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ON THE MEIJER TRANSFORMATION

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ABSTRACT. Recently [8], an operational calculus for the operator $B_\mu = t^{-\mu}Dt^{1+\mu}D$ with $-1 < \mu < \infty$ was developed via the algebraic approach [4], [13], [15]. This paper gives the integral transform version. In particular, a differentiation theorem and a convolution theorem are proved.

1. INTRODUCTION.

Ditkin [4], and later with Prudnikov [6], developed an operational calculus for the operator $\frac{d}{dt} t \frac{d}{dt}$ similar to the algebraic approach of Mikusinski [15]. Meller [13], [14] generalized Ditkin's calculus to operators $B_\alpha = t^{-\alpha}Dt^{1+\alpha}D$ with $-1 < \alpha < 1$. Krätzel [9], [10], [11], [12] gave an integral transform version to Meller's calculus and also generalized the calculus to operators containing n^{th} order derivatives. He developed a theory of integral transforms of the form

$$\mathcal{L}_v^{(n)}\{f\}(s) = \int_0^\infty w_v^{(n)}(n(st)^{1/n})f(t)dt,$$

where $n = 1, 2, \ldots, \mathrm{Re}(v) > \frac{1}{n} - 1$, and

$$w_v^{(n)}(z) = \frac{(2\pi)^{\frac{n-1}{2}}\sqrt{n}\,(\frac{z}{n})^{nv}}{\Gamma(v+1-1/n)} \int_1^\infty (y^n-1)^{v-\frac{1}{n}} \exp(-zy)dy.$$

Here, $\mathcal{L}_v^{(1)}$ is the Laplace transform and $\mathcal{L}_v^{(2)}$ is the Meijer transform of the form

$$\mathcal{L}_v^{(2)}\{f\}(s) = 2\int_0^\infty (st)^{v/2} K_v(2\sqrt{st})f(t)dt, \tag{1}$$

where $K_v(z)$ is the MacDonald function of order v. Dimovski [1], [2], [3] developed an operational calculus for the operator

$$B = t^{\alpha_0} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} \cdots t^{\alpha_{n-1}} \frac{d}{dt} t^{\alpha_n},$$

using an integral transform that for $n = 2$ reduces to the Meijer transform of the form

$$\tilde{k}_v\{f\}(s) = 2s^{-v}\int_0^\infty (st)^{v/2} K_v(2\sqrt{st})f(t)dt. \tag{2}$$

In [8], Koh reconsidered Meller's operator $B_\mu = t^{-\mu}\frac{d}{dt}t^{1+\mu}\frac{d}{dt}$ but with $\mu\varepsilon(-1,\infty)$. Following Mikusinski, Ditkin, et. al., he constructed an operational calculus through the field extension of a commutative convolution ring without zero divisors. His calculus reduces to Ditkin's when $\mu = 0$ and Meller's when $\mu\varepsilon(-1,1)$.

In this paper, we give an integral transform analogue of [8] via the Meijer transform of the form

$$k_\mu\{f\}(p) = \frac{2p}{\Gamma(\mu+1)} \int_0^\infty (pt)^{\mu/2} K_\mu(2\sqrt{pt}) f(t)\, dt \tag{3}$$

for $\mathrm{Re}(\mu) > -1$. In particular, we prove a differentiation theorem and a convolution theorem. The presence of a factor $\frac{2p}{\Gamma(\mu+1)}$ in (3), as opposed to those in (1) and (2), is essential in our convolution theorem.

2. <u>THE MAIN THEOREMS</u>.

We will define the convolution, *, of two functions, f, g by

$$f*g = \frac{1}{\Gamma(\mu+1)} D t^{1-\mu} D^{\mu+1} \int_0^t \xi^\mu (t-\xi)^\mu \int_0^1 f(x\xi) g[(1-x)(t-\xi)]\, dx\, d\xi, \tag{4}$$

see Koh [8], where D^λ is the Riemann-Liouville derivative of order λ, see Ross [17]. This convolution exists if, for example, f and g are in $C^\infty[0,\infty)$, the space of infinitely differentiable complex functions on $[0,\infty)$.

The following properties of K_μ will be used:

$$K_\mu(2\sqrt{pt}) = \tfrac{1}{2}(t/p)^{\mu/2} \int_0^\infty x^{-\mu-1} \exp(-px - \tfrac{t}{x})\, dx \tag{5.1}$$

$$= \tfrac{1}{2}(t/p)^{-\mu/2} \int_0^\infty x^{\mu-1} \exp(-px - \tfrac{t}{x})\, dx, \tag{5.2}$$

$\mathrm{Re}(\mu) > -1$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(t) > 0$.

$$2(pt)^{\mu}K_{\mu}(2\sqrt{pt}) \sim \left\{ \begin{array}{l} -\ln t+0(1),\ \mu = 0 \\ \Gamma(\mu)+0[t^{\min(1,\mu)}],\ \mu > 0 \\ -\frac{\Gamma(1-\mu)}{\mu}(pt)^{\mu}+0(1),\ -1 < \mu < 0 \end{array} \right\},\ t \to 0 \tag{6.1}$$

$$\sim \sqrt{\frac{\pi}{2}}\, t^{\frac{\mu}{2}-\frac{1}{4}} e^{-2\sqrt{pt}}\{1+0(|t|^{-\frac{1}{2}})\},\ t \to \infty . \tag{6.2}$$

$$\frac{d}{dt}\{(pt)^{\pm\mu/2}K_{\mu}(2\sqrt{pt})\} = -p(pt)^{\pm\mu-\frac{1}{2}}K_{\mu\mp 1}(2\sqrt{pt}). \tag{7}$$

In order that the Meijer transform (3) converges, it is sufficient for f(t) to be locally Lebesgue integrable on $(0,\infty)$ and $|f(t)| < Ce^{2\gamma\sqrt{t}}$ $(t \to \infty)$ for $\mu > 0$ and for f(t) to remain bounded in the neighbourhood of the origin for $-1 < \mu \leq 0$. The integral then converges absolutely within the parabolic region $\mathrm{Re}\sqrt{p} > \gamma$. This is clear from the asymptotic behaviors (6.1) and (6.2). Indeed,

$$|\int_0^{\infty} f(t)(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt| \leq \int_0^{\infty}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})\,dt$$

$$\leq \sup_{0<t<\varepsilon}|f(t)|\int_0^{\varepsilon}(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt + \int_{\varepsilon}^{T}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})dt$$

$$+ \int_T^{\infty}|f(t)|(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})\,dt, \text{ for some } 0 < \varepsilon < T < \infty. \tag{8}$$

The first integral on the right hand side of (8) exists because of (6.1); the second exists because of the local integrability of f(t) and the continuity of $(pt)^{\frac{\mu}{2}}K_{\mu}(2\sqrt{pt})$; and the last integral exists because of (6.2) provided $\operatorname{Re}\sqrt{p} > \gamma$. We state this result in

THEOREM 1. If $f(t)\varepsilon L_{\ell oc}(0,\infty)$, if there are constants C and γ such that $|f(t)| < Ce^{2\gamma\sqrt{t}}$ as $t \to \infty$, and if $\lim_{t\to 0^+} f(t) = f(0^+) < \infty$, then (3) converges absolutely in $\operatorname{Re}\sqrt{p} > \gamma$ for all $\mu\varepsilon(-1,\infty)$. Furthermore, the integral (3) as a function of p is analytic in the region of convergence.

The proof of the analyticity is standard and is omitted. When a function f(t) satisfies the hypothesis of theorem 1, we shall write, for brevity, $f\varepsilon \mathrm{HypI}$. Clearly, if a function f has continuous derivative on $[0,\infty)$ and $f'\varepsilon\mathrm{HypI}$, then $f\varepsilon\mathrm{HypI}$.

THEOREM 2. If $f\varepsilon C^2[0,\infty)$ and $f'\varepsilon\mathrm{HypI}$, then

$$k_\mu(B_\mu f) = p(k_\mu f) - pf(0^+).$$

PROOF.

$$k_\mu(B_\mu f) = \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \int_0^\infty [t^{-\mu}\frac{d}{dt}t^{1+\mu}\frac{d}{dt}f(t)]t^{\frac{\mu}{2}}K_\mu(2\sqrt{pt})dt$$

$$= \frac{2p^{\frac{\mu}{2}+1}}{\Gamma(\mu+1)} \{t^{-\frac{\mu}{2}}K_\mu(2\sqrt{pt})t^{\mu+1}\frac{df}{dt}\Big|_0^\infty - \int_0^\infty (t^{\mu+1}\frac{df}{dt})\frac{d}{dt}(t^{-\frac{\mu}{2}}K_\mu(2\sqrt{pt})dt\}.$$

The limit terms vanish because $f'\varepsilon\mathrm{HypI}$. We now use (7) and another integration by parts to yield

$$k_\mu(B_\mu f) = \frac{2p^{\frac{\mu}{2}+\frac{3}{2}}}{\Gamma(\mu+1)} \int_0^\infty \left(\frac{df}{dt}\right) t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt})dt$$

$$= \frac{2p^{\frac{\mu}{2}+\frac{3}{2}}}{\Gamma(\mu+1)} \{-\lim_{t\to 0^+} [f(t)t^{\frac{\mu+1}{2}} K_{\mu+1}(2\sqrt{pt})]+p^{\frac{1}{2}}\int_0^\infty f t^{\frac{\mu}{2}} K_\mu(2\sqrt{pt})dt\}$$

$$= pk_\mu(f)(p)-pf(0^+). \quad \text{QED}$$

This result immediately generalizes to the next theorem by induction.

THEOREM 3. If $f\varepsilon C^{2k}[0,\infty)$ and $f^{(2k-1)}\varepsilon \text{HypI}$, then

$$k_\mu(B_\mu{}^k f) = p^k k_\mu[f] - \sum_{j=1}^{k} p^j B_\mu^{k-j} f(0).$$

Note that this theorem is the integral transform version of Lemma 1 of [8]. The operational calculus for the operator B_μ is now effected through this formula. To solve the initial value problem

$$Q(B_\mu)f(t) = g(t)$$

$$f(0) = C_0,\ B_\mu f(0) = C,\ \ldots,\ B^{k-1}f(0) = C_{k-1} \tag{9}$$

where Q(z) is a polynomial, we transform (9) into

$$Q(p)k_\mu f = P(p) + k_\mu g$$

where P(p) is a polynomial of degree less than or equal to that of Q(p). Therefore

$$k_\mu f = \frac{P(p)}{Q(p)} + \frac{1}{Q(p)}(k_\mu g)(p)$$

and f(t) is retrieved by means of an inversion formula and possibly a convolution theorem.

The following inversion theorem is obtained from Meijer's Theorem [18] through a simple change of variables, viz. $x \to \sqrt{t}$ and $y \to 2\sqrt{p}$.

THEOREM 4. Let μ be a complex number whose real part is not less than $-\frac{1}{2}$. Assume that in $\mathrm{Re}\sqrt{p} > \gamma_0 \geq 0$, $F(p)$ is an analytic function and is bounded according to $|F(p)| < M|p|^{-q}$ where $q > \frac{3}{2}\mathrm{Re}\mu+2$. Then for real $c > \gamma_0$ and for $\mathrm{Re}\sqrt{p} > c$, $F(p) = k_\mu(f)$ where

$$f(t) = \frac{\Gamma(\mu+1)t^{-\frac{\mu}{2}}}{2\pi i} \int_{\mathrm{Re}\sqrt{p}=c} F(p)p^{-\frac{\mu}{2}-1} I_\mu(2\sqrt{pt})\,dp. \tag{10}$$

The following lemma will be used in proving a convolution theorem for k_μ.

LEMMA. Letting D_∞^μ denote the Weyl derivative of order μ, we have

$$D_\infty^\mu \{(z/t)^{\frac{\mu}{2}} K_\mu(2\sqrt{zt})\} = (-z/t)^\mu K_{2\mu}(2\sqrt{zt}).$$

PROOF. By definition, $D_\infty^\mu \{f(t)\} = (d/dt)^k W^{k-\mu}\{f(t)\}$, where $k-1 < \mu < k$, and where

$$W^\nu\{f(t)\} = [-1/\Gamma(\nu)] \int_t^\infty f(s)(t-s)^{\nu-1}ds.$$

Since

$$K_\mu(2\sqrt{zt}) = \frac{1}{2}(t/z)^{\mu/2} \int_0^\infty \exp\{-(zy+t/y)\}y^{-\mu-1}dy \tag{11}$$

we have

$$W^{k-\mu}\{t^{-\mu/2}K_\mu(2\sqrt{zt})\} =$$

$$= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_t^\infty (t-s)^{k-\mu-1} ds \int_0^\infty \exp\{-(zy+s/y)\} y^{-\mu-1} dy$$

$$= \{-z^{-\mu/2}/2\Gamma(k-\mu)\} \int_0^\infty \exp(-zy) y^{-\mu-1} dy \int_t^\infty \exp(-s/y)$$

$$\cdot (t-s)^{k-\mu-1} ds.$$

On putting $s-t = y\lambda$, $ds = yd\lambda$ and using the definiton of the gamma function, this becomes

$$(-1)^{k-\mu}\{z^{-\mu/2}/2\} \int_0^\infty \exp\{-(zy+t/y)\} y^{k-2\mu-1} dy.$$

Differentiating k times with respect to t, we get

$$D_\infty^\mu\{t^{-\mu/2}K_\mu(2\sqrt{zt})\} =$$

$$= (-1)^{-\mu}\{z^{-\mu/2}/2\} \int_0^\infty \exp\{-(zy+t/y)\} y^{-2\mu-1} dy,$$

and using (11), with μ replaced by 2μ, completes the proof.

THEOREM 5. (Convolution theorem) If f and g belong to $C^\infty[0,\infty)$ and $f^{(n)}$ and $g^{(n)}$ satisfy HypI for every n, then $k_\mu(f*g)$ converges absolutely in $\mathrm{Re}\sqrt{p} > \gamma_f+\gamma_g$ and

$$k_\mu(f*g) = (k_\mu f)(k_\mu g) \text{ for } \mu\varepsilon(-1,\infty).$$

PROOF. We use the two-dimensional Laplace convolution theorem (pp. 26-29, [5]): Let

$$F_i(p,q) \equiv L(f_i) = \int_0^\infty\int_0^\infty e^{-px-qy} f_i(x,y)dxdy, \ i = 1,2$$

and

$$f^*(x,y) = \int_0^x\int_0^y f_1(\xi,\eta)f_2[x-\xi,y-\eta]\,d\eta\,d\xi. \tag{12}$$

If F_1 converges absolutely, then so does $F^*(p,q) \equiv L(f^*)$ and

$$F^*(p,q) = F_1(p,q)F_2(p,q). \tag{13}$$

In (12), let $f_1(x,y) = x^\mu f(xy)$ and $f_2(x,y) = x^\mu g(xy)$. Then $f^*(x,y) = \frac{1}{y^{2\mu}}\zeta(xy)$ where

$$\zeta(t) = \int_0^t\int_0^1 w^\mu(t-w)^\mu f(wv)g\,[(t-w)(1-v)]\,dv\,dw, \text{ and}$$

$$L[f^*] = \int_0^\infty\int_0^\infty e^{-px-qy}\zeta(xy)y^{-2\mu}dx\,dy$$

$$= \int_0^\infty \zeta(t)\int_0^\infty e^{-qy-pty^{-1}}y^{-2\mu-1}dy\,dt. \tag{14}$$

Similarly,

$$L[x^\mu f(xy)] = \int_0^\infty f(t)dt \int_0^\infty \exp(-py-qty^{-1})y^{\mu-1}dy$$

$$= 2\int_0^\infty f(t)p^{-\mu}(pqt)^{\frac{\mu}{2}}K_\mu(2\sqrt{pqt})dt, \tag{15}$$

on using the integral representation (5.2). Set $p = 1$ in (14) and (15) and invoke (13); we have

$$\int_0^\infty \zeta(t)dt\int_0^\infty \exp(-qy-ty^{-1})y^{-2\mu-1}dy = \frac{\Gamma^2(\mu+1)}{q^2}(k_\mu f)(k_\mu g). \tag{16}$$

It remains to show that the left hand side of (16) is equal to $\frac{\Gamma^2(\mu+1)}{q^2}k_\mu(f*g)$. Indeed, letting R_λ denote the Riemann-Liouville integral of order λ,

$$k_\mu(f*g) = k_\mu[\frac{1}{\Gamma(\mu+1)} B_\mu t^{-\mu}D^\mu\zeta(t)] \tag{17}$$

$$= \frac{q}{\Gamma(\mu+1)}\, k_\mu\, [t^{-\mu}D^\mu\zeta(t)] \tag{18}$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^\infty (qt)^{\mu/2}K_\mu(2\sqrt{qt})t^{-\mu}(\frac{d}{dt})^k R_{k-\mu}\zeta(t)dt \tag{19}$$

$$= \frac{(-1)^k 2q^2}{\Gamma(k-\mu)\Gamma^2(\mu+1)} \int_0^\infty (\frac{d}{dt})^k\{(q/t)^{\mu/2}K_\mu(2\sqrt{qt})\}dt \int_0^t \zeta(s) \cdot (t-s)^{k-\mu-1}ds \tag{20}$$

$$= \frac{(-1)^\mu 2q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t)D_\infty^\mu\{(q/t)^{\mu/2}K_\mu(2\sqrt{qt})\}dt \tag{21}$$

$$= \frac{2q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t)(q/t)^\mu K_{2\mu}(2\sqrt{qt})dt \tag{22}$$

$$= \frac{q^2}{\Gamma^2(\mu+1)} \int_0^\infty \zeta(t)dt \int_0^\infty \exp(-qy-t/y)y^{-2\mu-1}dy \tag{23}$$

which proves our assertion concerning the left hand side of (16). Equation (17) follows from the definition of convolution. Equation (18) follows from theorem 2 since $Dt^{-\mu}D^\mu\zeta(t)\varepsilon\mathrm{HypI}$ for $f^{(n)}$ and $g^{(n)}\varepsilon\mathrm{HypI}$. Furthermore, from a theorem of Ritt [16], we have $f,g = O(1) \Rightarrow \zeta(t) = O(t^{2\mu+1}) \Rightarrow t^{-\mu}D^\mu\zeta(t) = O(t)$ as $t \to 0^+$. Thus $\lim_{t\to 0^+} t^{-\mu}D^\mu\zeta(t) = 0$. Equation (20) follows from (19) by the definition of the Riemann-Liouville integral, $R_{k-\mu}$, and integrating by parts k-times. The integrated terms vanish at $t = 0$ and $t = \infty$ by (6.1), (6.2), and the fact that in the definition of $\zeta(t)$, the functions f and g satisfy the hypotheses of theorem 5. Equation (22) is due to the preceding lemma. That

$k_\mu(f*g)$ converges absolutely follows from the absolute convergence of (13). This completes the proof.

3. SOME OPERATIONAL FORMULAS.

$$\text{Let } F(y) = \int_0^\infty f(x)K_\mu(xy)^{1/2}dx \equiv \hat{f} \tag{24}$$

If we set $y = 2\sqrt{p}$ and $x = \sqrt{t}$, we get

$$k_\mu[f(\sqrt{t})t^{-1/4-\mu/2}] = \frac{2^{2/3}p^{3/4+\mu/2}}{\Gamma(\mu+1)} F(2\sqrt{p}) \ . \tag{25}$$

From Erdélyi [7] p. 137 (16),

$$[x^{\beta+\mu-1/2}J_\beta(ax)]^\wedge = 2^{\beta+\mu}a^\beta y^{\mu+1/2}\Gamma(\beta+\mu+1)(y^2+a^2)^{-\beta-\mu-1} \tag{26}$$

$\mathrm{Re}\beta > |\mathrm{Re}\mu|-1$, $\mathrm{Re}y > |\mathrm{Im}p|$.

By (25),

$$k_\mu[t^{\beta/2}J_\beta(2\sqrt{at})] = \frac{a^{\beta/2}p^{\mu+1}}{(p+a)^{\beta+\mu+1}} \cdot \frac{\Gamma(\beta+\mu+1)}{\Gamma(\mu+1)} \ , \tag{27}$$

$\mathrm{Re}\sqrt{p} > |\mathrm{Im}\sqrt{a}|$.

Letting $\beta = v-\mu$, this becomes, for $\mathrm{Re}v > \mathrm{Re}\mu$,

$$k_\mu[t^{(v-\mu)/2}J_{v-\mu}(2\sqrt{at})] = \frac{a^{(v-\mu)/2}p^{\mu+1}\Gamma(v+1)}{(p+a)^{v+1}\Gamma(\mu+1)} \ . \tag{28}$$

Since $K_\mu(\cdot) = K_{-\mu}(\cdot)$, we have from (26), for $\mathrm{Re}\sqrt{p} > |\mathrm{Im}a|$,

$$[x^{\beta-\mu+1/2}J_\beta(ax)]^\wedge = \frac{2^{\beta-\mu}a^\beta y^{-\mu+1/2}\Gamma(\beta-\mu+1)}{(y^2+a^2)^{-\mu+\beta+1}}. \tag{29}$$

From (25),

$$k_\mu[t^{\beta/2-\mu}J_\beta(2\sqrt{at})] = \frac{a^{\beta/2}\Gamma(\beta-\mu+1)p}{\Gamma(\mu+1)(p+a)^{\beta-\mu+1}} \tag{30}$$

Setting $\beta = v+\mu$,

$$k_\mu[t^{(v-\mu)/2}J_{v+\mu}(2\sqrt{at})] = \frac{\Gamma(v+1)a^{(v+\mu)/2}p}{\Gamma(\mu+1)(p+a)^{v+1}} . \tag{31}$$

If $v = 0$ in (31),

$$k_\mu[\Gamma(\mu+1)(at)^{-\mu/2}J_\mu(2\sqrt{at})] = \frac{p}{p+a} . \tag{32}$$

Letting $a \to -a$, and using $I_\mu(z) = e^{-\mu\pi i/2}J_\mu(iz)$,

$$k_\mu[\Gamma(\mu+1)(at)^{-\mu/2}I_\mu(2\sqrt{at})] = \frac{p}{p-a} . \tag{33}$$

Equation (31) can be written as

$$k_\mu[\frac{\Gamma(\mu+1)}{\Gamma(v+1)}t^v(at)^{-(v+\mu)/2}J_{v+\mu}(2\sqrt{at})] = \frac{p}{(p+a)^{v+1}} . \tag{34}$$

Again letting $a \to -a$, and $I_\mu(z) = e^{-\mu\pi i/2}J_\mu(iz)$, this gives

$$k_\mu[\frac{\Gamma(\mu+1)}{\Gamma(v+1)}t^v(at)^{-(v+\mu)/2}I_{v+\mu}(2\sqrt{at})] = \frac{p}{(p-a)^{v+1}}. \tag{35}$$

These expressions are useful in inverting rational functions. As an application, consider the problem of solving

$$(B_\mu^2+3B_\mu+2)\phi(t) = f(t),$$

$$\phi(0) = \phi_0,$$

$$(B_\mu\phi)(0) = \phi_1.$$

One gets

$$(p^2+3p+2)k_\mu(\phi) = \phi_0 p^2+(3\phi_0+\phi_1)p+k_\mu f,$$

whence

$$k_\mu(\phi) = -\frac{(\phi_0+\phi_1)p}{p+2} + \frac{(2\phi_0+\phi_1)p}{p+1} + (\frac{1}{2} + \frac{p}{2(p+2)} - \frac{p}{p+1})k_\mu f.$$

Therefore

$$\phi = -(\phi_0+\phi_1)\Gamma(\mu+1)(2t)^{-\mu/2}J_\mu(2\sqrt{2t})$$
$$+(2\phi_0+\phi_1)\Gamma(\mu+1)t^{-\mu/2}J_\mu(2\sqrt{t})+\frac{1}{2}f(t)$$
$$+\{\frac{1}{2}\Gamma(\mu+1)(2t)^{-\mu/2}J_\mu(2\sqrt{2t})-\Gamma(\mu+1)t^{-\mu/2}J_\mu(2\sqrt{t})\}*f(t).$$

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REFERENCES

1. Dimovski, I.H. An Explicit Expression for the Convolution of the Meijer Transformation, Comptes Rendus de l'Academie Bulgare des Sciences, Tome 26, N° 10, 1973, 1293-1296.
2. Dimovski, I.H. On a Bessel-type Integral Transformation due to N. Obrechkoff, Comptes Rendus de l'Academie Bulgare des Sciences, Tome 27, N° 1, 1974, 23-26.
3. Dimovski, I.H. Foundations of Operational Calculi for the Bessel-type Differential Operators, SERDICA, Bulgaricae Mathematicae Publicationes 1 (1975) 51-63.
4. Ditkin, V.A. Operational Calculus Theory, Dokl. Akad. Nauk. SSSR 116 (1957) 15-17.
5. Ditkin, V.A., and A.P. Prudnikov. Operational Calculus in Two Variables, Pergamon Press, New York, 1962.
6. Ditkin, V.A., and A.P. Prudnikov. Integral Transforms and

Operational Calculus, Pergamon Press, New York, 1965.

7. Erdélyi, A., et. al. Tables of Integral Transforms, Vol. II, McGraw-Hill, 1956.

8. Koh, E.L., A Mikusinski Calculus for the Bessel Operator B_μ, Proc. of the Conf. on Differential Equations at Dundee, Springer Verlag Lecture Notes #564, 1976, 291-300.

9. Krätzel, E. Eine Verallgemeinerung der Laplace und Meijer Transformation, Wiss. Z. Univ. Jena, Math. Naturw. Reihe, Heft 5, 1965, 369-381.

10. Krätzel, E. Die Faltung fur $\mathcal{L}$ Transf., Wiss. Z. Univ. Jena, Math. Naturw. Reihe, Heft 5, 1965, 383-390.

11. Krätzel, E. Bemerkungen zur Meijer Transf. und Andwendungen, Math. Nachr. 30 (1965) 327-334.

12. Krätzel, E. Differentiationssatze der $\mathcal{L}$ Transformation und Differentialgleichungen nach dem Operator $\frac{d}{dt} t^{\frac{1}{n}-v} (t^{1-\frac{1}{n}} \frac{d}{dt})^{n-1} t^{v-1-\frac{2}{n}}$, Math. Nachr. 35 (1967) 105-114.

13. Meller, N.A. On an Operational Calculus for the Operator $B_\alpha = t^{-\alpha} \frac{d}{dt} t^{\alpha+1} \frac{d}{dt}$, Vichislitelnaya Matematika 6 (1960) 161-168.

14. Meller, N.A. Some Applications of Operational Calculus to Problems in Analysis, Zh. Vichysl. Mat. i. Mat. Fiz. 3(1) (1963) 71-89.

15. Mikusinski, J.G. Operational Calculus, Pergamon Press, Oxford, 1959.

16. Ritt, J.F. On the Differentiability of Asymptotic Series,

Bull. Amer. Math. Soc. 24 (1918) 225-227.

17. Ross, B. (ed). Fractional Calculus and its Applications, Springer-Verlag, 1975.

18. Zemanian, A.H. Generalized Integral Transformations, Interscience, New York, 1968.

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