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GENERALIZED DISTRIBUTIONS OF ORDER k ASSOCIATED WITH SUCCESS RUNS IN BERNOULLI TRIALS

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In a sequence of independent Bernoulli trials, by counting multidimensional lattice paths in order to compute the probability of a first-passage event, we derive and study a generalized negative binomial distribution of order k, type I, which extends to distributions of order k, the generalized negative binomial distribution of Jain and Consul (1971), and includes as a special case the negative binomial distribution of order k, type I, of Philippou et al. (1983). This new distribution gives rise in the limit to generalized logarithmic and Borel-Tanner distributions and, by compounding, to the generalized Pólya distribution of the same order and type. Limiting cases are considered and an application to observed data is presented.

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1. Introduction. In five pioneering papers, Philippou and Muwafi [22], Philippou et al. [21], Philippou [17], and Philippou et al. [19, 20] introduced the study of univariate and multivariate distributions of order k. Since then, the subject matter received a lot of attention from many researchers. For comprehensive reviews at the time of publication, we refer to Johnson et al. [9, 10].

Consider a sequence of Bernoulli trials with success (S) probability p $(0 < p < 1)$, and let $T_{k,r}$ denote the number of trials until r $(r \geq 1)$ nonoverlapping success runs of length k $(k \geq 1)$ appear. Philippou et al. [21] (see also Philippou [18]) derived, for $x = kr, kr+1, \dots$, the following exact formula for the probability distribution of $T_{k,r}$:

$$P(T_{k,r} = x) = \sum_{\Sigma_j j x_j = x-kr} \binom{x_1+\cdots+x_k+r-1}{x_1,\dots,x_k,r-1} p^x \left(\frac{q}{p}\right)^{\Sigma_j x_j}. \tag{1.1}$$

The probability distribution (1.1) is known as negative binomial distribution of order k, type I, with parameters r and p and it is denoted by $\mathrm{NB}_{k,I}(r;p)$. A different sampling derivation of the (suitably shifted) negative binomial distribution of order k, type I, say $\overline{\mathrm{NB}}_{k,I}(r;p)$, was given by Antzoulakos and Philippou [2] (see also Tripsiannis and Philippou [25]). The Poisson and the logarithmic series distributions of order k, say type I, were obtained as limiting cases of $\mathrm{NB}_{k,I}(r;p)$, by Philippou et al. [21] and Aki et al. [1], respectively. Panaretos and Xekalaki [15] derived and studied a hypergeometric, a negative hypergeometric, and a generalized Waring distribution of order k by means of the methodology of Philippou and Muwafi [22], and Philippou et al. [21]. They also stated, without further details, the derivation of a Polya and an inverse Polya distribution of order k, of which the preceding three are proper special cases. Ling [11] rederived the above-mentioned inverse Polya distribution of order k, say type I, and introduced, allowing runs of length k to overlap, a new inverse Polya distribution of order k, say type III. A different sampling derivation of the (shifted) inverse Polya distribution of order k, type I, was given by Tripsiannis and Philippou [25].

Jain and Consul [6] introduced and studied the generalized negative binomial distribution (see also Mohanty [13]). The generalized logarithmic series distribution and the Borel-Tanner or generalized Poisson distribution were obtained as limiting cases of the generalized negative binomial distribution by Jain and Consul [6], and Jain and Singh [8], respectively (see also Jain [5], Jain and Gupta [7], and Haight and Breuer [4]). Recently, Sen and Mishra [23] obtained the generalized Polya distribution, which unifies the usual Polya and inverse Polya distributions.

In this paper, we extend the above-mentioned generalized distributions to distributions of order k. In Section 2, we derive a generalized negative binomial distribution of order k, type I, say $\mathrm{GNB}_{k,I}(\cdot)$, which includes as a special case the negative binomial distribution of order k, type I, of Philippou et al. [21]. We do it by counting multidimensional lattice paths in a generalized sampling scheme employing a first passage approach (see Theorem 2.1 and Definition 2.2). Another genesis scheme of $\mathrm{GNB}_{k,I}(\cdot)$ is given next (see Proposition 2.3), which indicates potential applications and provides its probability generating function (PGF), mean and variance (see Proposition 2.4). We next obtain two limiting cases of $\mathrm{GNB}_{k,I}(\cdot)$ (see Propositions 2.5 and 2.7), which provide, respectively, a generalized logarithmic series distribution of order k, type I, say $\mathrm{GLS}_{k,I}(\cdot)$, and a Borel-Tanner distribution of the same order and type, say $\mathrm{BT}_{k,I}(\cdot)$ (see Definitions 2.6 and 2.8). By means of a generalized sampling scheme and a first-passage approach (see Theorem 2.9), and by compounding the $\mathrm{GNB}_{k,I}(\cdot)$ with the Beta distribution (see Proposition 2.11), we introduce a generalized Polya distribution of order k, type I, say $\mathrm{GP}_{k,I}(\cdot)$ (see Definition 2.10), which includes as a special case the inverse Polya distribution of order k, type I, of Panaretos and Xekalaki [15]. In Section 3, we introduce, as special cases of $\mathrm{GP}_{k,I}(\cdot)$, several distributions of order k, most of which are new. In Section 4, we relate asymptotically $\mathrm{GNB}_{k,I}(\cdot)$ to the Poisson distribution $(P_{k,I}(\lambda))$ of order k, type I, of Philippou et al. [21] (see Proposition 4.1), and $\mathrm{GP}_{k,I}(\cdot)$ to $\mathrm{GNB}_{k,I}(\cdot)$, $\mathrm{BT}_{k,I}(\cdot)$, $P_{k,I}(\lambda)$ and to the negative binomial distribution $(\mathrm{NB}_{k,I}(n;p))$ of order k, type I, of Philippou et al. [21] (see Propositions 4.2, 4.3, 4.4, and 4.5). An application to observed data is also presented.

In order to avoid unnecessary repetitions, we mention here that in this paper, $x_1,\dots,x_k$ are nonnegative integers as specified. In addition, whenever sums and products are taken over j, ranging from 1 to k, we will omit these limits for notational simplicity.

2. Generalized negative binomial distribution of order k, type I. In this section, we introduce a new distribution of order k, type I, by means of a generalized sampling scheme and a first-passage approach, and we obtain its PGF, mean and variance. Furthermore, we derive three new distributions of order k: the generalized logarithmic series, Polya distributions of order k, type I, and the Borel-Tanner distribution of the same order and type. First, we consider the following theorem.

Theorem 2.1. *In a sequence of independent Bernoulli trials with success* (S) *probability* p $(0 < p < 1)$, *consider the random variables* X_j $(1 \le j \le k)$ *and* L_k $(k \ge 1)$ *denoting, respectively, the number of events*

$$e_j = \underbrace{S\cdots S}_{j-1}F, \qquad \widetilde{e}_k = \underbrace{S\cdots S}_{k}. \tag{2.1}$$

Let X *be a random variable denoting the number of occurrences of failures and the total number of successes which precede directly the occurrences of failures but do not belong to any success run of length* k, *that is,* $X = \Sigma_j jX_j$. *Trials are continued until* $n + \mu\Sigma_j X_j$ $(n > 0$ *and* $\mu \ge -1)$ *nonoverlapping success runs of length* k *appear for the first time, that is, at any trial* t $(1 \le t \le \Sigma_j jX_j + k(n + \mu\Sigma_j X_j) - 1)$, *the condition* $A = \{L_k^{[t]} < n + \mu\Sigma_j X_j^{[t]}$, *where* $X_j^{[t]}$ *and* $L_k^{[t]}$ *are the numbers of events* e_j *and* $\widetilde{e}_k$, *respectively, in the first* t *trials*$\}$, *is satisfied. Then, for* $x = 0, 1, \ldots$,

$$P(X = x) = \sum_{\Sigma_j jx_j = x} \frac{n}{n + (1+\mu)\Sigma_j x_j} \binom{n + (1+\mu)\Sigma_j x_j}{x_1, \ldots, x_k, n + \mu\Sigma_j x_j} p^{k(n+\mu\Sigma_j x_j)+x} \left(\frac{q}{p}\right)^{\Sigma_j x_j}, \tag{2.2}$$

where $q = 1 - p$.

Proof. For any fixed nonnegative integer x, a typical element of the event $(X = x)$ is a sequence of outcomes $\Sigma_j jx_j + k(n + \mu\Sigma_j x_j)$ of the letters F and S, such that the event e_j appears x_j $(1 \le j \le k)$ times and the event $\widetilde{e}_k$ appears $n + \mu\Sigma_j x_j$ times, satisfying the condition A and $\Sigma_j jx_j = x$.

Fix x_j $(1 \le j \le k)$ (n and μ are fixed), and denote the event e_j $(1 \le j \le k)$ by a unit step in Z_j-direction and the event $\widetilde{e}_k$ by a unit step in Z_0-direction. Therefore, we represent a sequence of x_j events e_j $(1 \le j \le k)$ and $n + \mu\Sigma_j x_j$ events $\widetilde{e}_k$ by a $(k+1)$-dimensional lattice path from the origin to $(n + \mu\Sigma_j x_j, x_1, \ldots, x_k)$, which does not touch the hyperplane $z_0 = n + \mu\Sigma_j x_j$ except at the point

$(n+\mu\Sigma_j x_j, x_1, \ldots, x_k)$. Then, the number of such lattice paths is

$$\frac{n}{n+(1+\mu)\Sigma_j x_j}\binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} \tag{2.3}$$

(see [14, Example 10, page 25]) and each one of them has probability

$$p^{k(n+\mu\Sigma_j x_j)+x}\left(\frac{q}{p}\right)^{\Sigma_j x_j}. \tag{2.4}$$

Then the theorem follows since the nonnegative integers $x_1, \ldots, x_k$ may vary subject to $\Sigma_j j x_j = x$. □

By means of the transformations $x_j = r_j$ $(1 \le j \le k)$ and $x = r + \Sigma_j (j-1) r_j$, and by the multinomial theorem and relation (2.3) of Jain and Consul [6], it may be seen that the above derived probability function is a proper probability distribution.

For $k = 1$, this distribution reduces to the generalized negative binomial distribution of Jain and Consul [6]; with $\beta = \mu + 1$, and for $\mu = 0$, it reduces to the (shifted) negative binomial distribution of order k, type I, of Philippou et al. [21]. We, therefore, introduce the following definition.

Definition 2.2. A random variable (RV) X is said to have the generalized negative binomial distribution of order k, type I, with parameters n, μ, and p $(n > 0$ and $\mu \ge -1$, both integers, and $0 < p < 1)$ to be denoted by $\mathrm{GNB}_{k,I}(n;\mu;p)$ if, for $x = 0, 1, \ldots$, $P(X = x)$ is given by (2.2).

The following proposition, which is a direct consequence of Definition 2.2, indicates that $\mathrm{GNB}_{k,I}(\cdot)$ may have potential applications whenever there are multiple groups of items and we are interested in the distribution of the total number of items.

Proposition 2.3. *Let X_j, $1 \le j \le k$, be random variables and set $X = \Sigma_j j X_j$. Then, X is distributed as $\mathrm{GNB}_{k,I}(n;\mu;p)$ if and only if $X_1, \ldots, X_k$ are jointly distributed as multivariate generalized negative binomial distribution with parameters $n, \mu, \ldots, \mu$ and $Q_1, \ldots, Q_k$, where $Q_j = p^{j-1} q$ $(1 \le j \le k)$.*

The PGF, mean and variance, of the generalized negative binomial distribution of order k, type I, may be readily obtained by means of Proposition 2.3 or by means of Definition 2.2, the transformations $x_j = r_j$ $(1 \le j \le k)$ and $x = r + \Sigma_j (j-1) r_j$, and simple expectation properties.

Proposition 2.4. *Let X be a RV following the generalized negative binomial distribution of order k, type I. Then,*

(i) *the PGF of X is given by:*

$$g_X(t) = \left(\frac{1-\vartheta(y_1,\ldots,y_k)}{1-\vartheta(y_1 t,\ldots,y_k t^k)}\right)^n, \quad |t| \le 1, \tag{2.5}$$

where

$$\vartheta(y_1,\dots,y_k)=\sum_{x_1=0}^{\infty}\cdots\sum_{x_k=0}^{\infty}\frac{\Gamma((1+\mu)\Sigma_j x_j-1)}{\Gamma(\mu\Sigma_j x_j)x_1!\cdots x_k!}\Pi_j y_j^{x_j},$$
$$y_j=qp^{k\mu+j-1}\quad(1\le j\le k). \tag{2.6}$$

(ii) *The mean and variance of X are given by:*

$$E(X)=\frac{nq}{p^k-\mu(1-p^k)}\Sigma_j jp^{j-1};$$
$$\operatorname{Var}(X)=\frac{nq}{p^k-\mu(1-p^k)}\times\left(\Sigma_j j^2p^{j-1}+\frac{q}{p^k-\mu(1-p^k)}\left[2\mu+1+\frac{\mu(1+\mu)(1-p^k)}{p^k-\mu(1-p^k)}\right](\Sigma_j jp^{j-1})^2\right). \tag{2.7}$$

For $k=1$, Proposition 2.4(i) reduces to the PGF of the generalized negative binomial distribution (see Jain [5]) and (ii) reduces to its mean and variance (see Jain and Consul [6]).

Jain and Consul [6] obtained the generalized logarithmic series distribution as a limit of the generalized negative binomial distribution. We extend this result to the generalized negative binomial distribution of order k, type I, and we name the limit distribution accordingly.

Proposition 2.5. *Let X_n $(n>0)$ be a RV distributed as* $\mathrm{GNB}_{k,I}(n;\mu;p)$, *and assume that $n\to 0$. Then, for $x=1,2,\dots,$*

$$P(X_n=x\mid X_n\ge 1)\longrightarrow\alpha\sum_{\Sigma_j jx_j=x}\frac{1}{(1+\mu)\Sigma_j x_j}\binom{(1+\mu)\Sigma_j x_j}{x_1,\dots,x_k,\mu\Sigma_j x_j}p^{k\mu\Sigma_j x_j+x}\left(\frac{q}{p}\right)^{\Sigma_j x_j}, \tag{2.8}$$

where $\alpha=-(k\log p)^{-1}$.

Proof. For $x=1,2,\dots,$ we have

$$\begin{aligned}P(X_n=x\mid X_n\ge 1)&=\frac{P(X_n=x,\ X_n\ge 1)}{1-P(X_n=0)}\\&=\frac{np^{kn}}{1-p^{kn}}\sum_{\Sigma_j jx_j=x}\frac{1}{n+(1+\mu)\Sigma_j x_j}\frac{(n+(1+\mu)\Sigma_j x_j)!/n!}{(n+\mu\Sigma_j x_j)!/n!}\frac{1}{\Pi_j x_j!}p^{k\mu\Sigma_j x_j+x}\left(\frac{q}{p}\right)^{\Sigma_j x_j}\\&\longrightarrow\frac{1}{-k\log p}\sum_{\Sigma_j jx_j=x}\frac{1}{(1+\mu)\Sigma_j x_j}\binom{(1+\mu)\Sigma_j x_j}{x_1,\dots,x_k,\mu\Sigma_j x_j}p^{k\mu\Sigma_j x_j+x}\left(\frac{q}{p}\right)^{\Sigma_j x_j},\end{aligned} \tag{2.9}$$

which establishes the proposition. □

For $k = 1$, the above derived distribution reduces to the generalized logarithmic series distribution of Jain and Consul [6] and for $\mu = 0$, it reduces to the logarithmic series distribution of order k, type I, of Aki et al. [1]. We, therefore, introduce the following definition.

Definition 2.6. A RV X is said to have the generalized logarithmic series distribution of order k, type I, with parameters μ and p ($\mu \geq -1$ is an integer, and $0 < p < 1$) to be denoted by $\mathrm{GLS}_{k,I}(\mu;p)$, if for $x = 1,2,\ldots$, and $\alpha = -(k\log p)^{-1}$,

$$P(X = x) = \alpha \sum_{\Sigma_j jx_j = x} \frac{1}{(1+\mu)\Sigma_j x_j} \binom{(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,\mu\Sigma_j x_j} p^{k\mu\Sigma_j x_j + x} \left(\frac{q}{p}\right)^{\Sigma_j x_j}. \tag{2.10}$$

It is well known that the usual Borel-Tanner distribution may be obtained as a limit of the generalized negative binomial distribution (see Jain and Singh [8]). We extend this result to the generalized negative binomial distribution of order k, type I, and we name the limit distribution accordingly.

Proposition 2.7. *Let $X_{n,\mu,q}$ ($n > 0$ and $\mu \geq -1$, both integers, and $0 < q < 1$) be a RV distributed as $\mathrm{GNB}_{k,I}(n;\mu;p)$, and assume that $nq \to r\alpha\beta$ ($\alpha > 0$, $\beta > 0$, and $r = 1,2,\ldots$) and $\mu q \to \alpha d\beta$ ($d = 1,2,\ldots$), as $n \to \infty$, $\mu \to \infty$, and $q \to 0$. Then, for $x = 0,1,\ldots$,*

$$P(X_{n,\mu,q} = x) \longrightarrow \sum_{\Sigma_j jx_j = x} \frac{r(\alpha\beta)^{\Sigma_j x_j}}{\Pi_j x_j!} (r + d\Sigma_j x_j)^{\Sigma_j x_j - 1} e^{-(k\alpha\beta)(r + d\Sigma_j x_j)}. \tag{2.11}$$

Proof. For $x = 0,1,\ldots$, we have

$$\begin{aligned} &P(X_{n,\mu,q} = x) \\ &\quad = \sum_{\Sigma_j jx_j = x} \frac{1}{n^{\Sigma_j x_j - 1}} \frac{(n + (1+\mu)\Sigma_j x_j - 1)!}{(n + \mu\Sigma_j x_j)!} p^{k(n+\mu\Sigma_j x_j) + x - \Sigma_j x_j} \frac{(nq)^{\Sigma_j x_j}}{\Pi_j x_j!} \\ &\quad \longrightarrow \sum_{\Sigma_j jx_j = x} \frac{r(\alpha\beta)^{\Sigma_j x_j}}{\Pi_j x_j!} (r + d\Sigma_j x_j)^{\Sigma_j x_j - 1} e^{-(k\alpha\beta)(r + d\Sigma_j x_j)}, \end{aligned} \tag{2.12}$$

which establishes the proposition. □

For $k = 1$, this distribution reduces to the (shifted) Borel-Tanner distribution of Haight and Breuer [4] with parameters $r = r/d$ and $\alpha = \alpha d\beta$. We, therefore, introduce the following definition.

Definition 2.8. A RV X is said to have the Borel-Tanner distribution of order k, type I, with parameters α, β, d, and r ($\alpha > 0$, $\beta > 0$, $d = 1,2,\ldots$, and

$r = 1,2,\ldots$) to be denoted by $\mathrm{BT}_{k,I}(\alpha;\beta;d;r)$ if, for $x = 0,1,\ldots$,

$$P(X=x) = \sum_{\Sigma_j jx_j = x} \frac{r(\alpha\beta)^{\Sigma_j x_j}}{\Pi_j x_j!}(r+d\Sigma_j x_j)^{\Sigma_j x_j - 1} e^{-(k\alpha\beta)(r+d\Sigma_j x_j)}. \tag{2.13}$$

We observe that this distribution is essentially the generalized Poisson distribution of order k, type I, of Tripsiannis et al. [26], with $\theta = \alpha\beta r$ and $\lambda = \alpha\beta d$.

We proceed now to derive a generalized Polya distribution of order k, type I, using a first-passage approach in a generalized sampling scheme, which follows along the same lines as those in the proof of Theorem 2.1.

Theorem 2.9. *An urn contains $c_0 + c_1 (= c)$ balls of which c_1 bear the letter F and c_0 bear the letter S. A ball is drawn at random from the urn, its letter is recorded, and it is replaced into the urn, together with s balls bearing the same letter. Consider the random variables X_j $(1 \le j \le k)$, L_k, and X as in Theorem 2.1. Then, for $x = 0,1,\ldots$,*

$$P(X=x)$$

$$= \sum_{\Sigma_j jx_j = x} \frac{n}{n+(1+\mu)\Sigma_j x_j}\binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} \frac{c_0^{(k(n+\mu\Sigma_j x_j)+\Sigma_j (j-1)x_j,s)} c_1^{(\Sigma_j x_j,s)}}{c^{(k(n+\mu\Sigma_j x_j)+x,s)}}, \tag{2.14}$$

where $a^{(b,d)} = a(a+d)\cdots(a+(b-1)d)$, for $b > 0$, and $a^{(0,d)} = 1$.

For $k = 1$, this distribution reduces to the generalized Polya distribution of Sen and Mishra [23] and for $\mu = 0$, it reduces to the inverse Polya distribution of order k, type I, of Panaretos and Xekalaki [15]. We, therefore, introduce the following definition.

Definition 2.10. A RV X is said to have the generalized Polya distribution of order k, type I, with parameters n,μ,s,c, and c_1 (s, $\mu \ge -1$, both integers, and n, c, and c_1, positive integers) to be denoted by $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$ if, for $x = 0,1,\ldots$, $P(X=x)$ is given by (2.14).

The next proposition provides another derivation of the generalized Polya distribution of order k, type I, by compounding the generalized negative binomial distribution of the same order and type.

Proposition 2.11. *Let X and P be two RVs such that $(X \mid P = p)$ is distributed as $\mathrm{GNB}_{k,I}(n;\mu;p)$, and P is distributed as $B(\alpha,\beta)$ (the Beta distribution with positive real parameters α and β). Then, for $x = 0,1,\ldots$,*

$$\begin{aligned} P(X=x) = \sum_{\Sigma_j jx_j = x} &\frac{n}{n+(1+\mu)\Sigma_j x_j}\binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} \\ &\times \frac{B(\alpha+k(n+\mu\Sigma_j x_j)+\Sigma_j (j-1)x_j, \beta+\Sigma_j x_j)}{B(\alpha,\beta)}. \end{aligned} \tag{2.15}$$

Proof. For $x = 0,1,\ldots$, we get

$$P(X=x) = \frac{1}{B(\alpha,\beta)} \sum_{\Sigma_j jx_j = x} \frac{n}{n+(1+\mu)\Sigma_j x_j} \binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} \times \int_0^1 p^{\alpha+k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j-1}(1-p)^{\beta+\Sigma_j x_j-1}dp$$
$$= \sum_{\Sigma_j jx_j=x} \frac{n}{n+(1+\mu)\Sigma_j x_j}\binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} \times \frac{B(\alpha+k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j,\beta+\Sigma_j x_j)}{B(\alpha,\beta)}, \tag{2.16}$$

which establishes the proposition. □

We note that relation (2.15) reduces to (2.14) if $\alpha = c_0/s$ and $\beta = c_1/s$ $(s \neq 0)$, which indicates that (2.15) may be considered as another form of $\mathrm{GP}_{k,I}(\cdot)$.

For $k = 1$, the above proposition provides a new derivation of the generalized Polya distribution of Sen and Mishra [23].

3. Special cases of $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$.

In this section, we consider the following special cases of the generalized Polya distribution of order k, type I, eleven of which are new.

Class 3.1. In $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$, let $\mu = -1$. Then, for $x = 0,1,\ldots,kn$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n}{x_1,\ldots,x_k,n-\Sigma_j x_j} \frac{c_0^{(k(n-\Sigma_j x_j)+\Sigma_j(j-1)x_j,s)} c_1^{(\Sigma_j x_j,s)}}{c^{(k(n-\Sigma_j x_j)+x,s)}}, \tag{3.1}$$

which reduces to the usual Polya distribution (see, e.g., Patil et al. [16, page 51]), for $k = 1$. We say that the RV X has the Polya distribution of order k, type I, with parameters n,s,c, and c_1, and we denote it by $P_{k,I}(n;s;c,c_1)$.

Case 1. In $P_{k,I}(n;s;c,c_1)$, let $s = -1$. Then, for $x = 0,1,\ldots,kn$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n}{x_1,\ldots,x_k,n-\Sigma_j x_j} \frac{c_0^{(k(n-\Sigma_j x_j)+\Sigma_j(j-1)x_j)} c_1^{(\Sigma_j x_j)}}{c^{(k(n-\Sigma_j x_j)+x)}}, \tag{3.2}$$

where $\alpha^{(r)} = \alpha(\alpha-1)\cdots(\alpha-r+1)$ and $\alpha^{(0)} = 1$, which reduces to the hypergeometric distribution (see, e.g., Patil et al. [16, page 47]) for $k = 1$. We say that the RV X has the hypergeometric distribution of order k, type I, with parameters n,c, and c_1, and we denote it by $H_{k,I}(n;c,c_1)$.

Case 2. In $P_{k,I}(n;s;c,c_1)$, let $s = 1$. Then, for $x = 0,1,\ldots,kn$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n}{x_1,\ldots,x_k,n-\Sigma_j x_j} \frac{c_0^{[k(n-\Sigma_j x_j)+\Sigma_j(j-1)x_j]} c_1^{[\Sigma_j x_j]}}{c^{[k(n-\Sigma_j x_j)+x]}}, \tag{3.3}$$

where $\alpha^{[r]} = \alpha(\alpha+1)\cdots(\alpha+r-1)$ and $\alpha^{[0]} = 1$, which reduces to the negative hypergeometric distribution (see, e.g., Patil et al. [16, page 49]), for $k = 1$. We say that the RV X has the negative hypergeometric distribution of order k, type I, with parameters n, c, and c_1, and we denote it by $\mathrm{NH}_{k,I}(n;c,c_1)$.

Case 3. In $P_{k,I}(n;s;c,c_1)$, let $s = c_0 = c_1$. Then, for $x = 0,1,\dots,kn$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n}{x_1,\dots,x_k,n-\Sigma_j x_j} \left\{ \binom{k(n-\Sigma_j x_j)+x+1}{\Sigma_j x_j+1} (\Sigma_j x_j+1) \right\}^{-1}, \tag{3.4}$$

which reduces to the uniform distribution (see, e.g., Patil et al. [16, page 82]), for $k = 1$. We say that the RV X has the uniform distribution of order k, type I.

Case 4. In $P_{k,I}(n;s;c,c_1)$, let $s = 0$, and interchange p and q. Then, for $x = 0,1,\dots,kn$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n}{x_1,\dots,x_k,n-\Sigma_j x_j} q^{k(n-\Sigma_j x_j)+x} \left(\frac{p}{q}\right)^{\Sigma_j x_j}, \tag{3.5}$$

which reduces to the usual binomial distribution with parameters n and p (see, e.g., Patil et al. [16, page 14]), for $k = 1$. We say that the RV X has the binomial distribution of order k, type I, with parameters n and p and we denote it by $B^*{}_{k,I}(n;p)$.

Class 3.2. In $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$, let $\mu_i = 0$. Then, for $x = 0,1,\dots$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n+\Sigma_j x_j-1}{x_1,\dots,x_k,n-1} \frac{c_0^{(kn+\Sigma_j(j-1)x_j,s)} c_1^{(\Sigma_j x_j,s)}}{c^{(kn+x,s)}}, \tag{3.6}$$

which is the inverse Polya distribution of order k, type I, of Panaretos and Xekalaki [15] with parameters $c-c_1$, c_1, n, and s. We denote it by $\mathrm{IP}_{k,I}(n;s;c,c_1)$.

Case 1. In $\mathrm{IP}_{k,I}(n;s;c,c_1)$, let $s = -1$. Then, for $x = 0,1,\dots$,

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \binom{n+\Sigma_j x_j-1}{x_1,\dots,x_k,n-1} \frac{c_0^{(kn+\Sigma_j(j-1)x_j)} c_1^{(\Sigma_j x_j)}}{c^{(kn+x)}}, \tag{3.7}$$

which reduces to the inverse hypergeometric distribution (see, e.g., Patil et al. [16, page 49]), for $k = 1$. We say that the RV X has the inverse hypergeometric distribution of order k, type I, with parameters n, c, and c_1, and we denote it by $\mathrm{IH}_{k,I}(n;c,c_1)$.

Case 2. The $\mathrm{IP}_{k,I}(n;s;c,c_1)$, for $s = 1$, reduces to the generalized Waring distribution of order k of Panaretos and Xekalaki [15], with parameters $c-c_1$, c_1, and n.

Case 3. The $\mathrm{IP}_{k,I}(n;s;c,c_1)$, for $s = 0$, reduces to the negative binomial distribution of order k, type I, of Philippou et al. [21] with $p = (c-c_1)/c$.

Class 3.3. The $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$, for $s=0$, reduses to the generalized negative binomial distribution of order k, type I, with $p=(c-c_1)/c$.

Case 1. In $\mathrm{GNB}_{k,I}(n;\mu;p)$, let $\mu=1$ and $q/p=P$ so that $q=P/Q$ and $p=1/Q$, where $Q=1+P$, and replace n by nk and x_j by x_j-n. Then, for $x=k(k+1)n/2$, $k(k+1)n/2+1,\dots,$

$$P(X=x) = \sum_{x_j\ge n,\,\Sigma_j jx_j=x} \frac{nk}{\Sigma_j x_j}\binom{2\Sigma_j x_j-nk-1}{x_1-n,\dots,x_k-n,\Sigma_j x_j-1}\frac{P^{\Sigma_j x_j-nk}}{Q^{x+k\Sigma_j x_j-[nk(k+1)]/2}}, \tag{3.8}$$

which reduces to the Haight distribution (see Haight [3]), for $k=1$. We say that the RV X has the Haight distribution of order k, type I, with parameters n and P.

Case 2. In $\mathrm{GNB}_{k,I}(n;\mu;p)$, let $q/p=P$ so that $q=P/Q$ and $p=1/Q$, where $Q=1+P$, and replace n by $k\mu$ and x_j by x_j-1. Then, for $x_i=k(k+1)/2$, $k(k+1)/2+1,\dots,$

$$P(X=x) = \sum_{\substack{x_j\ge 1\\ \Sigma_j jx_j=x}} \frac{k\mu}{(1+\mu)\Sigma_j x_j-k}\binom{(1+\mu)\Sigma_j x_j-k}{x_1-1,\dots,x_k-1,\mu\Sigma_j x_j}\frac{P^{\Sigma_j x_j-k}}{Q^{x+k[\mu\Sigma_j x_j-(k+1)/2]}}, \tag{3.9}$$

which reduces to the Takács distribution (see Takács [24]), for $k=1$. We say that the RV X has the Takács distribution of order k, type I, with parameters μ and P.

Case 3. In $\mathrm{GNB}_{k,I}(n;\mu;p)$, let $\mu=d-1$, and replace n by nkd and x_j by x_j-n. Then, for $x=k(k+1)n/2, k(k+1)n/2+1,\dots,$

$$\begin{aligned}P(X=x) = \sum_{x_j\ge n,\,\Sigma_j jx_j=x} &\frac{nk}{\Sigma_j x_j}\binom{d\Sigma_j x_j}{x_1-n,\dots,x_k-n,(d-1)\Sigma_j x_j+nk}\\ &\times p^{k[n(k+1)/2+(d-1)\Sigma_j x_j]+\Sigma_j(j-1)x_j}q^{\Sigma_j x_j-nk},\end{aligned} \tag{3.10}$$

which reduces to the binomial-delta distribution (see Johnson et al. [10, page 143]), for $k=1$. We say that the RV X has the binomial delta distribution of order k, type I, with parameters n, d, and p.

Case 4. In $\mathrm{GNB}_{k,I}(n;\mu;p)$, let $q/p=P$ so that $q=P/Q$ and $p=1/Q$, where $Q=1+P$, and replace n by $nk\mu$ and x_j by x_j-n. Then, for $x=k(k+1)n/2$, $k(k+1)n/2+1,\dots,$

$$P(X=x) = \sum_{\substack{x_j\ge n\\ \Sigma_j jx_j=x}} \frac{nk}{\Sigma_j x_j}\binom{(1+\mu)\Sigma_j x_j-nk-1}{x_1-n,\dots,x_k-n,\mu\Sigma_j x_j-1}\frac{P^{\Sigma_j x_j-nk}}{Q^{x+k[\mu\Sigma_j x_j-n(k+1)/2]}}, \tag{3.11}$$

which reduces to the negative binomial-delta distribution (see Johnson et al. [10, page 144]), for $k = 1$. We say that the RV X has the negative binomial-delta distribution of order k, type I, with parameters n, μ, and P.

Case 5. In $\mathrm{GNB}_{k,I}(n;\mu;p)$, let $q/p = P$ so that $q = P/Q$ and $p = 1/Q$, where $Q = 1+P$. Then, for $x = 0,1,\ldots,$

$$P(X=x) = \sum_{\Sigma_j jx_j=x} \frac{n}{n+(1+\mu)\Sigma_j x_j} \binom{n+(1+\mu)\Sigma_j x_j}{x_1,\ldots,x_k,n+\mu\Sigma_j x_j} Q^{-k(n+\mu\Sigma_j x_j)-x} P^{\Sigma_j x_j}, \tag{3.12}$$

which reduces to the negative binomial-negative binomial distribution (see Johnson et al. [10, page 145]), for $k = 1$. We say that the RV X has the negative binomial-negative binomial distribution of order k, type I, with parameters n, μ, and P.

4. Limiting cases of $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$—applications. In this section, we establish five propositions, which relate asymptotically $\mathrm{GNB}_{k,I}(\cdot)$ to the Poisson distribution $(P_{k,I}(\lambda))$ of order k, type I, of Philippou et al. [21] and $\mathrm{GP}_{k,I}(\cdot)$ to $\mathrm{GNB}_{k,I}(\cdot)$, $\mathrm{BT}_{k,I}(\alpha;\beta;d;r)$, $P_{k,I}(\lambda)$ and to the negative binomial distribution $(\mathrm{NB}_{k,I}(n;p))$ of order k, type I, of Philippou et al. [21]. An application to observed data is also presented.

Proposition 4.1. *Let $X_{n,q}$ (n is a positive integer, $0 < q < 1$) and X be two RVs distributed as $\mathrm{GNB}_{k,I}(n;\mu;p)$ and $P_{k,I}(\lambda)$, respectively, and assume that $nq \to \lambda$ $(\lambda > 0)$ as $n \to \infty$ and $q \to 0$. Then, for $x = 0,1,\ldots,$*

$$P(X_{n,q} = x) \longrightarrow P(X = x). \tag{4.1}$$

Proof. For $x = 0,1,\ldots,$ we have

$$\begin{aligned} &P(X_{n,q} = x) \\ &\quad= \sum_{\Sigma_j jx_j=x} \frac{(n+(1+\mu)\Sigma_j x_j-1)!}{n^{\Sigma_j x_j-1}(n+\mu\Sigma_j x_j)!}\left(1-\frac{nq}{n}\right)^{nk} p^{k\mu\Sigma_j x_j+\Sigma_j(j-1)x_j}\frac{(nq)^{\Sigma_j x_j}}{\Pi_j x_j!} \\ &\quad\longrightarrow \sum_{\Sigma_j jx_j=x} e^{-k\lambda}\frac{\lambda^{\Sigma_j x_j}}{\Pi_j x_j!}, \quad \text{as } n \longrightarrow \infty,\ q \longrightarrow 0, \end{aligned} \tag{4.2}$$

which establishes the proposition. □

Proposition 4.2. *Let X_{c_0,c_1} (c_0 and c_1, positive integers) and X be two RVs distributed as $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$ and $\mathrm{GNB}_{k,I}(n;\mu;p)$, respectively, and assume that $c_0/(c_0+c_1) \to p$ $(0 < p < 1)$ as $c_0 \to \infty$ and $c_1 \to \infty$. Then,*

$$P(X_{c_0,c_1} = x) \longrightarrow P(X = x), \quad x = 0,1,\ldots. \tag{4.3}$$

Proof. We observe that

$$\frac{c_0^{(k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j,s)}c_1^{(\Sigma_j x_j,s)}}{(c_0+c_1)^{(k(n+\mu\Sigma_j x_j)+x,s)}} \longrightarrow p^{k(n+\mu\Sigma_j x_j)+x}\left(\frac{1-p}{p}\right)^{\Sigma_j x_j}, \quad \text{as } c_0 \longrightarrow \infty,\ c_1 \longrightarrow \infty, \tag{4.4}$$

from which the proof follows. □

Proposition 4.3. *Let X_{n,μ,c,c_1} ($\mu \geq -1$ is an integer and n, c, and c_1 are positive integers) and X be two RVs distributed as in (2.14) and in the Borel-Tanner distribution of order k, type I, respectively, and assume that $c_1/c \to 0$, $nc_1/c \to ra\beta$ ($a > 0$, $\beta > 0$, and $r = 1,2,\ldots$), and $\mu c_1/c \to ad\beta$ ($d = 1,2,\ldots$) as $c \to \infty$, $c_1 \to \infty$, $n \to \infty$, and $\mu \to \infty$. Then,*

$$P(X_{n,\mu,c,c_1} = x) \longrightarrow P(X = x), \quad x = 0,1,\ldots. \tag{4.5}$$

Proof. We observe that

$$\frac{n(n+(1+\mu)\Sigma_j x_j - 1)!}{(n+\mu\Sigma_j x_j)!}\frac{c_0^{(k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j,s)}c_1^{(\Sigma_j x_j,s)}}{c^{(k(n+\mu\Sigma_j x_j)+x,s)}} \longrightarrow r(a\beta)^{\Sigma_j x_j}(r+d\Sigma_j x_j)^{\Sigma_j x_j - 1}e^{-(k\alpha\beta)(r+d\Sigma_j x_j)} \quad \text{as } c \longrightarrow \infty,\ c_1 \longrightarrow \infty,\ n \longrightarrow \infty,\ \mu \longrightarrow \infty, \tag{4.6}$$

from which the proof follows. □

Proposition 4.4. *Let X_{n,c,c_1} (n, c, and c_1, positive integer) and X be two RVs distributed as $\mathrm{GP}_{k,I}(n;\mu;s;c,c_1)$ and $P_{k,I}(\lambda)$, respectively, and assume that $c_1/c \to 0$, $n/c \to 0$, and $nc_1/c \to \lambda$ ($\lambda > 0$) as $c \to \infty$, $c_1 \to \infty$, and $n \to \infty$. Then,*

$$P(X_{n,c,c_1} = x) \longrightarrow P(X = x), \quad x = 0,1,\ldots. \tag{4.7}$$

Proof. We observe that

$$\frac{n(n+(1+\mu)\Sigma_j x_j - 1)!}{(n+\mu\Sigma_j x_j)!}\frac{c_0^{(k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j,s)}c_1^{(\Sigma_j x_j,s)}}{c^{(k(n+\mu\Sigma_j x_j)+x,s)}} \longrightarrow e^{-k\lambda}\lambda^{\Sigma_j x_j} \quad \text{as } c \longrightarrow \infty,\ c_1 \longrightarrow \infty,\ n \longrightarrow \infty, \tag{4.8}$$

which establishes the proposition. □

Proposition 4.5. *Let X_n (n is a positive integer) and X be two RVs distributed as in (2.14) and $\mathrm{NB}_{k,II}(\beta, c/(c+k))$, respectively, and assume that $n^{-1}\alpha = c_n \to c$ ($0 < c < \infty$) as $n \to \infty$. Then,*

$$P(X_n = x) \longrightarrow P(X = x), \quad x = 0,1,\ldots. \tag{4.9}$$

TABLE 4.1. Distribution of the counts of bacteria in leucocytes.

No. of bacteria per leucocyte	Observed frequency	Expected frequencies Poisson	GP	$GP_{2,I}$	McKendrick
0	269	245.62	259.21	264.74	268
1	4	49.12	28.94	15.55	7
2	26	4.91	7.57	16.46	23
3	0	0.33	2.56	1.88	0.6
4	1	0.02	1.72	1.08	1.1
Total	300	300	300	299.71	299.7
χ^2-value		133.52	69.25	15.47	1.97
d.f.	1	1	1	1	1

PROOF. We observe that

$$\begin{aligned}&\frac{n(n+(1+\mu)\Sigma_j x_j-1)!}{(n+\mu\Sigma_j x_j)!}\frac{B(\alpha+k(n+\mu\Sigma_j x_j)+\Sigma_j(j-1)x_j,\beta+\Sigma_j x_j)}{B(a,\beta)}\\&\qquad\longrightarrow\frac{(\Sigma_j x_j+\beta-1)!}{(\beta-1)!}\left(\frac{c}{c+k}\right)^{\beta}\left(\frac{1}{c+k}\right)^{\Sigma_j x_j},\quad\text{as } n\longrightarrow\infty,\end{aligned}\tag{4.10}$$

from which the proof follows. □

For $k=1$, Propositions 4.1, 4.2, 4.3, 4.4, and 4.5 reduce to new results on univariate generalized distributions of order k.

Finally, to illustrate the fit of the generalized distributions of order k, type I, to observed data, we consider the following application of the generalized Poisson (or Borel-Tanner) distribution of order k, type I.

McKendrick [12] considered the distribution of the sum of two correlated Poisson variables, which he applied to the counts of bacteria in leucocytes. Table 4.1 shows the expected frequencies of the counts of bacteria in leucocytes estimated by the generalized Poisson distribution of order 2, type I ($GP_{2,I}(\cdot)$), using the following moment estimators of the parameters θ and λ:

$$\hat{\theta}=\frac{\overline{x}}{3A},\qquad\hat{\lambda}=\frac{A-1}{2A},\quad A=\sqrt{\frac{2[s^2-(5/3)\overline{x}]}{3\overline{x}}+1},\tag{4.11}$$

with $\overline{x}=0{,}2$ and $s^2=0{,}374582$. An ordinary Poisson ($P(\cdot)$) and a generalized Poisson distribution ($GP(\cdot)$) (see, e.g., Jain [5]) have been fitted for comparison.

We observe that the generalized Poisson distribution of order 2, type I, gives a good fit to the data.

REFERENCES

[1] S. Aki, H. Kuboki, and K. Hirano, *On discrete distributions of order k*, Ann. Inst. Statist. Math. **36** (1984), no. 3, 431–440.

[2] D. L. Antzoulakos and A. N. Philippou, *On multivariate distributions of various orders obtained by waiting for the rth success run of length k in trials*

with multiple outcomes, Advances in Combinatorial Methods and Applications to Probability and Statistics (N. Balakrishnan, ed.), Stat. Ind. Technol., Birkhäuser Boston, Massachusetts, 1997, pp. 411–426.

[3] F. A. Haight, *A distribution analogous to the Borel-Tanner*, Biometrika **48** (1961), 167–173.

[4] F. A. Haight and M. A. Breuer, *The Borel-Tanner distribution*, Biometrika **47** (1960), 143–150.

[5] G. C. Jain, *On power series distributions associated with Lagrange expansion*, Biometrische Z. **17** (1975), 85–97.

[6] G. C. Jain and P. C. Consul, *A generalized negative binomial distribution*, SIAM J. Appl. Math. **21** (1971), 501–513.

[7] G. C. Jain and R. P. Gupta, *A logarithmic series type distribution*, Trabajos Estadíst. **24** (1973), no. 1-2, 99–105.

[8] G. C. Jain and N. Singh, *On bivariate power series distributions associated with Lagrange expansion*, J. Amer. Statist. Assoc. **70** (1975), no. 352, 951–954.

[9] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Discrete Multivariate Distributions*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, 1997.

[10] N. L. Johnson, S. Kotz, and A. W. Kemp, *Univariate Discrete Distributions*, 2nd ed., Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, 1992.

[11] K. D. Ling, *On discrete distributions of order k defined on Pólya-Eggenberger urn model*, Soochow J. Math. **14** (1988), no. 2, 199–210.

[12] A. G. McKendrick, *Applications of mathematics to medical problems*, Proc. Edinburgh Math. Soc. **44** (1926), 98–130.

[13] S. G. Mohanty, *On a generalized two coin tossing problem*, Biometrische Z. **8** (1966), 266–272.

[14] ———, *Lattice Path Counting and Applications*, Academic Press, New York, 1979.

[15] J. Panaretos and E. Xekalaki, *On some distributions arising from certain generalized sampling schemes*, Comm. Statist. A—Theory Methods **15** (1986), no. 3, 873–891.

[16] G. P. Patil, M. T. Boswell, S. W. Joshi, and M. V. Ratnaparkhi, *Dictionary and Classified Bibliography of Statistical Distributions in Scientific Work, Vol. 1: Discrete Models*, International Co-operative Publishing House, Maryland, 1984.

[17] A. N. Philippou, *Poisson and compound Poisson distributions of order k and some of their properties*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **130** (1983), 175–180 (Russian).

[18] ———, *The negative binomial distribution of order k and some of its properties*, Biometrical J. **26** (1984), no. 7, 789–794.

[19] A. N. Philippou, D. L. Antzoulakos, and G. A. Tripsiannis, *Multivariate distributions of order k*, Statist. Probab. Lett. **7** (1988), no. 3, 207–216.

[20] ———, *Multivariate distributions of order k. II*, Statist. Probab. Lett. **10** (1990), no. 1, 29–35.

[21] A. N. Philippou, C. Georghiou, and G. N. Philippou, *A generalized geometric distribution and some of its properties*, Statist. Probab. Lett. **1** (1983), no. 4, 171–175.

[22] A. N. Philippou and A. A. Muwafi, *Waiting for the Kth consecutive success and the Fibonacci sequence of order K*, Fibonacci Quart. **20** (1982), no. 1, 28–32.

[23] K. Sen and A. Mishra, *A generalized Pólya-Eggenberger model generating various discrete probability distributions*, Sankhyā Ser. A **58** (1996), no. 2, 243–251.

[24] L. Takács, *A generalization of the ballot problem and its application in the theory of queues*, J. Amer. Statist. Assoc. **57** (1962), 327–337.

[25] G. A. Tripsiannis and A. N. Philippou, *A new multivariate inverse Pólya distribution of order k*, Comm. Statist. Theory Methods **26** (1997), no. 1, 149–158.

[26] G. A. Tripsiannis, A. N. Philippou, and A. A. Papathanasiou, *Multivariate generalized distributions of order k*, Medical statistics technical report 41, Democritus University of Thrace, Greece, 2001.

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