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# MINIMAL CR-SUBMANIFOLDS OF A SIX-DIMENSIONAL SPHERE

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**ABSTRACT.** We establish several formulas for a 3-dimensional CR-submanifold of a six-dimensional sphere and state some results obtained by making use of them.

**KEY WORDS AND PHRASES.** CR-submanifold, D-minimal and $D^\perp$-minimal CR-submanifold.
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1. **INTRODUCTION.** Among all submanifolds of a Kaehler manifold there are three typical classes: the complex submanifolds, the totally real submanifolds and the CR-submanifolds. The notion of a CR-submanifold of a Kaehler manifold was introduced by Bejancu [1] and it includes the other two classes as special cases. A Riemannian submanifold $M$ of an almost Hermitian manifold $\bar{M}$ is called a CR-submanifold if there exists a pair of orthogonal complementary distribution $D$ and $D^\perp$ on $M$ satisfying $JD = D$ and $JD^\perp \subset \nu$, where $\nu$ is the normal bundle of $M$. If $M$ is a real hypersurface of a Kaehler manifold, then $M$ is obviously a CR-submanifold.

It is known that every Kaehler manifold is nearly Kaehler but the converse is not true in general. The most typical example of nearly Kaehler manifolds is a six-dimensional sphere $S^6$. It is because of this nearly Kaehler, non-Kaehler, structure that $S^6$ has attracted attention.

The object of the present paper is to establish several formulas for a 3-dimensional CR-submanifold of a six-dimensional sphere and state some result obtained by making use of them.

2. **PRELIMINARIES.**

Let $\bar{M}$ be an almost complex manifold with almost complex structure $J$, and Hermitian metric $g$. $\bar{M}$ is called a nearly Kaehler manifold if

$$(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_Y J)(X) = 0 \tag{2.1}$$

for $X, Y \in (\bar{M})$, where $\bar{\nabla}$ is Riemannian connection on $\bar{M}$.

In [5], K. Takamatsu and T. Sato proved the following theorem:

**THEOREM.** Let $\bar{M} = (\bar{M}, J, g)$ be a non-Kaehler, nearly Kaehler manifold of constant holomorphic sectional curvature. Then $\bar{M}$ is a six-dimensional space of positive constant sectional curvature.

If a nearly Kaehler manifold $\bar{M}$ is constant holomorphic sectional curvature $c$, then by the above result, the curvature tensor $\bar{R}$ of $\bar{M}$ is given by

$$R(X,Y,Z,W) = c(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)). \tag{2.2}$$

Let $M$ be an $m$-dimensional CR-submanifolds of a six-dimensional sphere $\bar{M}$ and let us denote by the same $g$ the Riemannian metric tensor field induced on $M$ from that of $\bar{M}$. Let $P$ and $Q$ be the projection operators corresponding to $D$ and $D^{\perp}$ respectively.

For a vector field $X$ tangent to $M$, we put

$$JX = PX + QX \tag{2.3}$$

where $PX$ (resp. $QX$) denote the tangent (resp. normal) component of $JX$.

We now denote by $\bar{\nabla}$ (resp. $\nabla$) the Riemannian connection in $\bar{M}$ (resp. $M$) with respect to the Riemannian metric $g$. The linear connection induced by $\bar{\nabla}$ on the normal bundle $T^{\perp}M$ is denoted by $\nabla^{\perp}$. Thus the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.4}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $h$ is the second fundamental form of $M$ and $A_N$ is the fundamental tensor with respect to the normal section $N$. These tensor fields are related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{2.5}$$

The equation of Gauss is given by

$$\bar{R}(X,Y,z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)). \tag{2.6}$$

**DEFINITION.** A CR-submanifold $M$ is called $D$-minimal (resp. $D$-minimal) if $\sum_{i=1}^{2p} h(E_i, E_i) = 0$ (resp. $\sum_{i=1}^{q} h(F_i, F_i) = 0$) where $\{E_1, E_2, \text{----}, E_{2p}\}$ (resp. $\{F_1, F_2, \text{----}, F_q\}$ is a local field of frames of$D$ (resp. $D^{\perp}$).

**DEFINITION.** A CR-submanifold $M$ is called $D$-totally geodesic (resp. $D^{\perp}$-totally geodesic) if $h(X,Y) = 0$ for each $X, Y \in D$ (resp. $X, Y \in D^{\perp}$). $M$ is called a mixed totally geodesic if $h(X,Z) = 0$ for each $X \in D, Z \in D^{\perp}$.

## 3. THREE-DIMENSIONAL CR-SUBMANIFOLDS OF $S^6$.

Let $M$ be a 3-dimensional CR-submanifold of $S^6$. It is known that $S^6$ is nearly Kaehler manifold of constant type 1. Suppose $dim\ D = 2, dim\ D^{\perp} = 1$, and $\{E_1, JE_1\}$ be a local frame in $D$ and $\{F\}$ that of $D^{\perp}$.

The mean curvature vector $H$ is defined by

$$H = \frac{1}{3}\left\{\sum_{i=1}^{2} h(E_i, E_i) + h(F,F)\right\}. \tag{3.1}$$

If $H = 0$, then $M$ is said to be minimal. Now we define

$$H_D = \frac{1}{2}\sum_{i=1}^{2} h(E_i, E_i),\ H_{D^{\perp}} = h(F,F). \tag{3.2}$$

If $H_D = 0$, then $M$ is said to be $D$-minimal and if $H_{D^{\perp}} = 0$, then $M$ is said to be $D^{\perp}$-minimal.

Let $U, V$ be any vector field tangent to CR-submanifold $M$. The Ricci tensor and the scalar curvature are respectively given by

$$S(U,V) = \sum_{i=1}^{2} g(R(E_i, U)V, E_i) + g(R(F,U)V,F). \tag{3.3}$$

$$\rho = \sum_{i=1}^{2} S(E_i, E_i) + S(F,F). \tag{3.4}$$

Also

$$S_D(U,V) = g(R(E_i,U)V,E_i), S_{D^\perp}(U,V) = g(R(F,U)V,F). \tag{3.5}$$

$$\rho_{DD} = \sum_{i=1}^{2} S_D(E_i,E_i), \rho_{DD^\perp} = S_D(F,F). \tag{3.6}$$

$$\rho_{D^\perp D} = \sum_{i=1}^{2} S_{D^\perp}(E_i,E_i), \qquad \rho_{D^\perp D^\perp} = S_{D^\perp}(F,F). \tag{3.7}$$

Now using (2.2) and (2.6), we have for $X,Y \in TM$

$$S_D(X,Y) = 2g(X,Y) - g(PX,PY) + 2g(H_D,h(X,Y)) - \sum_{i=1}^{2} g(h(E_i,X),h(E_i,Y)), \tag{3.8}$$

$$S_{D^\perp}(X,Y) = g(X,Y) - g(QX,QY) + g(H_{D^\perp},h(X,Y)) - g(h(F,X),h(F,Y)), \tag{3.9}$$

$$\rho_{DD} = 2 + 4g(H_D,H_D) - \sum_{i,j=1}^{2} \| h(E_i,E_j) \|^2, \tag{3.10}$$

$$\rho_{DD^\perp} = 2 + 2g(H_{D^\perp},H_D) - \sum_{i=1}^{2} \| h(E_i,F) \|^2, \tag{3.11}$$

$$\rho_{D^\perp D^\perp} = g(H_{D^\perp},H_{D^\perp}) - \| h(F,F) \|^2. \tag{3.12}$$

It is easy to see that

$$\rho_{DD^\perp} = \rho_{D^\perp D}$$

Now we prove

**THEOREM 1.** Let $M$ be a $D$-minimal CR-submanifold of a 6-dimensional sphere $S^6$. Then the following hold:

(a) $S_D(X,X) - 2\| X \|^2 + \| PX \|^2 \leq 0$, for $X \in TM$

(b) $\rho_{DD} \leq 2$

(b′) $\rho_{DD^\perp} \leq 2$.

The equality in (a) for $X \in D$, and in (b) holds if and only if $M$ is $D$-totally geodesic.

The equality in (a) for $X \in D^\perp$ and in (b′) holds if and only if $M$ is mixed totally geodesic.

**PROOF.** Since $M$ is $D$-minimal, from (3.8), we have

$$S_D(X,X) - 2\| X \|^2 + \| PX \|^2 = -\sum_{i=1}^{2} g(h(E_i,X),h(E_i,X)).$$

This proves (a) and (b), (b′) follow from (3.10) and (3.11). Similarly, we have

**THEOREM 2.** Let $M$ be a $D^\perp$-minimal CR-submanifold of a 6-dimensional sphere $S^6$. Then the following hold:

(a) $S_{D^\perp}(X,X) - \| X \|^2 + \| QX \|^2 \leq 0$, for $X \in TM$

(b) $\rho_{D^\perp D} \leq 2$,

(b′) $\rho_{D^\perp D^\perp} \leq 2$.

The equality for $X \in D^\perp$ in (a) and (b′) holds if and only if $M$ is $D^\perp$-totally geodesic.

The equality for $X \in D$ in (a) and in (b) holds if and only if $M$ is mixed totally geodesic.

**PROOF.** Since $M$ is $D^\perp$-minimal, so from (3.9), we have

$$S_{D^\perp}(X,X) - \|X\|^2 + \|QX\|^2 = -g(h(F,X),h(F,X)),$$

which proves (a) and (b), (b′) follows from (3.11) and (3.12).

**REMARKS.** The example given by Sekigawa [6] is an example of $D$-totally geodesic and $D^\perp$-totally geodesic (and hence minimal) proper CR-submanifold of a 6-dimensional sphere and this illustrates the Theorem 1 in the sense that $S^2_f \times S^1$, where $f$ is a function on $S^2$, is a $D$-minimal CR-submanifold of $S^6$ in which it is easily verified that $\rho_{DD} = 2$. The equality arises because it is also $D$-totally geodesic in $S^6$.

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