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A NOTE ON ABSOLUTE SUMMABILITY FACTORS

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ABSTRACT. In this paper, a generalization of a theorem of Mishra and Srivastava [4] on $|C,1|_k$ summability factors has been proved.

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1. INTRODUCTION.

Let Σa_n be a given infinite series with partial sums (s_n). We denote by u_n the n-th $(C,1)$ mean of the sequence (s_n). The series Σa_n is said to be summable $|C,1|_k, k \geq 1$, [2] if

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]).

The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all values of n, then $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$ summability.

2. PRELIMINARY RESULT.

Mishra and Srivastava [4] proved the following theorem for $|C,1|_k$ summability.

THEOREM A. Let (X_n) be a positive non-decreasing sequence and there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n \tag{2.1}$$

$$\beta_n \to 0 \text{ as } n \to \infty \tag{2.2}$$

$$|\lambda_n| X_n = 0(1) \text{ as } n \to \infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty. \tag{2.4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} |s_n|^k = 0(X_m) \text{ as } m \to \infty, \tag{2.5}$$

then the series $\Sigma a_n \lambda_n$ is summable $|C,1|_k$, $k \geq 1$.

3. **MAIN RESULT.**

The aim of this paper is to generalize Theorem A for $|\bar{N}, p_n|_k$ the summability in the form of the following theorem.

THEOREM. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) such that conditions (2.1) - (2.4) of Theorem A are satisfied. Furthermore, if (p_n) is a sequence of positive numbers such that

$$P_n = 0(np_n) \tag{3.1}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = 0(X_m) \text{ as } m \to \infty, \tag{3.2}$$

then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N} p_n|_k, k \geq 1$.

REMARK. It should be noted that if we take $p_n = 1$ for all values of n, then the condition (3.2) will be reduced to the condition (2.5). Also noticed that, in this case condition (3.1) is obvious.

4. We need the following lemma for the proof of our theorem.

LEMMA ([4]). Under the conditions on $(X_n), (\beta_n)$ and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (2.4) is satisfied

$$n\, \beta_n X_n = 0(1) \text{ as } n \to \infty \tag{4.1}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{4.2}$$

5. **PROOF OF THE THEOREM.** Let (T_n) be the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{w=0}^{v} a_w \lambda_w = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1}\, a_v \lambda_v.$$

Applying Abel's transformation, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) s_v + \frac{p_n}{P_n} s_n \lambda_n = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v\, \Delta\lambda_v s_v + \frac{p_n}{P_n} s_n \lambda_n = T_{n,1} + T_{n,2} + T_{n,3}, \text{ say.}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid T_{n,w} \mid^k < \infty \text{ for } w = 1,2,3.$$

Now, applying Hölder's inequality, we have ($k > 1$)

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,1} \mid^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v \mid s_v \mid^k \mid \lambda_v \mid^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= 0(1) \sum_{v=1}^{m} p_v \mid s_v \mid^k \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = 0(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \mid s_v \mid^k \mid \lambda_v \mid^k.$$

Since $\mid \lambda_n \mid = 0(1/X_n) = 0(1)$, by (2.3), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,1} \mid^k = 0(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \mid s_v \mid^k \mid \lambda_v \mid \mid \lambda_v \mid^{k-1} = 0(1) \sum_{v=1}^{m} \frac{p_v}{P_v} \mid s_v \mid^k \mid \lambda_v \mid$$

$$= 0(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{w=1}^{v} \frac{p_w}{P_w} \mid s_w \mid^k + 0(1) \mid \lambda_m \mid \sum_{v=1}^{m} \frac{p_v}{P_v} \mid s_v \mid^k$$

$$= 0(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + 0(1) \mid \lambda_m \mid X_m = 0(1) \sum_{v=1}^{m-1} \beta_v X_v + 0(1) \mid \lambda_m \mid X_m = 0(1)$$

as $m \to \infty$, by the hypotheses of the theorem and lemma.

Using the fact that $\mid \Delta\lambda_n \mid \leq \beta_n$ and $P_n = 0(np_n)$, and after applying the Hölder's inequality, we get that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,2} \mid^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P^k_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid \mid s_v \mid \right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P^k_{n-1}} \left\{ \sum_{v=1}^{n-1} v\beta_v p_v \mid s_v \mid \right\}^k$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} (v\beta_v)^k p_v \mid s_v \mid^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= 0(1) \sum_{v=1}^{m} (v\beta_v)^k p_v \mid s_v \mid^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = 0(1) \sum_{v=1}^{m} (v\beta_v)^k \frac{p_v}{P_v} \mid s_v \mid^k.$$

Since $n\beta_n = 0(1/X_n) = 0(1)$, by (4.1), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} \mid T_{n,2} \mid^k = 0(1) \sum_{v=1}^{m} (v\beta_v)^{k-1}\, v\, \beta_v \frac{p_v}{P_v} \mid s_v \mid^k = 0(1) \sum_{v=1}^{m} v\, \beta_v \frac{p_v}{P_v} \mid s_v \mid^k$$

$$= 0(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{w=1}^{v} \frac{p_w}{P_w} \mid s_w \mid^k + 0(1)\, m\beta_m \sum_{v=1}^{m} \frac{p_v}{P_v} \mid s_v \mid^k$$

$$= 0(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + 0(1) m\beta_m X_m = 0(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + 0(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v$$

$$+ 0(1)m\beta_m X_m = 0(1) \text{ as } m \to \infty,$$

by virtue of the hypothesis and lemma.

Finally, as in $T_{n,1}$, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \,|\, T_{n,3} \,|^k = \sum_{n=1}^{m} \frac{p_n}{P_n} \,|\, s_n \,|^k \,|\, \lambda_n \,|^k = 0(1) \text{ as } m \to \infty.$$

Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} \,|\, T_{n,w} \,|^k = 0(1) m \to \infty, \text{ for } w = 1, 2, 3.$$

This completes the proof of the theorem.

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