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RESEARCH NOTES

THE CLOSED GRAPH THEOREM FOR MULTILINEAR MAPPINGS

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ABSTRACT. A version of the Closed Graph Theorem for multilinear mappings in the context of Banach spaces is presented.

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1. INTRODUCTION.

The classical Closed Graph Theorem is considered one of the most profound contributions of S. Banach to Functional Analysis. In the context of Banach spaces it can be formulated as follows (cf. [1]):

(1) If E and F are Banach spaces and $u : E \to F$ is a linear mapping with closed graph, then u is continuous.

We have that (1) is equivalent to the following statement (cf. [4]):

(2) If E, F are Banach spaces and $u : E \to F$ is linear, surjective and continuous, then there is a constant $A > 0$ such that for any $y \in F$ with $\|y\| = 1$, there is an element $x \in E$ such that $u(x) = y$ and $\|x\| \leq A$.

The title of P. J Cohen's article [3] suggests that would be given a counterexample to the bilinear version of (1):

(1') If E_1, E_2, F are Banach spaces and $f : E_1 \times E_2 \to F$ is a bilinear mapping with closed graph, then f is continuous.

However, what is really presented is a counterexample to a certain bilinear version of (2); namely:

(2') If E_1, E_2, F are Banach spaces and $f : E_1 \times E_2 \to F$ is bilinear, surjective and continuous, then there is a constant $A > 0$ such that for any $y \in F$ with $\|y\| = 1$, there is an element $(x_1, x_2) \in E_1 \times E_2$ such that $f(x_1, x_2) = y$ and $\|x_1\| \, \|x_2\| \leq A$.

Nevertheless, (2') is not equivalent to (1'). More precisely, the Closed Graph Theorem is in fact true for multilinear mappings, as the theorem below shows.

2. MAIN RESULT.

THEOREM. *If $E_1, \ldots, E_m, F$ are Banach spaces and $f : E_1 \times \ldots \times E_m \to F$ is a multilinear mapping with closed graph, then f is continuous.*

PROOF. For each $1 \leq i \leq m$, fix an arbitrary $a_i \in E_i$. Consider the partial mapping

$$f_i : x_i \in E_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_m) \in F \qquad (1 \leq i \leq m) .$$

By the multilinearity of f, we see that each f_i is a linear mapping. Moreover, the graph $Gr(f_i)$ of f_i is closed because it is the image of the closed set

$$Gr(f) \cap (\{a_1\} \times ... \times \{a_{i-1}\} \times E_i \times \{a_{i+1}\} \times ... \times \{a_m\} \times F)$$

by the homeomorphism

$$\psi : (a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_m, y) \in \{a_1\} \times ... \times \{a_{i-1}\} \times E_i \times \{a_{i+1}\} \times ... \times \{a_m\} \times F \mapsto$$

$$(x_i, y) \in E_i \times F \ .$$

Hence, by the classical Closed Graph Theorem, each f_i is continuous. This shows that f is separately continuous, and therefore f is continuous ([2], Chap. III, §5, proposition 2 and exercise 14).

REFERENCES

[1] BANACH, S., *Théorie des Opérations Linéaires*, Warszawa, 1932.

[2] BOURBAKI, N., *Topological Vector Spaces*, Springer-Verlag, 1987.

[3] COHEN, P.J., A counterexample to the closed graph theorem for bilinear maps, *J. Func. Anal. 16* (1974), 235-240.

[4] FERNANDEZ, C.S., O Teorema do Gráfico Fechado e o Teorema de Goldstine para Aplicações m-Lineares, Dissertação de Mestrado, Instituto de Matemática, Universidade Federal do Rio de Janeiro, 1993.