Internat. J. Math. & Math. Sci.
VOL. 19 NO. 4 (1996) 797-800

NEIGHBORHOODS OF CERTAIN ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

OSMAN ALTINTAS
Department of Mathematics
Hacettepe University
Beytepe, Ankara, TURKEY

SHIGEYOSHI OWA
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577, JAPAN

(Received December 5, 1993 and in revised form December 6, 1995)

ABSTRACT. The object of the present paper is to derive some properties of neighborhoods of analytic functions with negative coefficients in the open unit disk

KEY WORDS AND PHRASES: Neighborhoods, analytic functions, and starlike functions
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 30C45

1. INTRODUCTION

Let $A(n)$ be the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0, n \in N = \{1, 2, 3, ...\}) \tag{1 1}$$

that are analytic in the open unit disk $U = \{z : |z| < 1\}$ For any $f(z) \in A(n)$ and $\delta \geq 0$, we define

$$N_{n,\delta}(f) = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\}, \tag{1 2}$$

which was called (n, δ)-neighborhood of $f(z)$ So, for $e(z) = z$, we see that

$$N_{n,\delta}(e) = \left\{ g \in A(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}.$$

The concept of neighborhoods was first introduced by A W Goodman [Proc Amer Math Soc 8 (1957), 598-601] and then generalized by Ruscheweyh [1]

In the present paper, we consider (n, δ)-neighborhoods for functions with negative coefficients in U

2. NEIGHBORHOODS FOR CLASSES $S_n^*(\alpha)$ AND $C_n(\alpha)$

Let $S_n^*(\alpha)$ denote the subclass of $A(n)$ consisting of functions which satisfy

$$\text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U) \tag{2 1}$$

for some $\alpha (0 \leq \alpha < 1)$ A function $f(z)$ in $S_n^*(\alpha)$ is said to be *starlike of order* α in U A function $f(z) \in A(n)$ is said to be *convex of order* α if it satisfies

$$\text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U) \tag{2 2}$$

for some $\alpha (0 \leq \alpha < 1)$ We denote by $C_n(\alpha)$ the subclass of $A(n)$ consisting of all such functions

For classes $S_n^*(\alpha)$ and $C_n(\alpha)$, we need the following lemmas by Chatterjea [2] (also, see Srivastava, Owa and Chatterjea [3])

LEMMA 2.1. A function $f(z) \in A(n)$ is in the class $S_n^*(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} (k-\alpha)a_k \leq 1-\alpha. \tag{2 3}$$

LEMMA 2.2. A function $f(z) \in A(n)$ is in the class $C_n(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} k(k-\alpha)a_k \leq 1-\alpha. \tag{2 4}$$

Applying the above lemmas, we prove

THEOREM 2.1. $S_n^*(\alpha) \subset N_{n,\delta}(e)$, where $\delta = (n+1)(1-\alpha)/(n+1-\alpha)$, and $S_n^*(0) = N_{n,1}(e)$

PROOF. It follows from (2 3) that if $f(z) \in S_n^*(\alpha)$, then

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)(1-\alpha)}{n+1-\alpha} = \delta. \tag{2 5}$$

Further, if $\alpha = 0$, then $f(z) \in S_n^*(0)$ if and only if

$$\sum_{k=n+1}^{\infty} ka_k \leq 1. \tag{2 6}$$

This gives that $f(z) \in N_{n,1}(e)$.

Letting $n = 1$ in Theorem 2 1, we have

COROLLARY 2.1. $S_1^*(\alpha) \subset N_{1,\delta}(e)$, where $\delta = 2(1-\alpha)/(2-\alpha)$, and $S_1^*(0) = N_{1,1}(e)$

THEOREM 2.2. $C_n(\alpha) \subset N_{n,\delta}(e)$, where $\delta = (1-\alpha)/(n+1-\alpha)$

PROOF. Noting that $f(z) \in C_n(\alpha)$ satisfies

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{1-\alpha}{n+1-\alpha}, \tag{2 7}$$

then $C_n(\alpha) \subset N_{n,\delta}(e)$

Making $n = 1$ in Theorem 2 2, we have

COROLLARY 2.2. $C_1(\alpha) \subset N_{1,\delta}(e)$, where $\delta = (1-\alpha)/(2-\alpha)$.

3. NEIGHBORHOODS FOR CLASSES $R_n(\alpha)$ AND $P_n(\alpha)$

A function $f(z) \in A(n)$ is said to be in the class $R_n(\alpha)$ if it satisfies

$$\operatorname{Re} f'(z) > \alpha \quad (z \in U) \tag{3 1}$$

for some $\alpha\,(0 \leq \alpha < 1)$. A function $f(z)$ in $R_n(\alpha)$ is said to be *close-to-convex of order* α in U (Duren [4], or Sarangi and Uralegaddi [5]).

Further, a function $f(z) \in A(n)$ is said to be a member of the class $P_n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \quad (z \in U) \tag{3 2}$$

for some $\alpha\,(0 \leq \alpha < 1)$

It is easy to see that

LEMMA 3.1. A function $f(z) \in A(n)$ is in the class $R_n(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} ka_k \leq 1-\alpha. \tag{3 3}$$

LEMMA 3.2. A function $f(z) \in A(n)$ is in the class $P_n(\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty} a_k \leq 1-\alpha. \tag{3 4}$$

From the above lemmas, we see that $R_n(\alpha) \subset P_n(\alpha)$

Now, we derive

THEOREM 3.1. $P_n(\alpha) = N_{n,\delta}(e)$, where $\delta = 1 - \alpha$

The proof of Theorem 3 1 is clear from Lemma 3 1

THEOREM 3.2. $N_{n,\delta}(e) \subset P_n(\alpha)$, where $\alpha = (n+1-\delta)/(n+1)$

PROOF. If $f(z) \in N_{n,\delta}(e)$, we have

$$\sum_{k=n+1}^{\infty} k a_k \leq \delta, \tag{3 5}$$

which gives that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\delta}{n+1} = 1 - \frac{n+1-\delta}{n+1}. \tag{3 6}$$

Thus we see that $f(z) \in P_n(\alpha)$

Making $n = 1$ in Theorem 3 2, we have

COROLLARY 3.1. $N_{1,\delta}(e) \subset P_1(\alpha)$, where $\alpha = (2-\delta)/2$

4. NEIGHBORHOODS FOR CLASSES $K_n(\alpha,\beta)$ AND $S_n(\alpha,\beta)$

Let $f(z)$ and $g(z)$ be given by (1 1) and

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \quad (b_k \geq 0). \tag{4 1}$$

If a function $f(z) \in A(n)$ satisfies

$$\text{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha \quad (z \in U) \tag{4 2}$$

for some $\alpha (0 \leq \alpha < 1)$ and $g(z) \in S_n^*(\beta)\,(0 \leq \beta < 1)$, then we say that $f(z) \in K_n(\alpha,\beta)$ If we take $g(z) = z$, then $K_n(\alpha,\beta)$ becomes $R_n(\alpha)$ Further, a function $f(z) \in A(n)$ is said to be in the class $S_n(\alpha,\beta)$ if it satisfies

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha \quad (z \in U) \tag{4 3}$$

for some $\alpha (0 \leq \alpha < 1)$ and $g(z) \in S_n^*(\beta)\,(0 \leq \beta < 1)$ If we put $g(z) = z$, then $S_n(\alpha,\beta)$ becomes $P_n(\alpha)$

For classes $K_n(\alpha,\beta)$ and $S_n(\alpha,\beta)$, we prove

THEOREM 4.1. $K_n(\alpha,\beta) \subset N_{n,\delta}(e)$, where

$$\delta = \{n(1-\alpha) + (1-\beta)\}/(n+1-\beta).$$

PROOF. If $f(z) \in K_n(\alpha,\beta)$, then we have

$$\text{Re}\left\{\frac{1 - \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} k b_k z^{k-1}}\right\} > \frac{1 - \sum_{k=n+1}^{\infty} k a_k}{1 - \sum_{k=n+1}^{\infty} k b_k} \geq \alpha. \tag{4 4}$$

It follows from (4 4) that

$$\sum_{k=n+1}^{\infty} ka_k \leq 1-\alpha+\alpha\sum_{k=n+1}^{\infty} kb_k$$
$$\leq 1-\alpha+\alpha\,\frac{1-\beta}{n+1-\beta}$$
$$\leq \frac{n(1-\alpha)+(1-\beta)}{n+1-\beta}=\delta. \qquad (4\ 5)$$

This gives $f(z) \in N_{n,\delta}(e)$

Putting $n = 1$ in Theorem 4 1, we have

COROLLARY 4.1. $K_1(\alpha,\beta) \subset N_{1,\delta}(e)$, where $\delta = (2-\alpha-\beta)/(2-\beta)$

Finally we derive

THEOREM 4.2. $N_{n,\delta}(g) \subset S_n(\alpha,\beta)$, where $g(z) \in S_n^*(\beta)$ and

$$\alpha = 1-\frac{(n+1-\beta)\delta}{n(n+1)}. \qquad (4\ 6)$$

PROOF. Let $f(z)$ be in $N_{n,\delta}(g)$ for $g(z) \in S_n^*(\beta)$

Then we know that

$$\sum_{k=n-1}^{\infty} k|a_k-b_k| \leq \delta \qquad (4\ 7)$$

and

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{1-\beta}{n+1-\beta}. \qquad (4\ 8)$$

Thus we have

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty} b_k} \leq \frac{\delta}{n+1}\cdot\frac{n+1-\beta}{n}$$
$$=\frac{(n+1-\beta)\delta}{n(n+1)}=1-\alpha. \qquad (4\ 9)$$

This implies that $f(z) \in S_n(\alpha,\beta)$.

Letting $n = 1$ in Theorem 4 2, we have

COROLLARY 4.2. $N_{1,\delta}(g) \subset S_1(\alpha,\beta)$, where $g(z) \in S_1^*(\beta)$ and $\alpha = 1-(2-\beta)\delta/2$

ACKNOWLEDGMENT. This work was supported, in part, by the Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research

REFERENCES

[1] RUSCHEWEYH, ST , Neighbourhoods of univalent functions, *Proc. Amer. Math. Soc.* **81** (1981), 521-527

[2] CHATTERJEA, S K , On starlike functions, *J. Pure Math.* **1** (1981), 23-26

[3] SRIVASTAVA, H M , OWA, S and CHATTERJEA, S K , A note on certain classes of starlike functions, *Rend. Semın. Mat. Unıv. Padova* 77 (1987), 115-124

[4] DUREN, P L , *Unıvalent Functıons,* Grundlehren der Mathematischen Wissenchaften 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983

[5] SARANGI, S M and URALEGADDI, B A , The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients, *Attı Accad. Nazıon. Lıncei* **65** (1978), 38-42