

$$\frac{\partial}{\partial x_2} R(x, x_1; i\sigma_k) = 0, \; (\sigma_k > 0). \tag{2.23}$$

To prove (2.21) we use the following asymptotic formulas for large μ:

$$\frac{\partial}{\partial x} P^{-\mu}_{-1/2+i\tau}(x) = -\frac{\mu}{\Gamma(1+\mu)} \frac{1}{(1-x)(1+x)} \left(\frac{1+x}{1-x}\right)^{-\mu/2} [1 + O(|\mu|^{-1})],$$

$$\frac{\partial}{\partial x} P^{-\mu}_{-1/2+i\tau}(-x) = -\frac{\mu}{\Gamma(1+\mu)} \frac{1}{(1+x)(1-x)} \left(\frac{1-x}{1+x}\right)^{-\mu/2} [1 + O(|\mu|^{-1})], \tag{2.24}$$

The proof of (2.21) is similar to the proof in the section 2, and we do not reproduce it. We note that (2.21) becomes a result in [5] when $x_1 = -1$ and $x_2 = 1$.

3. EXAMPLES.

We now give examples of expansions of some functions.

$$(1)\quad (1-x^2)^{\nu/2} = \sum_k \sigma_k \Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}$$

$$\cdot\frac{2^\nu\Gamma(1+\nu)}{(\nu^2+\sigma_k^2)}(\nu+i\sigma_k)\left[P_\nu^{-\nu}(x_1)M_1(x_1,x_1;i\sigma_k)-P_\nu^{-\nu}(x_2)\,M_1(x_2,x_1;i\sigma_k)\right]$$

$$+\frac{2^\nu\Gamma(1+\nu)}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma(\nu+i\sigma)}{\nu^2+\sigma^2}\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}$$

$$\cdot\left[P_\nu^{-\nu}(x_1)M_1(x_1,x_1;i\sigma)-P_\nu^{-\nu}(x_2)\,M_1(x_2,x_1;i\sigma)\right]d\sigma,$$

$$(-1<x_1<x<x_2<1)$$

where

$$M(x,y;i\sigma)=P_\nu^{i\sigma}(x)\,P_\nu^{i\sigma}(-y)-P_\nu^{i\sigma}(-x)\,P_\nu^{i\sigma}(y),$$

$$M_1(x,y;i\sigma)=P_{\nu-1}^{i\sigma}(x)\,P_\nu^{i\sigma}(-y)-P_{\nu-1}^{i\sigma}(-x)\,P_\nu^{i\sigma}(y)\text{ and }\nu=-1/2+i\tau.$$

$$(2)\quad P_\nu^\mu(x)=\sum_k\sigma_k\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma_k\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma_k\right)\frac{M(x,x_2;i\sigma_k)}{(\partial/\partial\sigma_k)M(x_2,x_1;i\sigma_k)}\;\frac{1}{(\mu^2+\sigma_k^2)}$$

$$\cdot\Big[(\nu+\mu)\,P_{\nu-1}^\mu(x_2)M(x_2,x_1;i\sigma_k)+(\nu+i\sigma_k)\,\{P_\nu^\mu(x_1)M_1(x_1,x_2;i\sigma_k)$$

$$-P_\nu^\mu(x_2)M_1(x_2,x_2;i\sigma_k)\}\Big]+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma}{\mu^2+\sigma^2}\Gamma\left(\tfrac{1}{2}+i\tau-i\sigma\right)\Gamma\left(\tfrac{1}{2}-i\tau-i\sigma\right)$$

$$\cdot\frac{M(x,x_2;i\sigma)}{M(x_2,x_1;i\sigma)}\Big[(\nu+\mu)\,P_{\nu-1}^\mu(x_2)M(x_2,x_1;i\sigma)+(\nu+i\sigma)\,\{P_\nu^\mu(x_1)$$

$$\cdot M_1(x_1,x_2;i\sigma)-P_\nu^\mu(x_2)M_1(x_2,x_1;i\sigma)\}\Big]\,d\sigma.$$

In all these results the conditions under which the expansion theorem hold are satisfied.

ACKNOWLEDGEMENT. This research is supported by CSIR, New Delhi, through a research project No. 25(41)/EMR-II/88 administered by the Calcutta Mathematical Society.

REFERENCES

1. ERDÉLYI, A.; MAGNUS, W.; OBERHITTINGER, F. & TRICOMI, F.G., *Higher Transcendental Functions, Vol. 1*, McGraw Hill Co., 1953.
2. FELSEN, L.B., Some new transform theorems involving Legendre functions, *J. Math. Phys.* **37** (1958), 188-191.
3. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Integral expansion of an arbitrary function in terms of spherical functions, *PMM* **30** (1966), 252-258.
4. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Expansion of an arbitrary function into an integral in terms of associated spherical functions, *PMM* **32** (1968), 421-427.
5. LEBEDEV, N.N. & SKAL'SKAYA, I.P., Integral representations related to Mehler-Fok transformations, *Differential Equations* **22** (1986), 1050-1056.
6. MANDAL, B.N., An integral transform associated with degree of Legendre functions, *Bull. Cal. Math. Soc.* **63** (1971), 1-6.

7. MANDAL, B.N., Note on an integral transform, *Bull. Math. de la Soc. Math. de la R.S. de Roumanie* **63** (1971), 87-93.

8. MANDAL, B.N. & GUHA ROY, P., On a Mehler-Fok type integral transform, *Appl. Math. Lett.* **4** (1991), 29-32.

9. MANDAL, N. & MANDAL, B.N., Integral representation of a function in terms of associated Legendre functions, *Bull. Math. de la Soc. Math. de la R.S. de Roumanie* **4** (1991), to appear.

10. SNEDDON, I.N., *The Use of Integral Transforms*, McGraw Hill Co., 1972.