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GRAPHS WHICH HAVE PANCYCLIC COMPLEMENTS

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ABSTRACT. Let p and q denote the number of vertices and edges of a graph G, respectively. Let $\Delta(G)$ denote the maximum degree of G, and $\overline{G}$ the complement of G. A graph G of order p is said to be pancyclic if G contains a cycle of each length n, $3 \leq n \leq p$. For a nonnegative integer k, a connected graph G is said to be of rank k if $q = p - 1 + k$. (For k equal to 0 and 1 these graphs are called trees and unicyclic graphs, respectively.)

In 1975, I posed the following problem: Given k, find the smallest positive integer p_k, if it exists, such that whenever G is a rank k graph of order $p \leq p_k$ and $\Delta(G) < p - 2$ then $\overline{G}$ is pancyclic. In this paper it is shown that a result by Schmeichel and Hakimi (2) guarantees that p_k exists. It is further shown that for $k = 0, 1,$ and 2, $p_k = 5, 6,$ and 7, respectively.

1. INTRODUCTION.

Throughout this paper the terminology of Behzad and Chartrand (1) will be followed. In particular, p and q shall denote the number of vertices and edges of a graph G, respectively. We let $\Delta(G)$ denote the maximum degree of G and $\overline{G}$ denote the complement of G.

A graph G of order p is called pancyclic if G contains a cycle of each length n, $3 \leq n \leq p$. For a nonnegative integer k, a connected graph G is said to be of rank k if $q = p - 1 + k$. Here the number k gives the number of independent cycles in G. When k equals 0 or 1 these graphs are called trees or unicyclic graphs, respectively.

In this paper we explore the following idea: if G is a graph having, in some sense, little cycle structure relative to its order, then perhaps $\overline{G}$ will have a great deal of cycle structure. As an example, consider the graph shown in Figure 1. This graph is a tree, i.e., a connected graph having no cycles. On the other hand note that its complement is pancyclic.

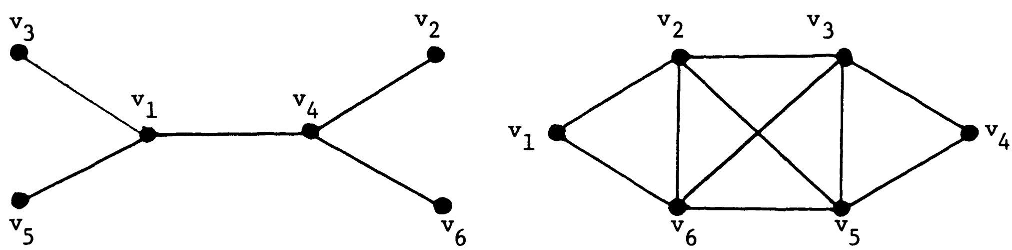


Figure 1. A tree...and its pancyclic complement

In 1975, after obtaining the results for k = 0, 1, and 2 which are presented here, I posed the following problem: Given k, find the smallest positive integer p_k, if it exists, such that whenever G is a graph of rank k of order $p \geq p_k$ and $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic. Recently, J. A. Bondy has pointed out that the existence of p_k is guaranteed by the following result due to Schmeichel and Hakimi [2].

THEOREM. Let G be a graph with p vertices, q edges, and minimum degree $\delta \geq 2$. If

$$q > \begin{cases} \frac{1}{2}(p^2 - (2\delta + 1)p + 3\delta^2 + \delta), & 2 \leq \delta \leq \frac{p+5}{6} \text{ and p odd} \\ \frac{1}{2}(p^2 - (2\delta + 1)p + 3\delta^2 + \delta), & 2 \leq \delta \leq \frac{p+8}{6} \text{ and p even} \\ \frac{1}{8}(3p^2 - 8p + 5) + \delta, & \frac{p+5}{6} \leq \delta \leq \frac{p-1}{2} \text{ and p odd} \\ \frac{1}{8}(3p^2 - 10p + 16) + \delta, & \frac{p+8}{6} \leq \delta \leq \frac{p-2}{2} \text{ and p even} \\ \frac{1}{4}p^2 , & \frac{p-1}{2} < \delta \end{cases}$$

then G is pancyclic.

COROLLARY. Let k be a nonnegative integer. Then there exists a positive integer p_k such that whenever G is a graph of rank k of order $p \geq p_k$ and $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. Let G be a graph of rank k with $\Delta(G) < p - 2$. If G has p vertices, then $\overline{G}$ has p vertices, $q = \frac{p^2 - 3p + 2}{2} - k$ edges and minimum degree $\delta \geq 2$. Depending on the value of δ, the requirements for q given by the theorem would yield the following inequalities:

$$(2\delta - 2)p > 3\delta^2 + \delta + 2k - 2$$
$$p^2 - 4p > 8k + 8\delta - 3$$
$$p^2 - 2p > 8k + 8\delta + 8$$
$$p^2 - 6p > 4k - 4$$

Note that each of the above inequalities is true provided that p is large enough. Hence we can choose p_k to be the least positive integer which makes all the above inequalities true.

The above theorem yields an upper bound for p_k; however, in the known cases it does not give us a very good bound. For example, the theorem would tell us that $p_2 \leq 10$, whereas we will show that $p_2 = 7$. In the remainder of the paper we show that for k = 0, 1, and 2, p_k = 5, 6, and 7, respectively.

2. MAIN RESULTS.

THEOREM 1. If G is a tree of order $p \geq 5$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. The proof is by induction on p. If p = 5, then G is a path on 5 vertices. Thus $\overline{G} = C_5 + e$ and so clearly $\overline{G}$ is pancyclic. Now let $p \geq 6$, and assume the result holds for all trees of order less than p. Let G be a tree of order p with $\Delta(G) < p - 2$. If $\Delta(G) < p - 3$, let v be an end vertex of G. If $\Delta(G) = p - 3$, then unless G is the graph of Figure 1, we may choose v to be an end vertex adjacent with the unique vertex of degree p - 3. Now consider G - v, which is a tree of order p - 1 with $\Delta(G - v) < (p - 1) - 2$. Hence by the induction hypothesis, $\overline{G - v}$ has a cycle of each length n, $3 \leq n \leq p - 1$. Therefore so does $\overline{G}$. Since $\deg_{\overline{G}} v = p - 2 > \frac{p-1}{2}$, v must be adjacent in $\overline{G}$ to two consecutive vertices on the (p-1)-cycle in $\overline{G - v}$. Thus this cycle can be extended in $\overline{G}$ to a cycle of length p. Therefore, $\overline{G}$ is pancyclic. Now by induction the proof is completed.

COROLLARY 1. If G is a forest of order $p \geq 5$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. Note that there exists a tree H with $\Delta(H) < p - 2$ containing G as a spanning subgraph. Since $\overline{H}$ is pancyclic and $\overline{H} \subseteq \overline{G}$, $\overline{G}$ is pancyclic.

THEOREM 2. If G is a unicyclic graph of order $p \geq 6$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. Let $u_1, u_2, \ldots, u_n$ denote the cycle vertices of G. Among the vertices u_i, we will choose one of minimum degree in G; call it u.

CASE 1. Suppose $n \geq 4$. Then $\deg u \leq 2 + \frac{p-4}{4} = \frac{p+4}{4}$. Note that $\frac{p+4}{4} < \frac{p-1}{2}$ provided that $p \geq 6$. If $\Delta(G) = p - 3$, notice that we can choose u so that $\Delta(G - u) = p - 4$ unless G is the graph of Figure 3a. Now since $G - u$ is a forest, $\overline{G - u}$ is pancyclic by Corollary 1. Since u is adjacent in $\overline{G}$ to two consecutive vertices on the (p-1)-cycle of $\overline{G - u}$, this cycle can be extended to a p-cycle in $\overline{G}$. Therefore, $\overline{G}$ is pancyclic.

CASE 2. Suppose $n = 3$. First, suppose that if $\Delta(G) = p - 3 = \deg v$, then v is one of the 3 cycle vertices. Then we know that $\Delta(G - u) \leq p - 4$. Now $\deg u \leq 2 + \frac{p-3}{3} = \frac{p+3}{3}$. If $\deg u < \frac{p-1}{2}$ we can proceed just as in Case 1. This will happen if $p > 9$ or if $p = 8$. If it does not happen then G must be one of the graphs shown in Figures 3b - d, all of which have pancyclic complements. Secondly suppose that $\Delta(G) = p - 3 = \deg_G v$ and v is not a cycle vertex. Then G is the graph of Figure 2.

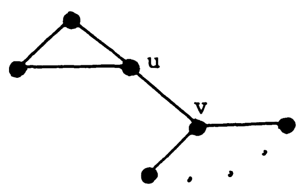


Figure 2

If $p \geq 8$, we can remove u and proceed as above. If $p = 6$ or 7, we again get special case graphs, which can be shown to have pancyclic complements.

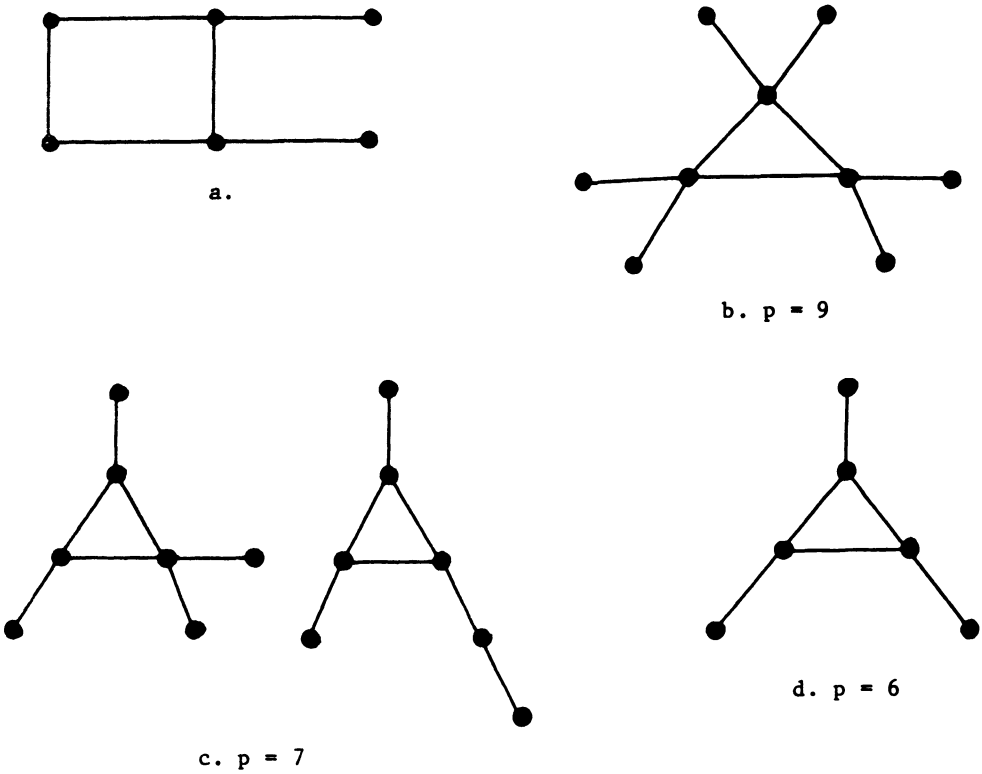


Figure 3

COROLLARY 2. If G is a graph of order $p \geq 6$ with $\Delta(G) < p - 2$ and G contains exactly one cycle, then $\overline{G}$ is pancyclic.

The five-cycle C_5 is a unicyclic graph on 5 vertices which does not have a pancyclic complement. This shows that $p_1 = 6$. The graph shown in Figure 4 is a rank 2 graph of order 6 whose complement does not contain a 3-cycle. Hence $p_2 \geq 7$. Our final result shows that indeed $p_2 = 7$.

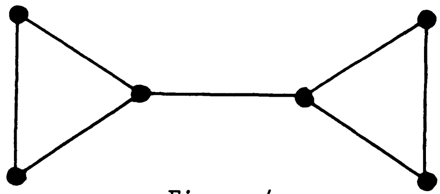


Figure 4

THEOREM 3. If G is a graph of rank 2 of order $p \geq 7$ with $\Delta(G) < p - 2$, then $\overline{G}$ is pancyclic.

PROOF. We consider three cases.

CASE 1. G has a cycle with a diagonal. Let $u_1, u_2, \ldots, u_n$ be the cycle vertices of G, $n \geq 4$, and suppose u_1u_i is the diagonal of the cycle, $3 \leq i \leq n - 1$. First, suppose $\Delta(G) < p - 3$. Choose a cycle vertex u which has the smallest degree in G among the cycle vertices. Then $\deg u \leq \frac{p+6}{4}$. Note that $\frac{p+6}{4} < \left\{\frac{p-1}{2}\right\}$ if $p \geq 8$. Also there does not exist a rank 2 graph on 7 vertices with $\Delta(G) = 3$ and $\deg u_i = 3$, $1 \leq i \leq n$. Thus $p \geq 7$ implies $\deg_G u < \left\{\frac{p-1}{2}\right\}$. Now by Corollary 2, $\overline{G - u}$ is pancyclic. Since u is adjacent in $\overline{G}$ to more than half the other vertices, $\overline{G}$ is pancyclic. Secondly, suppose $\Delta(G) = p - 3$ and $p \geq 8$. Then we either choose u as above or of degree 3 in G in such a way that $\Delta(G - u) = p - 4$. Since $3 < \frac{p-1}{2}$, $\deg_G u < \frac{p-1}{2}$ and so we argue as before. Lastly we must consider the case where $p = 7$, $\Delta(G) = 4$, and there does not exist a vertex u with $\deg_G u = 2$ and $\Delta(G - u) = 3$. In this case G must be one of the graphs shown in Figure 5, all of which have pancyclic complements.

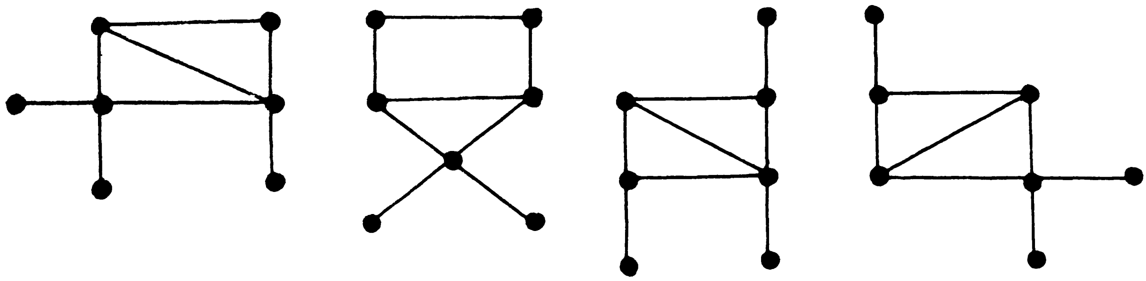
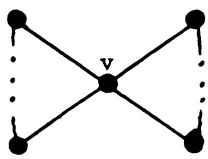


Figure 5

CASE 2. G has the following configuration as a subgraph.



Again let u be a cycle vertex of smallest degree. Then

$$\deg_G u \leq \begin{cases} 2 + \frac{p-3}{5}, & u \neq v \\ 4 + \frac{p-13}{5}, & u = v \end{cases} = \frac{p+7}{5} < \frac{p-1}{2}.$$

Also if $\Delta(G) = p - 3$, then clearly it is possible to choose u so that $\Delta(G - u) = p - 4$. Hence we may argue as before.

CASE 3. G contains the configuration of Figure 6.

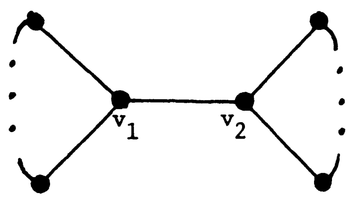


Figure 6

Choose u as before. Then

$$\deg_G u \leq \begin{cases} 2 + \frac{p-4}{6}, & u \notin \{v_1, v_2\} \\ 3 + \frac{p-10}{6}, & u \in \{v_1, v_2\} \end{cases} = \frac{p+8}{6} < \frac{p-1}{2}.$$

If $\Delta(G) = p - 3$, then $\deg v_i = p - 3$ for $i = 1$ or 2 but not both, and all other cycle vertices have degree 2. Hence we may choose u so that $\Delta(G - u) = p - 4$ and proceed as before.

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2. Schmeichel, E. F. and S. L. Hakimi. Pancyclic Graphs and a Conjecture of Bonday and Chvatal, J. Combinatorial Theory (B) 17 (1974) 22-34.

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