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EXISTENCE THEOREMS FOR THE MATRIX RICCATI EQUATION

$W' + WP(t)W + Q(t) = 0$

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ABSTRACT. Sufficient conditions are established for the matrix Riccati equation to have a symmetric solution on a given interval. The criteria involve integral conditions on the coefficient matrices of the Riccati equation. The present results are compared with previously known results.

1. INTRODUCTION.

Consider the matrix Riccati equation

$$W' + WP(t)W + Q(t) = 0 \qquad (' = \tfrac{d}{dt}), \tag{1.1}$$

where P(t) and Q(t) are n x n real symmetric matrix functions of the real variable t. In what follows sufficient conditions will be established for (1.1) to have a symmetric solution on a given interval. The criteria involve integral conditions on the matrices P(t) and Q(t). There are very few sufficiency criteria in the literature and even fewer integral criteria. The results will be compared with previously known results.

As for notation, capital letters will denote matrices and $P > 0$ ($P \geq 0$) indicates that the matrix P is positive definite (non-negative semi-definite). The notation $W < P$ or $P > W$ indicates $P - W > 0$ and similarly for $W \leq P$ or $P \geq W$. The transpose of a matrix P will be denoted by P^T and the $n \times n$ identity matrix will be denoted by I_n. When considering the real interval (a,b) we shall mean that $-\infty \leq a < b \leq \infty$; when considering the real interval $[a,b)$ we shall mean that $-\infty < a < b \leq \infty$ and similarly for $(a,b]$ and $[a,b]$.

2. MAIN RESULTS.

THEOREM 1. Let $P(t)$, $Q(t)$, and $N(t)$ be $n \times n$ real symmetric and continuous matrices on $I = (a,b)$ (or any interval). Suppose for each t in I that $P(t) \geq 0$, $\int_a^t (N(s) + Q(s))ds$ exists and is finite, and

$$N(t) \geq \left[\int_a^t (N(s) + Q(s))ds\right] P(t) \left[\int_a^t (N(s) + Q(s))ds\right].$$

Then (1.1) has a symmetric solution defined on I.

PROOF. Let $V(t) = -\int_a^t (N(s) + Q(s))ds$ for each t in $I = (a,b)$ then $V'(t) = -N(t) - Q(t)$ and hence on I

$$V'(t) + V(t)P(t)V(t) + Q(t)$$

$$= \left[\int_a^t (N(s) + Q(s))ds\right] P(t) \left[\int_a^t (N(s) + Q(s))ds\right] - N(t) \leq 0.$$

Let $c \in I = (a,b)$ and let $W(t)$ be a symmetric solution of (1.1) such that $W(c) = V(c)$. By [1;p. 52] (or [4;p.122]) $W(t)$ is defined and satisfies $W(t) \geq V(t)$ on $[c,b)$. An extension of the results of [1;p.52] yields $W(t)$ defined and satisfying $W(t) \leq V(t)$ on $(a,c]$. Thus $W(t)$ exists on $I = (a,b)$.

The validity of the theorem on intervals $[a,b)$, $(a,b]$, or $[a,b]$ is now evident. QED

Setting $N(t) = Q(t)$ in Theorem 1 we have the following known sufficiency criterion, [3;p.344].

COROLLARY TO THEOREM 1. Let $P(t) = I_n$ and let $Q(t) = Q^T(t)$ be an $n \times n$ real continuous matrix on $I = [0,1]$ with $Q(t) \geq 4 \left[\int_0^t Q(s)ds \right]^2$ for each t in I. Then (1.1) has a symmetric solution defined on I.

The following example illustrates that the matrix $Q(t)$ in Theorem 1 need not be non-negative semi-definite.

EXAMPLE. Let $p(t) = 1$, $q(t) = t - 1$, and $n(t) = t + 1$ on $I = [0,r]$ where r is the positive real root of $t^4 - t - 1 = 0$. Note that $1 < r$ and on $[0,r]$

$$0 \geq t^4 - t - 1.$$

Hence

$$t + 1 \geq t^4 = \left[\int_0^t 2sds \right]^2 = \left[\int_0^t ((s + 1) + (s - 1))ds \right]^2$$

or

$$n(t) \geq \left[\int_0^t (n(s) + q(s))ds \right]^2.$$

Thus by Theorem 1, $w' + w^2 + q(t) = 0$ has a solution defined on $I = [0,r]$.

A companion result to Theorem 1 is the following.

THEOREM 2. Let $P(t)$, $Q(t)$, and $N(t)$ be $n \times n$ real symmetric and continuous matrices on $I = (a,b)$ (or any interval). Suppose for each t in I that $P(t) \geq 0$, $\int_t^b (N(s) + Q(s))ds$ exists and is finite, and

$$N(t) \geq \left[\int_t^b (N(s) + Q(s))ds \right] P(t) \left[\int_t^b (N(s) + Q(s))ds \right].$$

Then (1.1) has a symmetric solution defined on I.

PROOF. Let $V(t) = \int_t^b (N(s) + Q(s))ds$ for each t in $I = (a,b)$. Then $V' + VP(t)V + Q(t) = VP(t)V - N(t) \leq 0$ on I. The proof now proceeds as in the proof of Theorem 1. QED

The following corollary of Theorem 2 is also a known sufficiency criterion, [3;p.344].

COROLLARY TO THEOREM 2: Let $P(t) = I_n$ and let $Q(t) = Q^T(t)$ be an $n \times n$ real continuous matrix on $I = (0,\infty)$. Suppose $\int_1^\infty Q(s)ds = \lim_{t\to\infty} \int_1^t Q(s)ds$ exists and is finite, and either

$$(\alpha) \quad Q(t) \geq 4\left[\int_t^\infty Q(s)ds\right]^2 \quad \text{on } I \text{ or}$$

$$(\beta) \quad -3I_n \leq 4t\int_t^\infty Q(s)ds \leq I_n \text{ on } I.$$

Then (1.1) has a symmetric solution defined on I.

PROOF. (α) In Theorem 2 take $a = 0$, $b = \infty$, $P(t) = I_n$ and $N(t) = Q(t)$.

(β) Since $-\frac{3}{4t} I_n \leq \int_t^\infty Q(s)ds \leq \frac{1}{4t} I_n$ then

$$0 \leq -\int_t^\infty Q(s)ds + \frac{1}{4t} I_n \text{ and } 0 \leq \frac{3}{4t} I_n + \int_t^\infty Q(s)ds.$$

Hence

$$0 \leq \frac{3}{16t^2} I_n + \frac{1}{4t}\int_t^\infty Q(s)ds - \frac{3}{4t}\int_t^\infty Q(s)ds - \left[\int_t^\infty Q(s)ds\right]^2,$$

or

$$\left[\int_t^\infty Q(s)ds + \frac{1}{4t} I_n\right]^2 \leq \frac{1}{4t^2} I_n.$$

Thus

$$\left[\int_t^\infty \left(Q(s) + \frac{1}{4s^2} I_n\right)ds\right]^2 \leq \frac{1}{4t^2} I_n.$$

In Theorem 2 take $a = 0$, $b = \infty$, $P(t) = I_n$, and $N(t) = \frac{1}{4t^2} I_n$. QED

THEOREM 3. Let $P(t)$, $Q(t)$, and $M(t)$ be $n \times n$ real symmetric and continuous matrices on $I = (a,b)$ (or $I = (a,b]$). Suppose for each t in I that $P(t) \geq 0$,

$\int_a^t (M(s) + P(s))ds$ exists and is finite, is invertible, and

$$Q(t) \le \left[\int_a^t (M(s) + P(s))ds \right]^{-1} M(t) \left[\int_a^t (M(s) + P(s))ds \right]^{-1}.$$

Then (1.1) has a symmetric solution defined on I.

PROOF. Let $V(t) = \left[\int_a^t (M(s) + P(s))ds \right]^{-1}$ for each t in I = (a,b) and proceed as in the proof of Theorem 1. QED

As an application of the above result we have the following corollary. The proof follows from Theorem 3 with $M(t) = \alpha t^r I_n$ and a well-known comparison theorem, [3;p.340].

COROLLARY TO THEOREM 3. Let P(t) and Q(t) be n × n real symmetric and continuous matrices on I = (0,b) (or I = (0,b]). Suppose α and $r(\neq -1)$ are real numbers such that for each t in I $1 + \alpha t^r > 0$, $0 \le P(t) \le P(t) \le I_n$, and

$$Q(t) \le \frac{\alpha t^{r-2}}{(1 + \frac{\alpha}{r+1} t^r)^2} I_n.$$

Then (1.1) has a symmetric solution defined on I.

The following example provides a comparison of the criterion given in Theorem 3 and the known result found in [3;p.344].

EXAMPLE. Consider

$$w' + w^2 + \frac{3}{2t^{3/2}(1 + t^{1/2})^2} = 0. \qquad (2.2)$$

By Corollary 6 with b = 1, r = 1/2 and α = 3/2 (2.2) has a solution defined on I = (0,1] and hence on $[1/10^4,1]$. Now

$$\int_{1/10^4}^1 \frac{3dt}{2t^{3/2}(1 + t^{1/2})^2} \ge \frac{3}{2} \int_{1/10^4}^1 \frac{dt}{4t^{3/2}} = \frac{3}{4}(99) > 8.$$

Also

$$4\left[\int_{1/10^4}^{1} dt\right]^{-1} = \frac{4}{1 - 1/10^4} < 8.$$

Thus

$$4\left[\int_{1/10^4}^{1} dt\right]^{-1} \not\leq \int_{1/10^4}^{1} \frac{3dt}{2t^{3/2}(1 + t^{1/2})^2}$$

and hence the sufficient condition as stated in [3;p.344] cannot be satisfied on $[1/10^4,1]$.

Before stating the final result we note two additional applications of Theorem 3. Setting $\alpha = 1$, $r = 0$ and $b = \infty$ in the Corollary to Theorem 3 we obtain Kneser's criterion ($Q(t) \leq 1/4t^2 I_n$). Then a case when $P(t)$ is not dominated by I_n is the following.

COROLLARY TO THEOREM 3. Let $P(t) = \beta t^q I_n$ on $I = (0,b)$ (or $I = 0,b]$) where β and $q(\neq -1)$ are real constants. Let $Q(t) = Q^T(t)$ be an $n \times n$ real continuous matrix on I. Suppose for each t in I that $P(t) \geq 0$, $\beta t^q + \alpha t^r > 0$ where α and $r(\neq -1)$ are real constants. Suppose further that for each t in I

$$Q(t) \leq \alpha t^{r^{-2}} \left[\frac{\alpha t^r}{r + 1} + \frac{\beta t^q}{q + 1}\right]^{-2} I_n.$$

Then (1.1) has a symmetric solution defined on I.

Finally we have the following companion result to Theorem 3.

THEOREM 4. Let $P(t)$, $M(t)$, and $Q(t)$ be $n \times n$ real symmetric and continuous matrices on $I = (a,b)$ (or $I = [a,b)$). Suppose for each t in I that $P(t) \geq 0$, $\int_t^b (M(s) + P(s))ds$ exists, is finite, invertible, and

$$Q(t) \leq \left[\int_t^b (M(s) + P(s))ds\right]^{-1} P(t) \left[\int_t^b (M(s) + P(s))ds\right]^{-1}.$$

Then (1.1) has a symmetric solution defined on I.

PROOF. Let

$$V(t) = -\left[\int_t^b (M(s) + P(s))ds\right]^{-1}$$

for each t in I and proceed as in the proof of Theorem 3. QED

It should be noted that in the main results stated above the matrix Q(t) is not required to be non-negative semi-definite in contrast to that requirement on Q(t) in (α) of the Corollary to Theorem 2, in [3;p.344], [2;p.89] and [2;p.83].

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