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A REMARK ON CERTAIN p-VALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to prove an interesting result for certain analytic and p-valent functions in the unit disc $U = \{z : |z| < 1\}$.

KEY WORDS AND PHRASES. Analytic, p-valent, Ruscheweyh derivative.
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1. INTRODUCTION.

Let A(p) denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \tag{1.2}$$

we define the convolution product $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by,

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{1.3}$$

Using the above convolution product, we define

$$D^{n+p-1} f(z) = \left(\frac{z^p}{(1-z)^{n+p}} \right) * f(z) \quad (f(z) \in A(p)), \tag{1.4}$$

where n is any integer greater than - p. We note that

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!}. \tag{1.5}$$

The symbol D^{n+p-1} when $p = 1$ was introduced by Ruscheweyh [8], and the symbol D^{n+p-1} was introduced by Goel and Sohi [5]. This symbol was named the Ruscheweyh derivative of $(n + p - 1)$ - th order by Chen and Owa [4].

It follows from (1.5) that

$$z(D^{n+p-1} f(z))' = (n+p) D^{n+p} f(z) - n D^{n+p-1} f(z) \quad \text{(cf. [4] and [5]).} \tag{1.6}$$

Recently, Chen and Lan ([1], [2]), Chen, Lee and Owa [3], Chen and Owa [4] and Srivastava, Owa and Pashkouleva [9] have been proved some interesting results of analytic functions involving Ruscheweyh derivatives. In the present paper, we prove an interesting result for functions $f(z) \in A(p)$ satisfying

$$\mathrm{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p}\right\} > \alpha,\ \ 0 \le \alpha < 1 \text{ and } n \in N_0 = N \cup \{0\}.$$

2. MAIN RESULT.

In order to prove our main result, we recall here the following lemma:

LEMMA (Miller [6]; Miller and Mocanu [7]). Let $\varphi(u, v)$ be a complex - valued function,

$$\varphi: D \to C,\ D \subset C \times C \quad (C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u, v)$ satisfies the following conditions:

(i) $\varphi(u, v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \le -\frac{(1+u_2^2)}{2}$.

Let $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ be regular in the unit disc U such that $(q(z), zq'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\varphi(q(z), zq'(z))\} > 0 \qquad (z \in U),$$

then

$$\mathrm{Re}\{q(z)\} > 0 \qquad (z \in U).$$

Applying the above Lemma, we derive the following:

THEOREM. Let the function f(z) be in the class A(p) satisfy

$$\mathrm{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p}\right\} > \alpha, \quad (z \in U) \tag{2.1}$$

for $0 \le \alpha < 1$ and $n \in N_0$. Then

$$\mathrm{Re}\left\{\sqrt{\frac{D^{n+p}f(z)}{z^p}}\right\} > \beta, \quad (z \in U), \tag{2.2}$$

where

$$\beta = \frac{1 + \sqrt{1 + 4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)}. \tag{2.3}$$

PROOF. For f(z) in A(p), we define the function q(z) by

$$\sqrt{\frac{D^{n+p}f(z)}{z^p}} = \beta + (1-\beta)q(z), \tag{2.4}$$

where β is given by (2.3). Then q(z) is regular in U and $q(z) = 1 + q_1 z + q_2 z^2 + \ldots\ldots$. Taking the derivatives of both sides in (2.4), we have

$$\frac{z(D^{n+p}f(z))' - p(D^{n+p}f(z))}{z^p} = 2(1-\beta)[\beta + (1-\beta)q(z)]zq'(z). \tag{2.5}$$

Since the identity (1.5) implies

$$z(D^{n+p}f(z))' = (n+p+1)D^{n+p+1}f(z) - (n+1)D^{n+p}f(z), \tag{2.6}$$

(2.5) becomes

$$\frac{D^{n+p+1}f(z)}{z^p} = [\beta+(1-\beta)q(z)]^2 + \frac{2(1-\beta)[\beta+(1-\beta)q(z)]zq'(z)}{(n+p+1)}, \tag{2.7}$$

or

$$\mathrm{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p} - \alpha\right\} = \mathrm{Re}\left\{[\beta+(1-\beta)q(z)]^2 + \frac{2(1-\beta)[\beta+(1-\beta)q(z)]zq'(z)}{(n+p+1)} - \alpha\right\} > 0. \tag{2.8}$$

Taking $q(z) = u = u_1 + iu_2$ and $zq'(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by

$$\varphi(u, v) = [\beta+(1-\beta)u]^2 + \frac{2(1-\beta)[\beta+(1-\beta)u]v}{(n+p+1)} - \alpha. \tag{2.9}$$

Then it follows from (2.9) that

(i) $\varphi(u,v)$ is continuous in $D = C \times C$,

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} = 1-\alpha > 0$,

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\dfrac{(1+u_2^2)}{2}$,

$$\begin{aligned}\mathrm{Re}\{\varphi(iu_2, v_1)\} &= \beta^2 - (1-\beta)^2 u_2^2 + \frac{2\beta(1-\beta)v_1}{(n+p+1)} - \alpha \\ &\leq \beta^2 - (1-\beta)^2 u_2^2 - \frac{\beta(1-\beta)(1+u_2^2)}{(n+p+1)} - \alpha \\ &< 0\end{aligned}$$

for $0 \leq \alpha < 1$, $n \in N_0$, $n > -p$ and β is given by (2.3). Therefore, the function $\varphi(u,v)$ satisfies the conditions in the lemma. Thus we have $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$, that is,

$$\mathrm{Re}\left\{\sqrt{\frac{D^{n+p}f(z)}{z^p}}\right\} > \beta = \frac{1+\sqrt{1+4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)} \tag{2.10}$$

which completes the proof of the Theorem.

Letting $\alpha = 0$, the theorem gives:

COROLLARY 1. Let the function f(z) be in the class A(p) satisfy

$$\mathrm{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p}\right\} > 0 \qquad (z \in U) \tag{2.11}$$

for $n \in N_0$ and $n > -p$. Then

$$\mathrm{Re}\left\{\sqrt{\frac{D^{n+p}f(z)}{z^p}}\right\} > \frac{1}{(n+p+2)} \quad (z \in U). \tag{2.12}$$

Taking $n = 1 - p$ in the above theorem, we have

COROLLARY 2. Let the function f(z) be in the class A(p) satisfy

$$\operatorname{Re}\left\{\frac{D^2 f(z)}{z^p}\right\} > \alpha \quad (z \in U) \tag{2.13}$$

for $0 \le \alpha < 1$ and $p \in N$. Then

$$\operatorname{Re}\left\{\sqrt{\frac{zf'(z)+(1-p)f(z)}{z^i}}\right\} > \frac{1+\sqrt{1+24\alpha}}{6} \tag{2.14}$$

REMARK 1. Putting $p = 1$ in the above results, we get the results obtained by Chen, Lee and Owa [3].

REMARK 2. Using the same technique as in the theorem (or putting $\frac{zf'(z)}{p}$ instead of f(z) in the theorem), we have the following result :

COROLLARY 3. Let the function f(z) be in the class A(p) satisfy

$$\operatorname{Re}\left\{\frac{(D^{n+p+1}f(z))'}{pz^{p-1}}\right\} > \alpha \quad (z \in U)$$

for $0 \le \alpha < 1$ and $n \in N_0$. Then

$$\operatorname{Re}\left\{\sqrt{\frac{(D^{n+p}f(z))'}{pz^{p-1}}}\right\} > \beta$$

where β is given by (2.3)

REFERENCES

1. CHEN, M. -P. and LAN, I. -R. On certain inequalities for some regular functions defined on the unit disc, Bull. Austral. Math. Soc. 35 (1987), 387 - 396.
2. CHEN, M.-P. and LAN, I.-R. On α-convex functions of order β of Ruscheweyh type, Internat. J. Math. Math. Sci. 12 (1989), 107 - 112.
3. CHEN, M.-P., LEE, S.-K. and OWA, S. A remark on certain regular functions, Simon Stevin 65 (1991), no. 1-2, 23- 30.
4. CHEN, M.-P. and OWA, S. A property of certain analytic functions involving Ruscheweyh derivatives, Proc. Japan Acad. 65, Ser.A (1989), no.10,333-335.
5. GOEL, R. M. and SOHI, N. S. A new criterion for p - valent functions , Proc. Amer. Math. Soc. 78 (1980), 353- 357.
6. MILLER, S. S. Differential inequalities and Caratheodory function , Bull. Amer. Math. Soc. 8 (1975), 79- 81.
7. MILLER, S. S. and MOCANU, P T. Second order differential inequalities in the complex plane , J. Math. Anal. Appl. 65 (1978) , 289 - 305.
8. RUSCHEWEYH, St. New criteria for univalent functions , Proc. Amer. Math. Soc. 49 (1975) , 109 - 115.
9. SRIVASTAVA, H. M., OWA, S. and PASHKOULEVA, D. Z. Some inequalities associated with a class of regular functions, Utilitas Math. 34 (1988), 163-168.