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ABOUT THE EXISTENCE AND UNIQUENESS THEOREM FOR HYPERBOLIC EQUATION

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ABSTRACT. In this paper we prove the existence and uniqueness theorem for almost everywhere solution of the hyperbolic equation using the method of successive approximations [1].
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1. INTRODUCTION.

Mixed problems for partial differential equations have been investigated by a number of authors [2], [3], [4], [5]. In this case we investigate the almost everywhere solution for the hyperbolic equation that have been studied in [6]. Namely, the solution for the hyperbolic equation in the space $B^{2,1}_{2,2,T}$ with a nonlinear operator at the right hand side.

2. STATEMENT OF THE PROBLEM.

Consider the following system

$$u_{tt}(t,x) - Lu(t,x) = F(u(t,x)) \qquad \text{in } Q_T \tag{2.1}$$

subject to the initial conditions

$$u(0,x) = \phi(x) \qquad u_t(0,x) = \psi(x) \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$u(t,x) \mid \Gamma = 0 \qquad t \in [0,T] \tag{2.3}$$

where $Q_T = [0,T] \times \Omega, 0 < T < \infty, \Omega$ is a bounded domain in R^n and G is the boundary of O;

$$L(u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)u, \tag{2.4}$$

and moreover the functions $a_{ij}(x)$ have continuous $\bar{\Omega}$ and $\frac{\partial a_{ij}(x)}{\partial x_k}$, $a(x)$ are measurable and bounded in Ω and satisfy the following conditions in Ω :

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n} \xi_i^2 \,, \tag{2.5}$$

ξ_i are any real number; $\psi(\xi)$, $\varphi(x)$ are given functions in Ω; F is a nonlinear operator.

3. PRELIMINARIES.

DEFINITION 1. The almost everywhere solution for the problem (2.1)-(2.3) is the function $u(x,t)$, element of $W_2^2(Q_T)$, belongs to $D_1^0(Q_T)$ and satisfies (2.1) almost everywhere in Q_T and $t \to +0$ satisfies the following

$$\int_\Omega [u(t,x) - \phi(x)]^2 dx = 0 , \qquad \int_\Omega [\frac{\partial u(t,x)}{\partial t} - \psi(x)]^2 dx = 0 \tag{2.6}$$

DEFINITION 2. We define the space $B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$ of all functions $u(t,x) = \sum_{s=1}^{\infty} u_s(t)\vartheta_s(x)$ in $Q_T = [0,T]\times\Omega$, where $v_s(x)$ are eigenfunctions for the operator L with the boundary condition (2.3) corresponding to the eigenvalues λ_s

$$(0 < \lambda_s \to \text{ as } s\to\infty) \; [7] , \qquad u_s(t) \text{ are } \ell \geq 0$$

times continuously differentiable in $[0,T]$ and

$$\sum_{i=1}^{l} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0\leq t\leq T} |u_s^{(i)}(t)|\right]^{\beta_i}\right\}^{1/\beta_i} < +\infty \tag{2.7}$$

and has the norm

$$\| u \|_{B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}} = \sum_{i=1}^{\ell} \left\{ \sum_{s=1}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0\leq t\leq T} |u_s^{(i)}(t)|\right]^{\beta_i}\right\}^{1/\beta_i} \tag{2.8}$$

where $\alpha_i \geq 0, 1 \leq \beta_i \leq 2, (i = 0,\dots,\ell)$.

DEFINITION 3. The function $u_s(t)$ is called the s-component of the function

$$u,(t,x) = \sum_{s=1}^{\infty} u_s(t)\ell_s(x)$$

and $\mu_x(s = 1,2,\dots)$ is the set of all s-components of elements of μ where $\mu \subset B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$.

THEOREM 2.1. The necessary and sufficient conditions for μ to be compact in $B^{\alpha_0,\dots,\alpha_\ell}_{\beta_0,\dots,\beta_\ell,T}$ are

(a) for every $s(s = 1,2,\dots)$ the set μ_s is compact in $C^\ell[0,T]$; and

(b) for any given $\epsilon > 0$ there exists a natural number n_ϵ so that for all $u(t,x) = \sum_{s=1}^{\infty} u_s\ell_s(x) \in \mu$,

$$\sum_{i=1}^{\ell}\left\{ \sum_{s=n_\epsilon}^{\infty} \left[\lambda_s^{\alpha_i} \max_{0\leq t\leq T} |u_s^{(i)}(t)|\right]^{\beta_i}\right\}^{1/\beta_i} < \epsilon.$$

This theorem can be proved analogously as in ([9] page 277-278).

LEMMA 1. For any almost everywhere solution $u(t,x)$ of (2.1) - (2.3) functions $u_s(t) = \int_\Omega u(t,x)\ell_s(x)dx$ satisfy the following system ([7] , [8])

$$u_s(t) = \phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t +$$

$$\frac{1}{\lambda_s}\int_0^t \int_\Omega F(u(\tau,x)).\ell_s(x) \sin \lambda_s(t-\tau)dxd\tau, (s = 1,2,\dots), \tag{2.9}$$

where

$$\phi_s = \int_\Omega \varphi(x)\vartheta_s(x)dx, \qquad \psi_s = \int_\Omega \psi(x)\vartheta_s(x)dx.$$

3. ASSUMPTION AND RESULTS.

THEOREM 3.1. Let

1. $a_{ij}(x)$ are continuously differentiable on $\bar{\Omega}$ and $a(x)$ continuous on $\bar{\Omega}$;
2. The eigenfunctions ϑ_s are twice continuously differentiable on $\bar{\Omega}$;
3. $\varphi(x) \in W_2^2(\Omega) \cap D^o(\Omega)$, $\psi(x) \in D^o(\Omega)$;
4. $F: B^{2,1}_{2,2,T} \cup (W_2^2(Q_T) \cap B^{1,0}_{2,2,T}) \to W^{1,0}_{x,t,2}(Q_T)$ and satisfies

$$\| F(u(t,x)) \|_{W_2^1(\Omega)} \leq c(t) + d(t) \| u \|_{B^{2,1}_{2,2,t}} \tag{3.1}$$

for all $u \in B^{2,1}_{2,2,T}$; where $c(t), d(t) \in L_2(0,T)$.

5. For any $u, v \in \mathcal{K}_o$ (where $\mathcal{K}_o$ is the sphere $\| u \|_{B^{2,1}_{2,2,T}} \leq C_o$)

$$\| F(u,t,x)) - F(v(t,x) \|_{W_2^1(\Omega)} \leq g(t) \| u - v \|_{B^{2,1}_{2,2,t}}, \quad g(t) \in L_2(0,T), \tag{3.2}$$

where

$$C_o = \left\{ \left[2\| W(t,x)\|^2_{B^{2,1}_{2,2,T}} + 16Ta_o^2 \| c(t) \|^2_{L_2(0,T)} \right] \exp\left[16Ta_o^2 \| d(t) \|^2_{L_2(0,T)} \right] \right\}^{1/2} \tag{3.3}$$

and

$$a_o^2 = \max\left\{ n \cdot \max_{ij} \left\{ \| a_{ij}(x) \|_{C(\bar{\Omega})} \right\}, \| a(x) \|_{C(\bar{\Omega})} \right\}$$

6. For any $u \in B^{2,1}_{2,2T} \cup \left(W_2^2(Q_T) \cap B^{1,0}_{2,2,T} \right)$ and $t \in [0,T]\, F(u(t,x)) \in D^o(\Omega)$.

Then the problem (2.1) - (2.3) has a unique solution,

PROOF. Let

$$W(x,t) = \sum_{s=1}^{\infty} (\phi_s \cos \lambda_s t + \frac{\psi_s}{\lambda_s} \sin \lambda_s t)\vartheta_s(x), \tag{3.4}$$

and

$$PF(u) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s} \int_0^t \int_\Omega F(u(\tau,x)) \cdot \vartheta_s(x) \sin \lambda_s(t-\tau) dx\, d\tau \cdot \vartheta_s(x) \tag{3.5}$$

From (3.4) and (3.5) let us assume that

$$Q(u) = W + PF(u) \tag{3.6}$$

Then it is easy to see that the operator Q acts in $B^{2,1}_{2,2,T}$ and satisfies Lipschitz condition

$$\| Q(u) - Q(v) \|_{B^{2,1}_{2,2t}} \leq 2\sqrt{T} a_o \| g(t) \|_{L_2(0,T)} \| u - v \|_{B^{2,1}_{2,2,t}} \tag{3.7}$$

in the sphere $\mathcal{K}_o$.

Consider the sequence $u_k(t,x) = Q(u_{k-1}(t,x))$ in $B^{2,1}_{2,2,T}$ where $u_o(t,x) = 0$. Using (3.1) and the mathematical induction we get for any $k(k=1,2,3,...)$ and $t\in[0,T]$:

$$\| u_k \|^2_{B^{2,1}_{2,2,t}} \le 2 \| W \|^2_{B^{2,1}_{2,2,T}} + 8Ta_o^2 \int_0^t \| F(u_{k-1}(\tau,x)) \|^2_{W^1_2(\Omega)} d\tau$$

$$\le 2 \| W \|^2_{B^{2,1}_{2,2,T}} + 16Ta_o^2 \left\{ \int_0^T c^2(\tau) d\tau + \int_0^t d^2(\tau) \| u_{k-1} \|^2_{B^{2,1}_{2,2,t}} d\tau \right\} \tag{3.8}$$

$$= \mathcal{A}^2 + \int_0^t \mathcal{B}^2(\tau) \| u_{k-1} \|^2_{B^{2,1}_{2,2,t}} d\tau$$

$$\le \mathcal{A}^2 + \mathcal{A}^2 \int_0^t \mathcal{B}^2(\tau) d\tau + ... + \mathcal{A}^2 \frac{\left\{ \int_0^t \mathcal{B}^2(\tau) d\tau \right\}^{k-1}}{(k-1)!},$$

where

$$\mathcal{A}^2 = 2 \| W \|^2_{B^{2,1}_{2,2,T}} + 16Ta_o^2 \| c(t) \|^2_{L_2(0,T)}, \tag{3.9}$$

and

$$\mathcal{B}^2(t) = 16Ta_o^2 d^2(t)$$

From (3.8) for any $k(k=1,2,...)$, we get

$$\| u_k \|^2_{B^{2,1}_{2,2,t}} \le \mathcal{A}^2 \cdot \exp\left\{ \int_0^T \mathcal{B}^2(\tau) d\tau \right\} = C_o^2 \tag{3.10}$$

i.e., all $u_k(t,x)$ are contained in the sphere $\mathcal{K}_o$. Further, using (3.2) and (3.3) we get for any $t\in[0,T]$ and $k(k=1,2,3,...)$

$$\| u_{k+1} - u_k \|^2_{B^{2,1}_{2,2,t}} \le 4Ta_o^2 \| F(u_k(\tau,x)) - F(u_{k-1}(\tau,x)) \|^2_{L_2(\Omega)} d\tau$$

$$\le 4Ta_o^2 \int_0^t g^2(\tau) \| u_k - u_{k-1} \|^2_{B^{2,1}_{2,2,t}}$$

$$\le \| u_1 - u_o \|^2_{B^{2,1}_{2,2,T}} \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!}$$

$$= \| u_1 \|^2_{B^{2,1}_{2,2,T}} \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \le C_o^2 \frac{\left\{ 4Ta_o^2 \int_0^t g^2(\tau) d\tau \right\}^k}{k!} \tag{3.11}$$

Therefore,

$$\| u_{k+1} - u_k \|^2_{B^{2,1}_{2,2,T}} \le C_o^2 \frac{\left\{ 4Ta_o^2 \| g(t) \|^2_{L_2(0,T)} \right\}^k}{k!}, (k=1,2,...) \tag{3.12}$$

Then $\{u_k(t,x)\}$ is a fundamental sequence in $B^{2,1}_{2,2,T}$. Since $B^{2,1}_{2,2,T}$ complete, then

$$u_k(t,x) \quad \underset{\longrightarrow}{B^{2,1}_{2,2,T}} \quad u(t,x) \in \mathcal{K}_o \qquad \text{as } k \to \infty \tag{3.13}$$

Since Q is continuous in $\mathcal{K}_o$, then from the relation $u_k(t,x) = Q(u_{k=1}(t,x))$

we have

$$u,(t,x) = Q(u(t,x))$$

Therefore, as in (3.11), (3.12) the speed of convergence is governed by the following inequality

$$\| u_k - u \|^2_{B^{2,1}_{2,2,T}} \leq \| u_o - u \|^2_{B^{1,0}_{2,2,T}} \frac{\left\{4Ta_o^2 \| g(t) \|^2_{L_2(0,T)}\right\}^k}{k!}$$

$$\leq C_o^2 \frac{\left\{4Ta_o^2 \| g(t) \|^2_{L_2(0,T)}\right\}^k}{k!}, (k = 1,2,...). \tag{3.14}$$

Now to prove the uniqueness let us assume the $u(t,x) = \sum_{s=1}^{\infty} u_s(t)\ell_2(x)$ solution to (2.1) - (2.3) then $F(u(t,x)) \in L_2(Q_T)$. By Lemma (1) $u_s(t)$ satisfy (2.9); from (2.9) we get

$$\| u(t,x) \|_{B^{1,0}_{2,2,t}} \leq \| W(t,x) \|_{B^{1,0}_{2,2,T}} + 2\sqrt{T} \| F(u(t,x)) \|_{L_2(Q_T)} < +\infty \tag{3.15}$$

Therefore $u \in B^{1,0}_{2,2,t}$. Since $u(t,x) \in W^2_2(Q_T) \cap B^{1,0}_{2,2,T}$ then by (3.1) $F(u(t,x)) \in W^{1,0}_{x,t,2}(Q_T)$, but by condition 6 Theorem 2 for all $t \in [0,T], F(u(t,x)) \in \overset{o}{D}(\Omega)$. Thus using (2.9) with some manipulation

$$\| u(t,x) \|_{B^{2,1}_{2,2,t}} \leq \| W(t,x) \|_{B^{2,1}_{2,2,T}} + 2\sqrt{T}\, a_o \| F(u(t,x)) \|_{W^{1,0}_{x,t,2}(Q_T)} < +\infty \tag{3.16}$$

Therefore, $u \in B^{2,1}_{2,2,T}$. Then, using (3.1), (3.8), (3.10), we get $\| u(t,x) \|_{B^{2,1}_{2,2,t}} \leq C_o$. Thus, all almost everywhere solutions (2.1)-(2.3) belong to the sphere K_o and they are fixed points in $B^{2,1}_{2,2,T}$ for operator Q. Le u,v be two solutions to (2.1)-(2.3), then by (3.2) we get

$$\| u - v \|^2_{B^{2,1}_{2,2,t}} \leq 4Ta_o^2 \int_0^t \| F(u(\tau,x)) - F(v(\tau,x)) \|^2_{W^1_2(\Omega)} d\tau$$

$$\leq 4Ta_o^2 \int_0^t g^2(\tau) \| u - v \|^2_{B^{2,1}_{2,2,t}} d\tau \tag{3.17}$$

Therefore, using Belmann's inequality [10] we have

$\| u - v \|^2_{2,2,t} = 0$ in $[0,T]$. Therefore, $u = v$.

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