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# CONCERNING ROW OPERATIONS ON SYSTEMS OF LINEAR EQUATIONS WITH INTEGRAL COEFFICIENTS

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While trying to solve the problem stated in the introduction of [1] the author became aware of the theorem contained in this paper. Let

$$\begin{array}{c} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array} \qquad (1)$$

be a system of linear equations, where each $a_{ij}$ and $b_i$ is an integer. Define a row operation on (1) to be where $\sum_{j=1}^{n} a_{ij}x_j = b_i$ is replaced by $\sum_{j=1}^{n} (a_{ij} + ta_{kj})x_j = b_i + tb_k$, where $k \neq i$ and $t$ is an integer.

THEOREM 1. Such a system as (1) can be transformed by a finite sequence of row operations into

$$\begin{array}{c} c_{11}x_1 + \dots + c_{1n}x_n = d_1 \\ \vdots \\ c_{m1}x_1 + \dots + c_{mn}x_n = d_m, \end{array} \qquad (2)$$

where (1) $d_2 = \sum_{j=1}^{n} c_{2j}$, (2) $d_i = \sum_{j=1}^{n} c_{ij} = 0$ if $m \geq i > 2$, and (3) $|d_1| < |d_2|$ or $d_2 = 0$.

PROOF. Given such a system as (1), define $c_j = \sum_{p=1}^{n} a_{jp} - b_j (1 \leq j \leq m)$, and define $E(1) = \sum_{j=1}^{m} |c_j|$. Suppose we consider systems (1), (2),...,(q) where $E(1) > E(2) > \dots > E(q)$, but no row operation can be performed on (q) to find a smaller E. For convenience suppose the coefficients of (q) are renamed as those in (1).

Suppose there exist two integers $j_1$, $j_2$ in [1,m] such that $|c_{j_1}| \geq |c_{j_2}| > 0$. Thus $|c_{j_1}| > |c_{j_1} - c_{j_2}|$ or $|c_{j_1}| > |c_{j_1} + c_{j_2}|$. In the first case replace

$\sum_{j=1}^{n} a_{j_1 j} = b_{j_1}$ by $\sum_{j=1}^{n} (a_{j_1 j} - a_{j_2 j})x_j = b_{j_1} - b_{j_2}$, and in the other case use the sum. In either case the resulting system has a smaller E, so in all but one of the equations of (q) the righthand coefficient is the sum of the other coefficients. Using row operations we may suppose row one is the exception.

Still assuming the coefficients of (q) are the $a_{ij}$ and $b_i$ define $D_j = \sum_{p=1}^{n} a_{jp} + b_j$ and let $N(q) = \sum_{j=1}^{m} |D_j|$. We now suppose N has been reduced as far as possible by row operations which do not involve row one. Since $D_j = 0$ for all but one j in {2,...,m}, we suppose the exceptional row is two, and that our system is in the form of (2) where conclusions (1) and (2) hold.

To show (3), if $d_2 \neq 0$ and $|d_1| \geq |d_2|$, we may subtract some integer multiple of row two from row one to produce the desired result.

## REFERENCES

1. Borosh, I. and L. B. Treybig. Bounds on Positive Integral Solutions of Linear Diophantine Equations, Proc. Amer. Math. Soc. 55 (1976) 299-304.

*KEY WORDS AND PHRASES. Linear equations.*

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