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EQUIVALENCE CLASSES OF THE 3RD GRASSMAN SPACE OVER A 5-DIMENSIONAL VECTOR SPACE

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ABSTRACT. An equivalence relation is defined on $\Lambda^r V$, the r^{th} Grassman space over V and the problem of the determination of the equivalence classes defined by this relation is considered. For any r and V, the decomposable elements form an equivalence class. For r = 2, the length of the element determines the equivalence class that it is in. Elements of the same length are equivalent, those of unequal lengths are inequivalent. When $r \geq 3$, the length is no longer a sufficient indicator, except when the length is one. Besides these general questions, the equivalence classes of $\Lambda^3 V$, when dim V = 5 are determined.

KEY WORDS AND PHRASES. *Grassman space, equivalent classes, representation of equivalent classes.*

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Suppose V is a finite dimensional vector space over an arbitrary field F and r is a positive integer. Consider $\Lambda^r V$, the rth Grassman space over V. We define an equivalence relation on $\Lambda^r V$ as follows: If X and Y are in $\Lambda^r V$, we write $X \sim Y$ iff $\exists$ a non-singular linear transformation $T:V \longrightarrow V$ such that $C_r(T)X = Y$, where $C_r(T)$ is the rth exterior product of T. Using the facts, that if T and S are two linear transformations of V, then $C_r(T)C_r(S) = C_r(TS)$ and if T is non-singular, then $C_r(T^{-1}) = C_r(T)^{-1}$, it follows that the above relation is an equivalence relation.

We consider the problem of determining the number of equivalence classes, into which the set $\Lambda^r V$ is decomposed, along with a system of distinct representatives of these equivalence classes.

DEFINITIONS. 1. If $X \in \Lambda^r V$ and $X = x_1 \wedge \ldots \wedge x_r$, we say X is _decomposable_.

2. If $X \in \Lambda^r V$, we define its _length_, to be denoted by $\ell(X)$ as $\ell(X) = \min\{m \mid X$ is a sum of m decomposable elements of $\Lambda^r V\}$.

3. If $X \in \Lambda^r V$, we define a subspace [X] of V as $[X] = \cap\{U \mid U$ is a subspace of V and $X \in \Lambda^r U\}$.

4. If $X \in \Lambda^r V$, we define the rank of X to be denoted by $\rho(X)$ as $\rho(X) = \dim[X]$.

PROPOSITION 1. If $X, Y \in \Lambda^r V$ and $X \sim Y$, then (i) $\ell(X) = \ell(Y)$, (ii) $P(X) = P(Y)$.

PROOF. (i) Let $T:V \to V$ be a n.s.l.t. such that $C_r(T)X = Y$. If $\ell(X) = s$ $X = \sum_{i=1}^{s} X_i$, where $X_i \in \Lambda^r V$ and $\ell(X_i) = 1$.

Then $Y = C_r(T)X = \sum_{i=1}^{s} C_r(T)X_i$. This implies $\ell(Y) \le s = \ell(X)$. Similarly $Y \sim X$ implies $\ell(Y) \le \ell(X)$ and this proves (i).

(ii) We first remark that if U and W are subspaces of V, then $X \in \Lambda^r U$ implies $Y \in \Lambda^r T(U)$ and $Y \in \Lambda^r W$ implies $X \in \Lambda^r T^{-1}(W)$, where $T:V \to V$ is a n.s.l.t. such that $Y = C_r(T)X$. From this remark, it follows easily that $[Y] = T[X]$ and hence $P(X) = P(Y)$.

PROPOSITION 2. If U and W are subspaces of V, then $\Lambda^r U \cap \Lambda^r W = \Lambda^r (U \cap W)$.

PROOF. Clearly $\Lambda^r(U \cap W) \subseteq (\Lambda^r U) \cap (\Lambda^r W)$. To prove the inclusion in the other direction, let $x_1, x_2, \ldots, x_k$ be a basis of $U \cap W$ and extend it to a basis $x_1, \ldots, x_k, y_1, \ldots, y_s$ of U and a basis $x_1, \ldots, x_k, z_1, \ldots, z_t$ of W. Then $x_1, \ldots, x_k, y_1, \ldots, y_s, z_1, \ldots, z_t$ is a basis of U + W.
If $A = \{x_i \wedge x_j | 1 \le i < j \le k\}$, $B = \{y_i \wedge y_j | 1 \le i < j \le s\}$, $C = \{z_i \wedge z_j | 1 \le i < j \le t\}$, $D = \{x_i \wedge y_j | 1 \le i \le k;\ 1 \le j \le s\}$, $E = \{x_i \wedge z_j | 1 \le i \le k;\ 1 \le j \le t\}$, $F = \{y_i \wedge z_j | 1 \le i \le s;\ 1 \le j \le t\}$, then the sets A, $A \cup B \cup D$, $A \cup C \cup E$, and $A \cup B \cup C \cup D \cup E \cup F$ form bases of $\Lambda^r(U \cap W)$, $\Lambda^r U$, $\Lambda^r W$ and $\Lambda^r(U + W)$ respectively. If $X \in (\Lambda^r U) \cap (\Lambda^r W)$, then

$X = \sum_A a_{ij} x_i \wedge x_j + \sum_B b_{ij} y_i \wedge y_j + \sum_D d_{ij} x_i \wedge y_j$ and also

$X = \sum_A a_{ij} x_i \wedge x_j + \sum_C c_{ij} z_i \wedge z_j + \sum_E e_{ij} x_i \wedge z_j$. Hence $a_{ij} = a_{ij}$ and $b_{ij} = d_{ij} = c_{ij} = e_{ij} = 0$ for all the appropriate values of the indices i and j. Thus $X \in \Lambda^r(U \cap W)$.

REMARK 1. The result of Proposition 2 holds for any number of subspaces of V.

REMARK 2. If $X \in \Lambda^r V$ and $\mathcal{S} = \{U | U \text{ is a subspace of } V, X \in \Lambda^r U\}$, then $\Lambda^r[X] = \Lambda^r(\bigcap_{U \in \mathcal{S}} U) = \bigcap_{U \in \mathcal{S}} (\Lambda^r U)$. Thus $X \in \Lambda^r[X]$ and [X] is the smallest such subspace of V.

PROPOSITION 3. Let $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^{k} x_i \wedge y_i$, then $x_1, \ldots, x_k, y_1, \ldots, y_k$ are linearly independent.

PROOF. If not, then one of them (say) y_k is a linear combination of the remaining $x_1,\dots,x_k$, $y_1,\dots,y_{k-1}$. Let $y_k = \sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{k-1} b_j y_j$.

Then $x_k \wedge y_k = \sum_{i=1}^{k} a_i x_k \wedge x_i + \sum_{j=1}^{k-1} b_j x_k \wedge y_j$. Hence X can be written as

$X = \sum_{i=1}^{k-1} (x_i \wedge y_i + x_k \wedge z_i)$, where $z_i = a_i x_i + b_i y_i$, $1 \le i \le k-1$. If $z_i = 0$, then $\ell(x_i \wedge y_i + x_k \wedge z_i) = 1$. If $z_i \neq 0$, let $a_i \neq 0$, then $x_i \wedge y_i + x_k \wedge z_i = z_i \wedge (a_i^{-1} y_i - x_k)$, thus $\ell(x_i \wedge y_i + x_k \wedge z_i) \le 1$. Hence $\ell(X) \le k-1$, a contradiction.

REMARK 3. If $X \in \Lambda^2 V$, $\ell(X) = k$ and $X = \sum_{i=1}^{k} x_i \wedge y_i$, then $[X] = \langle x_1,\dots,x_k, y_1,\dots,y_k\rangle$.

PROOF. Let $U = \langle x_1,\dots,x_k, y_1,\dots,y_k\rangle$; then $[X] \subseteq U$. By Proposition 3, dim $U = 2k$. Also $X \in \Lambda^2[X]$; let $X = \sum_{i=1}^{k} x_i' \wedge y_i'$, $x_i', y_i' \in [X]$, $1 \le i \le k$. Again by Proposition 3, dim $[X] \ge 2k$. Thus $[X] = U = \langle x_1,\dots,x_k, y_1,\dots,y_k\rangle$.

PROPOSITION 4. If $X,Y \in \Lambda^2 V$, $P(X) = P(Y)$, then $X \sim Y$.

PROOF. Let $X = \sum_{i=1}^{k} x_i \wedge y_i$, $Y = \sum_{j=1}^{s} x_j' \wedge y_j'$; then by Remark 3, $[X] = \langle x_1,\dots,x_k, y_1,\dots,y_k\rangle$ and $[Y] = \langle x_1',\dots,x_s', y_1',\dots,y_s'\rangle$. Also by Proposition 3, $P(X) = 2k$, $P(Y) = 2s$. Thus $k = s$. Let T be a linear transformation of V $Tx_i = x_i'$, $Ty_i = y_i'$, $1 \le i \le k$; then $C_r(T)X = Y$. Thus $X \sim Y$.

PROPOSITION 5. If $X \in \Lambda^r V$, $\ell(X) = 2$, $X = x_1 \wedge \dots \wedge x_r + y_1 \wedge \dots \wedge y_r$, then $X = \langle x_1,\dots,x_r, y_1,\dots,y_r\rangle$.

PROOF. Let $U = \langle x_1,\dots,x_r, y_1,\dots,y_r\rangle$; then $[X] \subseteq U$. If $[X] \neq U$, then at least one element (say) x_1 is not in $[X]$. Let B be a basis of $[X]$ and extend $\{x\} \cup B$ to a basis of U. Let W be a complement of $\langle x_1\rangle$ in U, containing $[X]$, i.e., $U = \langle x_1\rangle \oplus W$, $[X] \subseteq W$. Let $x_i = a_i x_1 + w_i$, $2 \le i \le r$ and $y_j = b_j x_1 + w_j'$, $1 \le j \le r$, where $w_i, w_j' \in W$. Then $X = X_1 + X_2$, where $X_1 \in x_1 \wedge (\Lambda^{r-1} W)$ and $X_2 \in \Lambda^r W$, and $\ell(X_i) = 1$, $i = 1,2$. But $U = \langle x_1\rangle \oplus W \Rightarrow \Lambda^r U = x_1 \wedge (\Lambda^{r-1} W) \oplus \Lambda^r W$. Also $X \in \Lambda^r[X] \subseteq \Lambda^r W$, hence

$X_1 = X - X_2 \in \Lambda^r W$. Thus $X_1 = 0$ and $X = X_2 \Rightarrow \ell(X) = 1$, a contradiction. Hence $[X] = U = \langle x_1,\dots,x_r, y_1,\dots,y_r\rangle$.

Note: The above proposition is true also for $\ell(X) = k$.

PROPOSITION 6. If $X,Y \in \Lambda^r V$, $\ell(X) = \ell(Y) = 2$, $P(X) = P(Y)$, then $X \sim Y$.

PROOF. Let $X = x_1\wedge\dots\wedge x_r + y_1\wedge\dots\wedge y_r$, $U_1 = \langle x_1,\dots,x_r\rangle$, $U_2 = \langle y_1,\dots,y_r\rangle$, then by Proposition 4, $[X] = U_1 + U_2$. Let $z_1,\dots,z_k$ be a basis of $U_1 \cap U_2$, and extend it to a basis $z_1,\dots,z_k, u_1,\dots,u_s$, where $k + s = r$ of U_1 and to a basis $z_1,\dots,z_k, v_1,\dots,v_s$ of U_2. Then $P(X) = k + 2s$. Since $x_1,\dots,x_r$ and $z_1,\dots,z_k, u_1,\dots,u_s$ are two bases of U_1, hence $x_1\wedge\dots\wedge x_r = az_1\wedge\dots\wedge z_k\wedge u_1\wedge\dots\wedge u_s = z_1\wedge\dots\wedge z_k\wedge\bar{u}_1\wedge\dots\wedge u_s$, where $\bar{u}_1 = au_1$. Similarly $y_1\wedge\dots\wedge y_r = bz_1\wedge\dots\wedge z_k\wedge v_1\wedge\dots\wedge v_s = z_1\wedge\dots\wedge z_k\wedge\bar{v}_1\wedge\dots\wedge v_s$, where $\bar{v}_1 = bv_1$. Hence $X = z_1\wedge\dots\wedge z_k \wedge (\bar{u}_1\wedge u_2\wedge\dots\wedge u_s + \bar{v}_1\wedge v_2\wedge\dots\wedge v_s)$, where $z_1,\dots,z_k, \bar{u}_1,u_2,\dots,u_s, \bar{v}_1,v_2,\dots,v_s$ is a basis of $[X]$.

Similarly $Y = z_1'\wedge\dots\wedge z_k' \wedge (\bar{u}_1'\wedge u_2'\wedge\dots\wedge u_s' + \bar{v}_1'\wedge v_2'\wedge\dots\wedge v_s')$, where $z_1',\dots,z_k', \bar{u}_1',u_2',\dots,u_s', \bar{v}_1',v_2',\dots,v_s'$ is a basis of $[Y]$.

Define $T:V \longrightarrow V$, a linear transformation

$Tz_i = z_i'$, $T\bar{u}_1 = \bar{u}_1'$, $Tu_i = u_i'$, $T\bar{v}_1 = \bar{v}_1'$, $Tv_i = v_i'$, for $i = 2, 3,\dots,s$.

Then $C_r(T)X = Y$; hence $X \sim Y$.

REMARK 4. Let $X \in \Lambda^r V$, $\ell(X) = 2$, then $r + 1 \le \rho(X) \le 2r$.

PROOF. If $X = x_1\wedge\dots\wedge x_r + y_1\wedge\dots\wedge y_r$, then $[X] = \langle x_1,\dots,x_r, y_1,\dots,y_r\rangle = U_1 + U_2$, where $U_1 = \langle x_1,\dots,x_r\rangle$, $U_2 = \langle y_1,\dots,y_r\rangle$. $U_1 \neq U_2$, for otherwise $y_1\wedge\dots\wedge y_r = ax_1\wedge\dots\wedge x_r$, where a is a scalar and $\ell(X) = 1$.

$P(X) = 2r - \dim U_1 \cap U_2$. Hence $r+1 \le P(X) \le 2r$.

THEOREM 1. Let $E(2, s) = \{X|X \in \Lambda^r V, \ell(X) = 2, P(X) = s\}$, then $E(2, s)$, $s = r+1, r+2,\dots,2r$ are all the equivalence classes on the set of all vectors of $\Lambda^r V$, of length 2.

PROOF. Follows from Proposition 6 and Remark 4.

PROPOSITION 7. Let $0 \neq X \in \Lambda^r V$ and $x \in V$ such that $x \wedge X = 0$; then $x \in [X]$.

PROOF. Let $x_1, x_2, \ldots, x_m$ be a basis of $[X]$. Then $\{\hat{x}_\alpha \mid \alpha \in Q_{r,m}\}$ is a basis of $\Lambda^r[X]$, where $Q_{r,m}$ is a set of all the strictly decreasing sequences of length r on the integers 1, 2,...,m. det $X = \Sigma a_\alpha \hat{x}_\alpha$; then $x \wedge X = \Sigma_\alpha a_\alpha\, x \wedge \hat{x}_\alpha$. If $x \notin [X]$, then $\{x \wedge \hat{x}_\alpha \mid \alpha \in Q_{r,m}\}$ is a part of a basis of $\Lambda^{r+1}\langle x,[X]\rangle$. Thus $x \wedge X = 0 \Rightarrow a_\alpha = 0\ \forall\, \alpha \in Q_{r,m} \Rightarrow X = 0$, a contradiction.

PROPOSITION 8. If $0 \neq X \in \Lambda^r V$ and $x \notin [X]$, then $[x \wedge X] = \langle x\rangle \oplus [X]$.

PROOF. By Proposition 7, $x \wedge X \neq 0$. Again by Proposition 7, since $x \wedge (x \wedge X) = 0$, hence $x \in [x\wedge X]$. Clearly $[x\wedge X] \subseteq \langle x\rangle \oplus [X]$. Let $x, x_1, \ldots, x_k$ be a basis of $[x\wedge X]$ and extend it to a basis $x, x_1, \ldots, x_k, x_{k+1}, \ldots, x_m$ of $\langle x\rangle \oplus [X]$. If $U = \langle x_1, \ldots, x_k\rangle$, then $[x\wedge X] = \langle x\rangle \oplus U$, $U \subseteq [X]$. $\Lambda^{r+1}[x\wedge X] = x \wedge (\Lambda^r U) \oplus \Lambda^{r+1}U$. Let $x\wedge X = x\wedge u+v$, where $u \in \Lambda^r U$ and $v \in \Lambda^{r+1}U$. Thus $x\wedge v = 0$. If $v \neq 0$, then by Proposition 7, $x \in [v] \subset U$, a contradiction. Hence $v = 0$ and thus $x\wedge X = x\wedge u$. Then $x \wedge (X-u) = 0$. If $X-u \neq 0$, then by Proposition 7, $x \in [X-u]$. Now $X \in \Lambda^r[X]$ and $u \in \Lambda^r U \subseteq \Lambda^r[X]$; thus $X - u \in \Lambda^r[X]$. Hence $[X-u] \subseteq [X]$. Thus $x \in [X-u] \Rightarrow x \in [X]$, which is a contradiction and therefore $X-u = 0$; i.e., $X = u \in \Lambda^r U$. Hence $[X] \subseteq U$. Also $U \subseteq [X]$, hence $U = [X]$ and $[x\wedge X] = \langle x\rangle \oplus [X]$.

PROPOSITION 9. Suppose $X \in \Lambda^2 V$, $\ell(X) = 2$, x_1, x_2 are linearly independent vectors in $[X]$. Then $\exists\ y_1, y_2 \in [X]$ and $\lambda \in F \ni X$ has one and only one of the following representations: (i) $X = x_1 \wedge y_1 + x_2 \wedge y_2$, (ii) $X = \lambda x_1 \wedge x_2 + y_1 \wedge y_2$.

PROOF. $X \in \Lambda^2 V$, $\ell(X) = 2 \Rightarrow P(X) = 4$. Extend x_1, x_2 to a basis x_1, x_2, x_3, x_4 of $[X]$.

Then $X = \sum_{1\le i<j\le 4} a_{ij} x_i \wedge x_j$, $a_{ij} \in F$.

If $a_{34} = 0$, take $y_1 = a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ and $y_2 = a_{23}x_3 + a_{24}x_4$, then $X = x_1 \wedge y_1 + x_2 \wedge y_2$. If $a_{34} \neq 0$, then

$(-\lambda + a_{12})a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$ ———————— (1) has a solution in F.

Set $Y = (-\lambda+a_{12})x_1\wedge x_2 + a_{13}x_1\wedge x_3 + a_{14}x_1\wedge x_4 + a_{23}x_2\wedge x_3 + a_{24}x_2\wedge x_4 + a_{34}x_3\wedge x_4$.

Then $Y = -\lambda x_1\wedge x_2 + X$. Because of (1), $\ell(Y) = 1$; also $Y \in \Lambda^2[X]$.

Thus $y_1, y_2 \in [X]$ $Y = y_1 \wedge y_2$. Hence $X = \lambda x_1\wedge x_2 + y_1\wedge y_2$.

If $X = x_1\wedge y_1 + x_2\wedge y_2$ and also $X = \lambda x_1\wedge x_2 + z_1\wedge z_2$ then $x_1 \wedge X = x_1 \wedge x_2 \wedge y_2$ and also $x_1 \wedge X = x_1 \wedge z_1 \wedge z_2$. Thus $0 \neq x_1\wedge x_2\wedge y_2 = x_1\wedge z_1\wedge z_2$ and hence $\langle x_1,x_2,y_2\rangle = \langle x_1,z_1,z_2\rangle$. Let $z_1 = a_1x_1 + a_2x_2 + a_3y_2$ and $z_2 = b_1x_1 + b_2x_2 + b_3y_2$.

Then $z_1\wedge z_2 = (a_1b_2-a_2b_1)x_1\wedge x_2 + (a_1b_3-a_3b_1)x_1\wedge y_2 + (a_2b_3-a_3b_2)x_2\wedge y_2$.

Putting this expression for $z_1\wedge z_2$ in $X = \lambda x_1\wedge x_2 + z_1 \wedge z_2$, we get two different representations of X in the basis of $\Lambda^2[X]$, determined by the basis x_1,x_2,y_1,y_2 of $[X]$; thus X has precisely one of the two representations.

PROPOSITION 10. If $X,Y \in \Lambda^rV$ are decomposable, then X+Y is decomposable iff $\dim[X] \cap [Y] \geq r-1$.

PROOF. (=>) Let X+Y be decomposable, and $X+Y = Z$, $\ell(Z) \leq 1$. Let $X = x_1\wedge\ldots\wedge x_r$, $Y = y_1\wedge\ldots\wedge y_r$, $Z = z_1\wedge\ldots\wedge z_r$. If $[X] = [Z]$, then for any i, $1 \leq i \leq r$, $z_i\wedge X = z_i\wedge Z = 0$; but then $z_i\wedge Y = 0$, and thus $z_i \in [Y]$ by Proposition 7, and $[Z] = [Y]$. Hence $[X] = [Y]$, i.e., $\dim[X] \cap [Y] = r$. If $[X] \neq [Z]$, then for some i, $z_i \notin [X]$. But $z_i \wedge (X+Y) = 0 \Rightarrow z_i\wedge X = -z_i\wedge Y \Rightarrow \langle z_i,[X]\rangle = \langle z_i,[Y]\rangle$. Thus $[X]$, $[Y]$ are r-dimensional subspaces in an (r+1) - dim space $\langle z_i,[X]\rangle$. Hence $\dim[X] \cap [Y] \geq \dim[X] + \dim[Y] - (r+1) = r-1$. (<=) If $\dim[X] \cap [Y] \geq r-1$. Let $u_1,\ldots,u_{r-1}$ be l.i. vectors in $[X] \cap [Y]$ and extend these to a basis $x, u_1,\ldots,u_{r-1}$ and a basis $y, u_1,\ldots,u_{r-1}$ of $[X]$ and $[Y]$ respectively. Thus $X = ax\wedge u_1\wedge\ldots\wedge u_{r-1}$, $Y = by \wedge u_1\wedge\ldots\wedge u_{r-1}$ for some a and b.

Hence $X+Y = (ax+by)\wedge u_1\wedge \dots u_{r-1}$, i.e., $X+Y$ is decomposable.

THEOREM 2. If $\dim V = 5$, $X \in \Lambda^3 V$, then $\ell(X) \le 2$.

PROOF. We shall first prove that $\ell(X) \le 3$. Let x_1, x_2, x_3, x_4, x_5 be a basis of V. Then

$$X = \sum_{1\le i<j<k\le 5} a_{ijk}x_i\wedge x_j\wedge x_k = x_1\wedge x_2\wedge(a_{123}x_3 + a_{124}x_4 + a_{125}x_5)$$
$$+ x_1\wedge x_3\wedge(a_{134}x_4 + a_{135}x_5) + x_2\wedge x_3\wedge(a_{234}x_4 + a_{235}x_5)$$
$$+ (a_{145}x_1 + a_{245}x_2 + a_{345}x_3)x_4\wedge x_5.$$

Let $y_1 = a_{134}x_4 + a_{135}x_5$, $y_2 = a_{234}x_4 + a_{235}x_5$. If y_1,y_2 are l.d., then $\ell(X) \le 3$. So we assume y_1,y_2 are l.i.; then $\langle y_1,y_2\rangle = \langle x_4,x_5\rangle$, and thus $x_4\wedge x_5 = \lambda y_1\wedge y_2$, $\lambda \in F$. Let $a_{124}x_4 + a_{125}x_5 = b_1y_1 + b_2y_2$. Then

$$X = x_1\wedge x_2\wedge(a_{123}x_3 + b_1y_1 + b_2y_2) + x_1\wedge x_3\wedge y_1 + x_2\wedge x_3\wedge y_2$$
$$+ \lambda(a_{145}x_1 + a_{245}x_2 + a_{345}x_3)y_1\wedge y_2$$
$$= a_{123}x_1\wedge x_2\wedge x_3 + (x_1 + a_{345}\lambda y_2)\wedge y_1\wedge(-b_1x_2 - x_3 + a_{145}\lambda y_2)$$
$$+ (b_2x_1 - x_3 - (a_{245} - a_{345}b_1)\lambda y_1)\wedge x_2\wedge y_2.$$

Hence $\ell(X) \le 3$.

Let $X = X_1 + X_2 + X_3$, where X_1, X_2, X_3 are decomposable, $X_1 = x_1\wedge x_2\wedge x_3$, $X_2 = y_1\wedge y_2\wedge y_3$, $X_3 = z_1\wedge z_2\wedge z_3$. Then $1 \le \dim[X_1] \cap [X_2] \le 3$.

CASE 1. $\dim[X_1] \cap [X_2] = 3$. Then $X_2 = \lambda X_1$ for some λ and thus $\ell(X) \le 2$.

CASE 2. $\dim[X_1] \cap [X_2] = 2$. Let u_1,u_2,v and u_1,u_2,w be bases of $[X_1]$ and $[X_2]$ respectively. Then $X_1 = \lambda u_1\wedge u_2\wedge v$ and $X_2 = \lambda\ u_1\wedge u_2\wedge w$. Then $\ell(X) \le 2$.

CASE 3. $\dim[X_1] \cap [X_2] = 1$. det u_1, u_2, u_3 and u_1, u_4, u_5 be bases of $[X_1]$ and $[X_2]$ respectively. Then $X_1 = u_1\wedge u_2\wedge u_3$, $X_2 = u_1\wedge u_4\wedge u_5$; we have assumed the co-effs. to be absorbed with the vectors u_i's and v_i's. Then $X_1+X_2 = u_1 \wedge Y$, where $Y = u_2\wedge u_3 + u_4\wedge u_5$. Also $[X_1] + [X_2] = V$.

Since $\dim\langle u_2,u_3,u_4,u_5\rangle \cap [X_3] \geq 2$, we can take $X_3 = w_1 \wedge w_2 \wedge w_3$, where $w_1, w_2 \in \langle u_2,u_3,u_4,u_5\rangle$. By Proposition 9, v_1, v_2 and λ $Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2$ or $Y = w_1 \wedge v_1 + w_2 \wedge v_2$. If $Y = \lambda w_1 \wedge w_2 + v_1 \wedge v_2$, then $X = u_1 \wedge Y + w_1 \wedge w_2 \wedge w_3$ has length ≤ 2. If $Y = w_1 \wedge v_1 + w_2 \wedge v_2$, then since u_1, w_1, w_2, v_1, v_2 is also a basis of V, let $w_3 = a_1u_1 + a_2w_1 + a_3w_2 + a_4v_1 + a_5v_2$. Then $X = X_1 + X_2 + X_3 = (u_1 - a_4w_2) \wedge w_1 \wedge v_1 + u_1 \wedge w_2 \wedge v_2 + (a_5v_2 + a_1u_1) \wedge w_1 \wedge w_2$ has length ≤ 2, since $Z = u_1 \wedge w_2 \wedge v_2 + (a_5v_2 + a_1u_1) \wedge w_1 \wedge w_2$ and $\dim\langle u_1,w_2,v_2\rangle \cap \langle a_5v_2 + a_1u_1, w_1, w_2\rangle \geq 2$ implies $\ell(Z) \leq 1$.

REMARK. There exists $X \in \Lambda^3V$ with $\ell(X) = 2$; for if x_1,x_2,x_3,x_4,x_5 is a basis of V and $X = x_1 \wedge x_2 \wedge x_3 + x_1 \wedge x_4 \wedge x_5$, then $\ell(X) = 2$, by Proposition 10.

REMARK. If $X \in \Lambda^3V$, $\dim V = 5$, $\ell(X) = 2$, then $P(X) = 5$; for let $X = X_1 + X_2$, where $\ell(X_1) = \ell(X_2) = 1$. Since X is not decomposable, then by Proposition 10, $\dim[X_1] \cap [X_2] < 2$ and hence $\dim[X] > \dim[X_1] + \dim[X_2] - \dim[X_1] \cap [X_2] = 4$, i.e., $P(X) = 5$.

It follows from Proposition 6 that if $X,Y \in \Lambda^3V$ and $\ell(X) = \ell(Y)$, then $X \sim Y$. Hence all the equivalence classes of Λ^3V are given by

$$S_0 = \{X | X \in \Lambda^3V,\ \ell(X) = 0\} = \{0\}$$

$$S_1 = \{X | X \in \Lambda^3V,\ \ell(X) = 1\}$$

$$S_2 = \{X | X \in \Lambda^3V,\ \ell(X) = 2\}.$$

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