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A REPRESENTATION THEOREM FOR OPERATORS ON A SPACE OF INTERVAL FUNCTIONS

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ABSTRACT. Suppose N is a Banach space of norm $|\cdot|$ and R is the set of real numbers. All integrals used are of the subdivision-refinement type. The main theorem [Theorem 3] gives a representation of TH where H is a function from RxR to N such that $H(p^+, p^+)$, $H(p, p^+)$, $H(p^-, p^-)$, and $H(p^-,p)$ each exist for each p and T is a bounded linear operator on the space of all such functions H. In particular we show that

$$TH = (I)\int_a^b f_H d\alpha + \sum_{i=1}^{\infty} \left[H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)\right] \beta(x_{i-1})$$

$$+ \sum_{i=1}^{\infty} \left[H(x_i^-,x_i)-H(x_i^-,x_i^-)\right] \Theta(x_{i-1},x_i)$$

where each of α, β, and Θ depend only on T, α is of bounded variation, β and Θ are 0 except at a countable number of points, f_H is a function from R to N depending on H, and $\{x_i\}_{i=1}^{\infty}$ denotes the points p in $[a,b]$ for which $\left[H(p,p^+)-H(p^+,p^+)\right]\neq 0$ or $\left[H(p^-,p)-H(p^-,p^-)\right]\neq 0$. We also define an interior

interval function integral and give a relationship between it and the standard interval function integral.

1. INTRODUCTION.

Let N be a Banach space of norm $|\cdot|$ and R the set of real numbers. The purpose of this paper is to exhibit a representation of TH where H is a function from RxR to N such that $H(p^+,p^+)$, $H(p, p^+)$, and $H(p^-,p^-)$, and $H(p^-,p)$each exist for each p and T is a bounded linear operator on the space of all such functions H. Functions H for which each of the four preceding limits exist have been used extensively in the study of both sum integration and multiplicative integration, (for example see [2]). In particular we show that

$$TH = (I)\int_a^b f_H d\alpha + \sum_{i=1}^{\infty} [H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)]\beta(x_{i-1})$$
$$+ \sum_{i=1}^{\infty} [H(x_i^-,x_i)-H(x_i^-,x_i^-)]\,\Theta\,(x_{i-1},x_i),$$

where each of α, β, Θ depend only on T, α is of bounded variation, β and Θ are 0 except at a countable number of points, f_H is a function from R to N depending on H, and $\{x_i\}_{i=1}^{\infty}$ denotes the points p in [a,b] for which $H(p,p^+)-H(p^+,p^+) \neq 0$ or $[H(p^-,p)-H(p^-,p^-)] \neq 0$. We also define an interior interval function integral and give a relationship between it and the standard interval function integral.

2. DEFINITIONS.

If H is a function from RxR to N, then $H(p^+,p^+) = \lim_{x,y\to p} + H(x,y)$ and similar meanings are given to $H(p,p^+)$, $H(p^-,p^-)$, and $H(p^-,p)$. The set of all functions for which each of the preceding four limits exist will be denoted by OL^0. If H is a function from RxR to N then H is said to be (1) of bounded variation on the interval [a,b] and (2) bounded on [a,b] if there exists a number M and a subdivision D of [a,b] such that if $D' = \{x_i\}_{i=0}^n$ is a refinement of D then

(1) $\sum_{i=1}^{n} |H(x_{i-1},x_i)| < M$ and (2) if $0<i\leq n$, then $|H(x_{i-1},x_i)| < M$, respectively.

Further, H is said to be integrable on $[a,b]$ if there is a number A such that for each $\varepsilon > 0$ there is a subdivision D of $[a,b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D, then $|\sum_{\substack{i=1\\D'}}^{n} H(x_{i-1},x_i) - A| < \varepsilon$ and A is denoted by $\int_a^b H$ when such an A exists. In our development we will also find a slight modification of the preceding definition useful. If $H(x_{i-1},x_i)$ is replaced by $H(r_i,s_i)G(x_{i-1},x_i)$, $x_{i-1} < r_i < s_i < x_i$, in the approximating sum of the preceding definition then the number **A** is denoted by $(I_H)\int_a^b HG$ and termed the interior integral of H with respect to G on $[a,b]$. Also, if each of f and α is a function from R to N, then the interior integral of f with respect to α exists means there is a number A such that if $\varepsilon > 0$ then there is a subdivision D of $[a,b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D and for $0<i\leq n$, $x_{i-1} < t_i < x_i$, $|\sum_{\substack{i=1\\D'}}^{n} f(t_i)[\alpha(x_i)-\alpha(x_{i-1})] - A| < \varepsilon$ and A is denoted by $(I)\int_a^b f d\alpha$.

If α is a function from R to N, $\alpha(p^+) = \lim_{x\to p^+} \alpha(x)$, $\alpha(p^-) = \lim_{x\to p^-} \alpha(x)$, and $d\alpha$ denotes the function H from RxR to N such that for $x<y$, $H(x,y) = \alpha(y)-\alpha(x)$. If each of H, H_1, H_2, . . . is a function from RxR to N, then $\lim_{n\to\infty} H_n = H$ uniformly on $[a,b]$ means if $\varepsilon > 0$ there is a positive integer N and a subdivision $D = \{x_i\}_{i=0}^{n}$ of $[a,b]$ such that if $n > N$ and $x_{i-1} \leq r < s \leq x_i$ for some $0<i\leq n$, then $|H(r,s)-H_n(r,s)| < \varepsilon$. If H is a function from RxR to N, then H is bounded on $[a,b]$ means there is a number M and a subdivision $D = \{x_i\}_{i=0}^{\infty}$ of $[a,b]$ such that if $0<i\leq n$ and $x_{i-1} \leq r < s \leq x_i$, then $|H(r,s)| < M$. The norm of H on $[a,b]$ with respect to D, $||H||_D$ is then defined as the greatest lower bound of the set of all such M's.

T is a linear operator on OL^0 means T is a transformation from OL^0 to N such that if each of H_1 and H_2 are in OL^0 then

$$T[k_1H_1 + k_2H_2] = k_1TH_1 + k_2TH_2$$

for k_1, k_2 in R. T is bounded on $[a,b]$ means there is a number M such that $|TH| \le M||H||_D$ for some subdivision D of $[a,b]$.

For convenience we adopt the following conventions for a function from RxR to N and R to N for some subdivision $D = \{x_i\}_{i=0}^n$ of $[a,b]$:

(1) $H(a^-,a) = H(a^-,a^-) = H(b,b^+) = H(b^+,b^+) = 0$,

(2) $H(x_{i-1},x_i) = H_i$, $0<i\le n$,

(3) $\alpha(x_i)-\alpha(x_{i-1}) = \Delta\alpha_i$,

(4) $\sum_{\substack{i=1\\D}}^{n} H(x_{i-1},x_i) = \Sigma_D H_i$.

3. THEOREMS.

We will begin by establishing a relationship between $\int_a^b H d\alpha$ and $(I_H)\int_a^b H d\alpha$ which will require the following lemmas.

LEMMA 1. If H is in OL^0 and α is a function from R to N of bounded variation on $[a,b]$, then $\int_a^b H d\alpha$ exists.

This lemma is a special case of THEOREM 2 of [2].

LEMMA 2. Suppose H is in OL^0, $[a,b]$ is an interval, $\varepsilon > 0$, and S_1 and S_2 are sets such that p is in S_1 if and only if p is in $[a,b]$ and $|H(p,p^+)-H(p^+,p^+)| \ge \varepsilon$ and p is in S_2 if and only if p is in $[a,b]$ and $|H(p^-,p)-H(p^-,p^-)| \ge \varepsilon$. Then, each of S_1 and S_2 is a finite set. [2, lemma page 498].

We note that it follows from LEMMA 2 that if S is the set such that p is in S if and only if $H(p,p^+)-H(p^+,p^+) \ne 0$ or $H(p^-,p)-H(p^-,p^-) \ne 0$ then S is countable.

LEMMA 3. If H is in OL^0 and α is a function from R to N of bounded variation on $[a,b]$ then (1) if p is in $[a,b]$ each of $\alpha(p^+)$ and $\alpha(p^-)$ exists and (2) if $\{x_i\}_{i=1}^\infty$ is a sequence of numbers such that if p is in $[a,b]$ and $H(p,p^+)-H(p^+,p^+) \ne 0$

or $H(p^-,p)-H(p^-,p^-) \neq 0$, then there is an n such that $p=x_n$, then

(1) $\sum_{i=1}^{\infty} [H(x_i,x_i^+)-H(x_i^+,x_i^+)][\alpha(x_i^+)-\alpha(x_i)]$ exists

and (2) $\sum_{i=1}^{\infty} [H(x_i^-,x_i)-H(x_i^-,x_i^-)][\alpha(x_i^-)-\alpha(x_i)]$ exists.

INDICATION OF PROOF. It follows from the bounded variation of α that for p in $[a,b]$ each of $\alpha(p^+)$ and $\alpha(p^-)$ exists.

Since H is in OL^0, it follows from the covering theorem that H is bounded on $[a,b]$ and that there is a number M_1 such that for each positive integer i,

$$|H(x_i,x_i^+)-H(x_i^+,x_i^+)| < M_1,$$

and, furthermore, for n a positive integer and $0 < i \leq n$, let $x_{p_i} > x_i$ such that $\sum_{i=1}^{n} |\alpha(x_i^+)-\alpha(x_{p_i})| < 1$. Hence,

$$\sum_{i=1}^{n} |[H(x_i,x_i^+)-H(x_i^+,x_i^+)][\alpha(x_i^+)-\alpha(x_i)]|$$

$$\leq M_1 [\sum_{i=1}^{n} |\alpha(x_i^+)-\alpha(x_{p_i})| + \sum_{i=1}^{n} |\alpha(x_{p_i})-\alpha(x_i)|$$

$$< M_1 (1) + M_1 \sum_D |\alpha(x_i)-\alpha(x_{i-1})|,$$

where D is a subdivision of $[a,b]$ containing x_i and x_{p_i} as consecutive points in D for each $0 < i \leq n$. Hence, since α is of bounded variation there is a number M such that

$$\sum_{i=1}^{n} |[H(x_i,x_i^+)-H(x_i^+,x_i^+)][\alpha(x_i^+)-\alpha(x_i)]| < M.$$

Therefore,

$\sum_{i=1}^{\infty} [H(x_i,x_i^+)-H(x_i^+,x_i^+)][\alpha(x_i^+)-\alpha(x_i)]$ exists. In a similar manner it may be shown that

$\sum_{i=1}^{\infty} [H(x_i^-,x_i)-H(x_i^-,x_i^-)][\alpha(x_i^-)-\alpha(x_i)]$ exists.

THEOREM 1. If H is in OL^0 and α is a function from R to N of bounded variation on $[a,b]$, then $(I_H)\int_a^b H d\alpha$ exists.

PROOF. If $\varepsilon > 0$ then it follows from LEMMA 2 that each of the sets A_ε^+ and A_ε^- to which p belongs if and only if p is in $[a,b]$ and $|H(p,p^+)-H(p^+,p^+)| \geq \varepsilon$ or $|H(p^-,p)-H(p^-,p^-)| \geq \varepsilon$, respectively, is a finite set. Let $A_\varepsilon^+ = \{c_i\}_{i=1}^{m_1}$, $A_\varepsilon^- = \{d_i\}_{i=1}^{m_2}$, and A^+ and A^- denote the sets to which p belongs if and only if p is in $[a,b]$ and $H(p,p^+)-H(p^+,p^+)\neq 0$ or $H(p^-,p)-H(p^-,p^-)\neq 0$, respectively. Since each of A^+ and A^- is a countable set then let $A^+ + A^- = \{y_i\}_{i=1}^{\infty}$.

Since α is of bounded variation on $[a,b]$, then for each c_i, $0 < i \leq m_1$ and d_i, $0 < i \leq m_2$ there is an $e_i > c_i$ and an $f_i > d_i$ such that if $e_i \geq r_i > c_i$ and and $f_i \leq s_i < d_i$, then $|\alpha(c_i^+)-\alpha(r_i)| < \frac{\varepsilon}{16m_1}$ and $|\alpha(d_i^-)-\alpha(s_i)| < \frac{\varepsilon}{16m_2}$.

From LEMMA 3, it follows that there is a positive integer N such that if $n > N$, then

$$(1)\quad \Big|\sum_{i=1}^{n}\left[H(y_i,y_i^+)-H(y_i^+,y_i^+)\right]\left[\alpha(y_i^+)-\alpha(y_i)\right] - \sum_{i=1}^{\infty}\left[H(y_i,y_i^+)-H(y_i^+,y_i^+)\right]\left[\alpha(y_i^+)-\alpha(y_i)\right]\Big| < \frac{\varepsilon}{8}$$

and

$$(2)\quad \Big|\sum_{i=1}^{n}\left[H(y_i^-,y_i)-H(y_i^-,y_i^-)\right]\left[\alpha(y_i^-)-\alpha(y_i)\right] - \sum_{i=1}^{\infty}\left[H(y_i^-,y_i)-H(y_i^-,y_i^-)\right]\left[\alpha(y_i^-)- \alpha(y_i)\right]\Big| < \frac{\varepsilon}{8}.$$

Note that for some y_i's, $\left[H(y_i^-,y_i)-H(y_i^-,y_i^-)\right]$ or $\left[H(y_i,y_i^+)-H(y_i^+,y_i^+)\right]$ may be zero.

Since, from LEMMA 1, $\int_a^b H d\alpha$ exists, then there is a number M and a subdivision D_1 of $[a,b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D_1, then

$$(3)\quad \sum_{D'} |\Delta\alpha_i| < M,$$

$$(4)\quad \Big| \int_a^b H d\alpha - \sum_{D'} H_i\Delta\alpha_i \Big| < \frac{\varepsilon}{4},$$

and (5) if $0 < i \leq n$, then $|H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)| < M$ and $|H(x_i^-,x_i)-H(x_i^-,x_i^-)| < M$.

Further, since H is in OL^0, using the covering theorem we may obtain a subdivision D_2 of $[a,b]$ such that if $D' = \{x_i\}_{i=0}^{n}$ is a refinement of D_2, $0 < i \leq n$, and

$x_{i-1} < r < s < x_i$, then

(6) $|H(r,s)-H(x_i^-,x_i^-)| < \frac{\varepsilon}{64M}$,

(7) $|H(r,s)-H(x_{i-1},x_{i-1})| < \frac{\varepsilon}{6M}$,

(8) $|H(x_{i-1},x_{i-1}^+)-H(x_{i-1},x_i)| < \frac{\varepsilon}{64}$,

and (9) $|H(x_i^-,x_i)-H(x_{i-1},x_i)| < \frac{\varepsilon}{64M}$,

Let $D = D_1+D_2+A_\varepsilon^++A_\varepsilon^-+\sum_{i=1}^{m1}\{e_i\} + \sum_{i=1}^{m2}\{f_i\} + \sum_{i=1}^{N}\{y_i\}$, $D'=\{x_i\}_{i=0}^n$ be a refinement of D, and for each $0 < i \leq n$, $x_{i-1} < r_{j-1} < s_j < x_i$. Choose $m > N$ such that for each x_i, $0 < i \leq n$, in $D'\cdot(A^++A^-)$ there exists a positive integer $z < m$ such that $y_z = x_i$. Hence, for x_i, $0 < i \leq n$, in D' such that neither x_{i-1} nor x_i is in (A^++A^-), it follows from (6)-(9) that $|H(r_i,s_i)-H(x_{i-1},x_i)| < \frac{\varepsilon}{32M}$.

If $W_i=H(y_{i-1},y_{i-1}^+)-H(y_{i-1}^+,y_{i-1}^+)$ and $Q=\{y_1, y_2, \ldots y_m\}$ for $0 < i \leq m$ then

$$|\sum_{i=1}^m W_i[\alpha(y_{i-1}^+)-\alpha(y_{i-1})] - \sum_{D'\cdot[A_\varepsilon^++(A^+-A_\varepsilon^+)]}[H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)]\Delta\alpha_i|$$

$$= |\sum_{A_\varepsilon^+} W_1[\alpha(y_{i-1}^+)-\alpha(y_{i-1})] + \sum_{Q-A_\varepsilon^\pm} W_1[\alpha(y_{i-1}^+)-\alpha(y_{i-1})]$$

$$- \sum_{D'\cdot A_\varepsilon^+}[H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^-)]\Delta\alpha_i$$

$$- \sum_{D'\cdot(A^+-A_\varepsilon^+)}[H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)]\Delta\alpha_i|$$

$$\leq \sum_{A_\varepsilon^+}|W_i| \cdot |\alpha(y_{i-1}^+)-\alpha(y_i)| + \sum_{Q-A_\varepsilon^+}|W_i| \cdot |\alpha(y_{i-1}^+)-\alpha(y_{i-1})|$$

$$+ \sum_{D'\cdot(A^+-A_\varepsilon^+)}|H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)| \cdot |\Delta\alpha_i|$$

$$< M\sum_{A_\varepsilon^+}\frac{\varepsilon}{16Mm_1} + \frac{\varepsilon}{16M}\sum_{Q-A_\varepsilon^+}|\alpha(y_{i-1}^+)-\alpha(y_{i-1})| + \frac{\varepsilon}{16M}\sum_{D'\cdot(A^+-A_\varepsilon^+)}|\Delta\alpha_i|$$

$$< \frac{3\varepsilon}{16}.$$

Hence

$$(10)\quad \Big|\sum_{i=1}^{m} W_i\big[\alpha(y_{i-1}^+)-\alpha(y_{i-1})\big]-\sum_{D'\cdot A^+} H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)\ \Delta\alpha_i\Big| < \frac{3\varepsilon}{16}$$

and in a similar manner it may be shown that

$$(11)\quad \Big|\sum_{i-1}^{m} Z_i\big[\alpha(y_i^-)-\alpha(y_i)\big] - \sum_{D'\cdot A^-}\big[H(x_i^-,x_i^-)-H(x_i^-,x_i)\big]\Delta\alpha_i\Big| < \frac{3\varepsilon}{16},$$

where $Z_i = H(y_i^-,y_i^-)-H(y_i^-,y_i)$.

Using inequalities (10) and (11) we are now able to complete the proof of the theorem. In the following manipulations W_i and Z_i are as defined for (10) and (11) and $P_i=H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)$ and $Q_i=H(x_i^-,x_i^-)-H(x_i^-,x_i)$.

$$\Big|\sum_{D'} H_j\Delta\alpha_i-\int_a^b H d\alpha-\sum_{i=1}^{\infty} W_i\big[\alpha\ (P_{i-1}^+)-\alpha(P_{i-1})\big]-\sum_{i=1}^{\infty} Z_i\big[\alpha(y_i^-)-\alpha(y_i)\big]\Big|$$

$$< \Big|\sum_{D'}(H_j-H_i)\Delta\alpha_i-\sum_{i=1} W_i\big[\alpha(y_{i-1}^+)-\alpha(y_{i-1})\big]-\sum_{i=1}^{m} Z_i\big[\alpha(y_i^-)-\alpha(y_i)\big]\Big| + \frac{\varepsilon}{4}+\frac{\varepsilon}{16}+\frac{\varepsilon}{16}$$

$$\leq \Big|\sum_{D'}(H_j-H_i)\Delta\alpha_i-\sum_{D'\cdot A^+} P_i\Delta\alpha_i-\sum_{D'\cdot A^-} Q_i\Delta\alpha_i\Big| + \frac{3}{16}+\frac{3}{16}+\frac{3}{8}$$

$$\leq \sum_{D'-D'\cdot(A^++A^-)}|H_j-H_i|\ \cdot\ |\Delta\alpha_i| + \sum_{D'\cdot A^+}|H_j-H_i-P_i|\Delta\alpha_i + \sum_{D'\cdot A^-}|H_j-H_i-Q_i|\cdot|\Delta\alpha_i| + \frac{3\varepsilon}{4}$$

$$< \frac{\varepsilon}{32M}\cdot M + \frac{\varepsilon}{32M}\cdot M + \frac{\varepsilon}{32M}\cdot M$$

$$< \varepsilon.$$

Hence, we have a relationship established between $(I_H)\int_a^b H d\alpha$ and $\int_a^b H d\alpha$ which will be used in the proof of the principal theorem.

THEOREM 2. If $\{H_i\}_{i=0}^{\infty}$ is a sequence of functions from SxS to N, such that for each i, H_i is in OL^0, $\lim_{n\to\infty} H_n=H_0$ uniformly on $[a,b]$, and T is a bounded linear operator on OL^0 then $\lim_{n\to\infty} TH_n=TH_0$.

The proof of this theorem is straightforward and we omit it.

THEOREM 3. Suppose H is in OL^0, T is a bounded linear operator on OL^0. Then,

$$TH = (I)\int_a^b f_H d\alpha + \sum_{i=1}^{\infty}\big[H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)\big]\,\beta(x_{i-1})$$

$$+ \sum_{i=1}^{\infty}\big[H(x_i^-,x_i)-H(x_i^-,x_i^-)\big]\ \Theta(x_{i-1},x_i),$$

where each of α, β, and Θ depend only on T, α is of bounded variation, β and Θ are 0 except at a countable number of points, f_H is a function from R to N depending on H, and $\{x_i\}_{i=1}^{\infty}$ denote the points in $[a,b]$ for which $[H(x_i,x_i^+)-H(x_i^+,x_i^+)]\neq 0$ or $H(x_i^-,x_i)-H(x_i^-,x_i^-)\neq 0$, $i=1,2,\dots,n$.

PROOF. We first define a sequence of functions converging uniformly to a given function H in OL^0 and then apply THEOREM 2 to establish THEOREM 3. We first define functions g and h for each pair of numbers t,x, $a \le t \le b$, $a \le x \le b$ such that

$$g(t,x) = \begin{cases} 1, & \text{if } t=x \\ 0, & \text{if } t\neq x \end{cases} \qquad \text{and } h(t,x) = \begin{cases} 1, & \text{if } a \le t \le x \\ 0, & \text{if } x < t \le b, \end{cases}$$

and using these functions and the operator T define functions α, β, γ, and $\emptyset$ such that

$$\alpha(x)=TH(\cdot,x);\ \beta(x)=Tg(\cdot,x);\ \gamma(x)=Tg(x,\cdot);\ \emptyset(x,y)=Tg(\cdot,x)g(y,\cdot);\ \text{and}$$

$$\Theta(x,y)=\gamma(y)-\emptyset(x,y) \text{ for } x \text{ and } y \text{ in } [a,b].$$

Clearly, α is of bounded variation on $[a,b]$ and we see from

$$\begin{aligned} \sum_{D'} |\emptyset(x_{i-1},x_i)| &= \sum_{D'} \emptyset_i^2 \\ &= \sum_{D'} \text{sgn}\emptyset_i Tg(\cdot,x_{i-1})g(x_i,\cdot) \\ &\le M\Big\|\sum_{D'} \text{sgn}\emptyset_i g(\cdot,x_{i-1})g(x_i,\cdot)\Big\|_D \\ &= M, \end{aligned}$$

for D' a refinement of a subdivision D of $[a,b]$, it follows that $\sum_{i=1}^{\infty} |\emptyset(x_{i-1},x_i)|$ exists and in a similar manner that each of $\sum_{i=1}^{\infty} |\beta(x_i)|$ and $\sum_{i=1}^{\infty} |\ (x_i)\ |$ exists.

Hence, $\sum_{i=1}^{\infty} |\Theta(x_{i-1},x_i)|$ exists.

Each of our approximating functions H_n will be defined in terms of a subdivision D_n of $[a,b]$ determined in the following manner.

Since α is of bounded variation on $[a,b]$ and H is in OL^0 then from THEOREM 1, $(I_H)\int_a^b Hd$ exists and there is a subdivision K_n of $[a,b]$ such that if $K'=\{x_i\}_{i=1}^m$ is a refinement of K_n, then $|(I_H)\int_a^b Hd\alpha - \sum_{K'} H(r_i,s_i)\Delta\alpha_i| < \frac{1}{n}$ where for $0 < i \le m$, $x_{i-1} < r_i < s_i < x_i$. It follows from the covering theorem and the existence of the limits $H(p,p^+)$ and $H(p^+p^+)$ that there is a subdivision $I_n = \{x_i\}_{i=0}^m$ of $[a,b]$ such that if $x_{i-1} < x < r < s < y < x_i$, $0 < i \le m$, then $|H(x,y)-H(r,s)| < \frac{1}{n}$.
Further, let J_n denote the set such that p is in J_n if p is in $[a,b]$ and $|H(p,p^+)-H(p^+,p^+)| \ge \frac{1}{n}$ or $|H(p^-,p)-H(p^-,p^-)| \ge \frac{1}{n}$ and $D_n = K_n + J_n + I_n$. For each positive integer n, let H_n be a function from RxR to N determined by $D_n = \{x_i\}_{i=1}^m$ in the following manner:

$$H_n(x,y) = \sum_{i=1}^m H(r_i,s_i)[h(x,x_i)-h(x,x_{i-1})] + \sum_{i=1}^m [H(x_{i-1},x_{i-1}^+)-H(r_i,s_i)]\,[g(x,x_i)]$$
$$+ \sum_{i=1}^m [H(x_i^-,x_i)-H(r_i,s_i)]\,g(x_i,y)$$
$$- \sum_{i=1}^m [H(x_i^-,x_i)-H(r_i,s_i)]\,g(x,x_{i-1})g(x_i,y)$$

for each (x,y) such that $x_{i-1} \le x < y \le x_i$, for some $0 < i \le m$, and for each $[x_{i-1},x_i]$, $0 < i \le m$, $x_{i-1} < r_i < s_i < x_i$.

It is evident that $\lim_{n\to\infty} H_n = H$ uniformly on $[a,b]$ for if $\varepsilon > 0$, $\frac{1}{n} < \varepsilon$, $D = D_n = \{x_i\}_{i=0}^m$, and $x_{p-1} < x < r < s < y < x_p$ for some $0 < p \le m$, then $H_n(x_{p-1},x_p) = H(x_{p-1},x_p)$, $H_n(x,x_p) = H(x,x_p)$, $H_n(x_{p-1},y) = H(x_{p-1},y)$, and $H_n(x,y) = H(r,s)$. Hence $\lim_{n\to\infty} H_n = H$ uniformly on $[a,b]$.

Since $\lim_{n\to\infty} H_n = H$ uniformly on $[a,b]$, applying THEOREM 2, we have

$$TH = \lim_{n\to\infty} TH_n$$

$$= \lim_{n\to\infty} \sum_{D_n} H(r_i,s_i)[TH(\cdot,x_i) - TH(\cdot,x_{i-1})]$$

$$+ \lim_{n\to\infty} \sum_{D_n} [H(x_{i-1},x_{i-1}^+)-H(r_i,s_i)]\, Tg(\cdot,x_{i-1})$$

$$+ \lim_{n\to\infty} \sum_{D_n} [H(x_i^-,x_i)-H(r_i,s_i)]\, Tg(x_i,\cdot)$$

$$+ \lim_{n\to\infty} [-H(x_i^-,x_i)+H(r_i,s_i)]\, Tg(\cdot,x_{i-1})g(x_i,\cdot)$$

$$= (I_H)\int_a^b H d\alpha + \sum_{i=1}^{\infty} [H(x_{i-1},x_{i-1}^+)-H(x_{i-1}^+,x_{i-1}^+)]\beta(x_{i-1})$$

$$+ \sum_{i=1}^{\infty} [H(x_i^-,x_i)-H(x_i^-,x_i^-)]\, \gamma(x_i)$$

$$+ \sum_{i=1}^{\infty} [H(x_i^-,x_i^-)-H(x_i^-,x_i)]\, \phi(x_{i-1},x_i)$$

$$+ \sum_{i=1}^{\infty} H(x_i^-,x_i)-H(x_i^-,x_i^-)\, \Theta(x_{i-1},x_i)$$

where the existence of each of the infinite sums is assured by LEMMA 3 and the equality of the last two expressions follows from the definition of D_n.

All that remains to complete the proof of THEOREM 3 is to show that $I_H\int_a^b H d\alpha$ may be represented by $(I)\int_a^b f_H d$ where f_H is a function from R to N. If we let f_H be the function such that for each p in $[a,b]$ $f_H(p) = H(p^+,p^+)$ then it follows that $(I)\int_a^b f_H d\alpha$ exists and is $(I_H)\int_a^b H d\alpha$.

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