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COMPLETION OF PROBABILISTIC NORMED SPACES

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ABSTRACT. We prove that every probabilistic normed space, either according to the original definition given by Šerstnev, or according to the recent one introduced by Alsina, Schweizer and Sklar, has a completion.

KEY WORDS AND PHRASES. Probabilistic Normed Spaces, completion, triangle function, Lévy distance.
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1. INTRODUCTION.

As is well known, a real or complex normed linear space admits a completion, namely, given a normed linear space $(V, \|\cdot\|)$, there exists another linear space $(V', \|\cdot\|')$ such that V' is isometric to a dense subspace of V.

It was proved by Muštari [2], Sherwood ([7], [8]) and Sempi [5] that a probabilistic metric space has a completion. Here we answer in the positive the natural question of whether a probabilistic normed space has a completion. In fact, there are two definitions of probabilistic normed space (= PN-space): the original one by Šerstnev ([6], but see [3] for a presentation in agreement with our notation), and a more recent one by Alsina, Schweizer and Sklar (see [1]). The proof will be given in both cases. For the notation and the concepts used we refer to the book by Schweizer and Sklar [3]; we shall write d.f. for distribution function.

According to Šerstnev, a PN-space is a triple (V,v,τ), where V is a real linear space; τ is a ***triangle function*** ([3], section 7.1), i.e., a binary operation on Δ^+, the space of distance distribution functions, that is commutative, associative and nondecreasing in each variable and which has the d.f. ε_0 as identity, i.e.,

(a) $\forall F,G \in \Delta^+ \quad \tau(F,G) = \tau(G,F)$;

(b) $\forall F,G,H \in \Delta^+ \quad \tau(F,\tau(G,H)) = \tau(\tau(F,G),H)$;

(c) $\forall H \in \Delta^+ F \leq G \Rightarrow \quad \tau(F,H) \leq \tau(G.H)$;

(d) $\forall F \in \Delta^+ \quad \tau(F,\varepsilon_0) = F$.

Here ε_0 is the d.f. defined by

$$\varepsilon_0(x) := \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0; \end{cases}$$

ν is the probabilistic norm, i.e., ν is a map from V into Δ^+ that satisfies the following conditions:

(N.1) $\nu(p) = \varepsilon_0$, if, and only if, $p = \vartheta$, where ϑ is the null vector of V;

(N.2) $\forall x \in \mathbb{R}_+, a \in \mathbb{R}$, with $a \neq 0 \quad \nu(ap)(x) = \nu(p)(x/\mid a \mid)$;

(N.3) $\forall p, q \in V \quad \nu(p+q) \geq \tau[\nu(p), \nu(q)]$.

In both definitions the triangle function is assumed to be continuous.

The space Δ^+ can be metrized by different metrics ([9], [3], [4], [10]), but we shall use here the modified Lévy metric d_L [3].

2. MAIN RESULTS.

THEOREM 1. Every PN-space (V, ν, τ) has a completion, viz. is isometric to a dense linear subspace of a complete PN-space (V', ν', τ).

PROOF. Only the steps needed to complement the treatment in [7] and [8] will be given. Now V' is the set of equivalence classes of Cauchy sequences of elements of V. In order to prove that V' is a linear space, let p' and q' be elements of V' and let $\{p_n\}$ and $\{q_n\}$ be Cauchy sequences of elements of V with $\{p_n\} \in p'$ and $\{q_n\} \in q'$. Since V is a linear space, one has, for every $n \in \mathbb{N}, p_n + q_n \in V$. We wish to show that it is possible to define a sum of p' and q' in such a way that $p' + q' \in V'$. Since $(V, \mathcal{F}, \tau)$, with $\mathcal{F}(p,q) := \nu(p-q)$ is a probabilistic metric space ([3], Theorem 15.1.2), one has, if n and m are large enough,

$$\mathcal{F}(p_n + q_n, p_m + q_m) = \nu((p_n + q_n) - (p_m + q_m))$$

$$= \nu((p_n - p_m) + (q_n - q_m)) \qquad \text{(because of (N.3))}$$

$$\geq \tau[\nu(p_n - p_m),\ \nu(q_n - q_m)].$$

Taking into account Lemma 4.3.4 in [3], one has

$$d_L(\mathcal{F}(p_n + q_n, p_m + q_m), \varepsilon_0) \leq d_L(\tau[\nu(p_n - p_m), \nu(q_n - q_m)], \varepsilon_0)$$

$$= d_L(\tau[\mathcal{F}(p_n, p_m), \mathcal{F}(q_n, q_m)], \varepsilon_0).$$

The continuity of both d_L and τ ensures that, when both m and n tend to infinity, $\mathcal{F}(p_n + q_n, p_m + q_m) \xrightarrow{w} \varepsilon_0$. Thus $\{p_n + q_n\}$ is a Cauchy sequence and, as a consequence, it belongs to an element of V', which will be denoted by r'. Then we define $p' + q' = r'$. This definition does not depend on the elements of p' and q' selected, for, if $\{p_n\}, \{p_n^*\} \in p'$ and $\{q_n\}, \{q_n^*\} \in q'$, then

$$\mathcal{F}(p_n + q_n, p_n^* + q_n^*) = \nu(p_n - p_n^*, q_n - q_n^*) \geq \tau[\nu(p_n - p_n^*), \nu(q_n - q_n^*)] = \tau[\mathcal{F}(p_n, p_n^*), \mathcal{F}(q_n, q_n^*)],$$

so that

$$d_L(\mathcal{F}(p_n + q_n, p_n^* + q_n^*), \varepsilon_0) \leq d_L\left(\tau[\mathcal{F}(p_n, p_n^*), \mathcal{F}(q_n, q_n^*)], \varepsilon_0\right).$$

Since both d_L and τ are continuous we obtain $\mathcal{F}(p_n + q_n, p_n^* + q_n^*) \xrightarrow{w} \varepsilon_0$, i.e., $\{p_n + q_n\} \sim \{p_n^* + q_n^*\}$. Thus the sum defined above is a good definition, which immediately satisfies the properties of an abelian group.

For every $\alpha \in \mathbb{R}$, and for every Cauchy sequence $\{p_n\}$ of elements of V, also $\{\alpha\ p_n\}$ is a Cauchy sequence of elements of V. This is obvious if $\alpha = 0$. If $\alpha \neq 0$, one has, for every $x > 0$,

$$\mathcal{F}(\alpha p_n, \alpha p_m)(x) = \nu(\alpha p_n - \alpha p_m)(x) = \nu(p_n - p_m)(x/\mid \alpha \mid)$$

$$= \mathcal{F}(p_n, p_m)(x/\mid \alpha \mid),$$

and this tends to 1, for every $x > 0$, as n and m tend to infinity, i.e., $\mathcal{F}(\alpha p_n, \alpha p_m) \xrightarrow{w} \varepsilon_0$. Thus $\{\alpha p_n\}$ is a Cauchy sequence; let us denote by u' the element of V' to which it belongs. Then we define $\alpha p' = u'$. This is again a good definition; in fact, let $\{p_n\}, \{p_n^*\} \in p'$. Then

$$\mathcal{F}(\alpha p_n, \alpha p_n^*)(x) = \nu[\alpha(p_n - p_n^*)](x) = \nu(p_n - p_n^*)\left(\frac{x}{\mid \alpha \mid}\right) = \mathcal{F}(p_n, p_n^*)\left(\frac{x}{\mid \alpha \mid}\right),$$

which tends to 1 for all $x > 0$ when $n \to \infty$, whence $\{\alpha p_n\} \sim \{\alpha p_n^*\}$. Therefore it is immediate to conclude that V' is a linear space. All that is left to show is that the distance d.f. $\mathfrak{F}'$ derives from a probabilistic norm ν' on V'. For every $p' \in V'$, set, if $\{p_n\} \in p'$ with $p_n \in V$ for every $n \in \mathbb{N}$

$$\nu'(p') := \mathfrak{F}'(p', \vartheta') = lim_n\, \mathfrak{F}(p_n, \vartheta) = lim_n\, \nu(p_n). \tag{2.1}$$

Thus

$$\mathfrak{F}'(p', q') = lim_n\, \mathfrak{F}(p_n, q_n) = lim_n\, \nu(p_n - q_n) = \nu'(p' - q').$$

It is now an easy task to verify that ν' does indeed fulfill conditions (N.1), (N.2) and (N.3). □

We now turn to the proof of the analogous result for PN-spaces according to the definition given in [1]. This latter differs from the one given above in that condition (N.2) is replaced by the weaker one

(N.2′) $\quad \forall p \in V \;\; \nu(-p) = \nu(p)$;

and a new one is added:

(N.4) $\quad \forall \alpha \in [0,1] \forall p \in V \;\; \nu(p) \leq \tau^*[\nu(\alpha p), \nu((1-\alpha)p)]$.

Then a PN-space is a quadruple (V, ν, τ, τ^*), where V, as above, is a real linear space, τ, τ^* are continuous triangle functions and $\nu' : V \to \Delta^+$ is a map such that conditions (N.1), (N.2′), (N.3) and (N.4) are satisfied.

The last part of this note is entirely devoted to PN-spaces according to this latter definition.

LEMMA 2. Let (V, ν, τ, τ^*) be a PN-space and let h and k be two real constants such that $0 \leq h \leq k$; then

$$\forall p, q \in V \qquad \mathfrak{F}(kp, kq) \;\leq\; \mathfrak{F}(hp, hq),$$

where $\mathfrak{F}(p, q) := \nu(p - q)$.

PROOF. There is $\lambda \in [0,1]$ such that $h = \lambda k$. Then

$$\begin{aligned} \mathfrak{F}(kp, kq) = \nu(kp - kq) = \nu[k(p-q)] &\leq \\ \leq \tau^*\big[\nu[\lambda k(p-q)], \nu[(1-\lambda)k(p-q)]\big] &\leq \\ \leq \tau^*\big[\nu[\lambda k(p-q)], \varepsilon_0\big] = \nu[\lambda k(p-q)] = \nu[h(p-q)] &= \mathfrak{F}(hp, hq). \end{aligned}$$ □

THEOREM 3. Every PN-space (V, ν, τ, τ^*) has a completion, viz. is isometric to a dense linear subspace of a complete PN-space (V', ν', τ, τ^*).

PROOF. Exactly as in the proof of Theorem 1, one can prove that if both p' and q' belong to V', then $p' + q' \in V'$. However, one can no longer use the same proof of the fact that if $\alpha \in \mathbb{R}$ and $p' \in V'$ then $\alpha p' \in V'$, because recourse was made to property (N2) which now may well not hold.

Now assume $\alpha \in \mathbb{R}$ and $p' \in V'$, let $\{p_n\} \in p'$ and consider the sequence $\{\alpha p_n\}$. As a first step, we shall prove that it is a Cauchy sequence in V. This is obviously true for $\alpha = 0$ and $\alpha = 1$. Because of (N.2′), it suffices to consider only the case $\alpha > 0$. Now assume that $\{\alpha p_n\}$ is a Cauchy sequence for $\alpha = 0, 1, \cdots, k-1 (k \in \mathbb{N})$. Then

$$\begin{aligned} \mathfrak{F}(kp_n, kp_m) = \nu[k(p_n - p_m)] \geq \tau\big[\nu(p_n - p_m), \nu[(k-1)(p_n - p_m)]\big] = \\ = \tau\big[\mathfrak{F}(p_n, p_m), \mathfrak{F}((k-1)p_n, (k-1)p_m)\big]. \end{aligned}$$

Since τ is continuous and

$$\lim_{n, m \to \infty} \mathfrak{F}(p_n, p_m) = \lim_{n, m \to \infty} \mathfrak{F}((k-1)p_n, (k-1)p_m) = \varepsilon_0$$

it follows that also $\{\alpha p_n\}$ is a Cauchy sequence for every $\alpha \in \mathbb{Z}_+$. If α is positive, but not integer, there exists $k \in \mathbb{Z}_+$ such that $k < \alpha < k+1$. Lemma 2 now gives

$$\mathcal{F}((k+1)p_n, (k+1)p_m) \leq \mathcal{F}(\alpha p_n, \alpha p_m) \leq \mathcal{F}(kp_n, kp_m);$$

hence it is immediate to conclude that $\{\alpha p_n\}$ is a Cauchy sequence for every $\alpha \in \mathbb{R}_+$. Thus there exists an element $u' \in V'$ such that $\{\alpha p_n\} \in u'$. Let us define $u' = \alpha p'$. In order to check that this is a good definition, let $\{p_n\} \sim \{p_n^*\}$. If $\alpha \in [0,1]$, it follows from Lemma 2 that $\mathcal{F}(p_n, p_n^*) \leq \mathcal{F}(\alpha p_n, \alpha p_n^*)$; since, by assumption $\mathcal{F}(p_n, p_n^*) \xrightarrow{w} \varepsilon_0$, also $\mathcal{F}(\alpha p_n, \alpha p_n^*) \xrightarrow{w} \varepsilon_0$. If $\alpha = k \in \mathbb{Z}_+$, as above, one has

$$\mathcal{F}(kp_n, kp_n^*) = \nu[k(p_n - p_n^*)] \geq \tau[\nu(p_n - p_n^*), \nu[(k-1)(p_n - p_n^*)]] =$$
$$= \tau[\mathcal{F}(p_n, p_n^*), \mathcal{F}((k-1)p_n, (k-1)p_n^*)].$$

The same argument as above yields $\{kp_n\} \sim \{kp_n^*\}$ for every $k \in \mathbb{Z}_+$. Again, from this it is easy to obtain that, for every $\alpha \in \mathbb{R}$ one has $\{\alpha p_n\} \sim \{\alpha p_n^*\}$.

Therefore V' is a linear space. Only conditions (N.2′) and (N.4) remain now to be proved. Proceeding as above, let $p' \in V'$ and let $\{p_n\}$ be a Cauchy sequence of elements of V that belongs to p'; then $\{-p_n\} \in -p'$. Since ν' is defined by (2.1), one has, on account of (N.2′), which holds for ν,

$$\nu'(-p') = lim_n \ \nu(-p_n) = lim_n \ \nu(p_n) = \nu'(p').$$

Moreover, for every $\alpha \in [0,1]$, one has, because τ^* is continuous,

$$\nu'(p') = lim_n \ \nu(p_n) \leq lim_n \ \tau^*[\nu(\alpha p_n), \nu((1-\alpha)p_n)]$$
$$= \tau^*[\nu'(\alpha p'), \nu((1-\alpha)p')].$$

□

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