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COMMUTING IDEMPOTENTS OF AN H^*-ALGEBRA

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Commutative H^*-algebra is characterized in terms of idempotents. Here we offer three characterizations.

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1. Introduction. In the past, the author used commuting idempotents to characterize continuous functions defined on a certain space [3, 4]. For example, it was shown in [3] that a certain Banach algebra is isometrically isomorphic to the space $C(S)$ of all continuous complex-valued functions on a totally disconnected compact space S. In the sequel, we use idempotents to characterize commutative H^*-algebras. An interesting consequence of this results (Theorems 3.1, 3.2, and 3.3 below) is somewhat unusual forms of characterizations of Hilbert spaces.

2. Preliminaries. A proper H^*-algebra is a Hilbert algebra (a Banach algebra with a Hilbert space norm) which has an involution $x \to x^*$ such that $(xy,z) = (y,x^*z) = (x,zy^*)$ for all $x,y,z \in A$. An idempotent is a *nonzero* member e of A such that $e^2 = e$.

Definition 2.1. An idempotent e in an algebra a is said to be *primary* if $ef = e$ for any idempotent $f \in A$ such that $ef \neq 0$.

Note that the product of any two distinct primary idempotents is zero.

3. Main results. Let A be a complex Banach algebra. Let I be the set of idempotents in A, let P be the set of all primary idempotents, and let A_o be the set of all (complex) finite linear combinations of primary idempotents $A_o = \{\sum_{i=1}^n \lambda_i e_i : e_i \in P,\ i = 1,\dots,n$ and $\lambda_1,\dots,\lambda_n$ are complex numbers$\}$.

Theorem 3.1. *Let A be a complex Banach algebra such that all members of P commute and A_o is dense in A. Assume further that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ for all $x,y \in A_o$ such that $xy = 0$. Then, A is a proper commutative H^*-algebra.*

Proof. First, note that A is commutative since members of A_o commute. Condition "$\|x+y\|^2 = \|x\|^2+\|y\|^2$ if $xy = 0$" implies that $\|x\|^2 = \sum_{i=1}^n |\lambda_i|^2 \|e_i\|^2$

for any member $x=\sum_{i=1}^{n}\lambda_i e_i$ of A_o (it can be readily established using induction on n). This fact can be used to show that there is an inner product $(\,,)$ and an involution $x\to x^*$ such that $(x,x)=\|x\|^2=\|x^*\|^2$ and $(xy,z)=(y,x^*z)$ for all $x,y,z\in A_o$. In fact, all we have to do is to set $(x,y)=(\sum_i\lambda_i e_i,\sum_j\mu_j e_j)=\sum_{i,j}\lambda_i\bar{\mu}_j\|e_ie_j\|^2$ and $x^*=\sum\bar{\lambda}_i e_i$ for members $x=\sum\lambda_i e_i$ and $y=\sum\mu_j e_j$ of A_o (note that $e_ie_j=0$ if $i\neq j$).

We leave it to the reader to verify that A is isometrically isomorphic to the space $L^2(P,\mu)$ of all complex-valued functions $x(\ )$ on P such that $\sum_{e\in P}|x(e)|^2\|e\|^2<\infty$, with pointwise multiplication of members of $L^2(P,\mu)$ ($xy(e)=x(e)y(e)$ for all $x,y\in L^2(P,\mu)$). (Measure μ on P is the set function that associates with each member e of P the positive number $\|e\|$.) (One can interpret the expression "$\sum_{e\in P}|x(e)|^2\|e\|^2<\infty$" to mean "there exists a countable subset $P_x=\{e_1,e_2,\dots,e_n,\dots\}$ of P such that $x(e)=0$ if $x\notin P_x$ and $\sum_{i=1}^{\infty}|x(e_i)|^2\|e_i\|^2$ converges"). Obviously, $L^2(P,\mu)$ is a proper commutative H^*-algebra under the pointwise multiplication. □

Theorem 3.2. *Let A be a complex Banach algebra such that the members of I commute and that the set A_1 of finite linear combinations of I is dense in A ($A_1=\{\sum_{i=1}^{n}\lambda_i e_i, e_1\cdots e_n\in I$ and $\lambda_1,\dots,\lambda_n$ are complex numbers$\}$). Assume further that*

(i) *for each $e\in I$, there exists $f\in P$ such that $ef\neq 0$,*
(ii) *if $x,y\in A$ and $xy=0$, then $\|x+y\|^2+\|x\|^2+\|y\|^2$.*

Then, A is a commutative proper H^*-algebra.

Proof. First, note that A is commutative. Also, it follows from assumption (i) that, for each $e\in I$, there are primary idempotents $f_1\cdots f_n$ (a finite number) such that $e=f_1+f_2+\cdots+f_n$ and $f_if_j=0$ if $i\neq j$. To see this, let $e\in I$ and $f\in P$ be such that $ef\neq 0$. Then, $ef=f$ and $g=e-f$ is also an idempotent such that $fg=0$. This means that $\|e\|^2=\|f+g\|^2=\|f\|^2+\|g\|^2$, and so $\|g\|^2=\|e\|^2-\|f\|^2<\|e\|^2-1$ since $\|f\|>1$ ($\|f\|=\|f^2\|\leq\|f\|^2$ and $\|f\|\neq 0$). It follows that if n is any natural number such that $\|e\|^2<n$, then $\|g\|^2<n-1$. Now, we can use induction on n to see that each idempotent can be represented as a finite sum of primary idempotents. (Note that if g is a finite sum of mutually annihilating members of P, then so is $e=f+g$ since $f\in P$ and $fg=0$.)

But this means that A_o (the space of finite linear combinations of the members of P) is dense in A. Theorem 3.1 now implies that A is an H^*-algebra. □

Theorem 3.3. *Let A be a Banach algebra such that all members of I commute, that the space of all finite linear combinations of members of I is dense in A, and that $\|x+y\|^2=\|x\|^2+\|y\|^2$ if $xy=0$ for any $x,y\in A$.*

Assume further that, for each closed ideal J in A, there is an ideal J^1 such that $J\cap J^1=(0)$ and $J+J^1=A$, that is, for any $a\in A$, there are $a_1\in J$ and $a_2\in J^1$ such that $a=a_1+a_2$. Then, A is a commutative H^*-algebra.

Proof. We only need to show that, for each $e \in I$, there exists $f \in P$ such that $ef \neq 0$.

Let $e \in I$ and let N be the annihilator of e, $N = \{x \in A : xe = 0\}$. Then, $ex - x \in N$ for each $x \in A$, that is, e is a relative identity modulo N and N is a regular ideal [2, Section 20] (see also [2, Subsection 22D and Subsection 22E]). Let M be the maximal regular ideal such that $M \supset N$ [2, Subsection 20B] and let M^1 be an ideal such that $M + M^1 = A$ and $M \cap M^1 = \{0\}$. Write $e = f + u$ with $f \in M^1$, $u \in M$. Then, f is also a relative identity modulo M ($fx - x = (e-u)x - x = ex - x - ux \in M$). Also, f is an idempotent since $ff - f \in M^1 \cap M = 0$ and $f \neq 0$ (otherwise $e \in M$).

Now, we show that $f \in P$, that is, f is primary. Let $h \in I$ be such that $fh \neq 0$. If $fh \neq f$, then $f - fh \neq 0$, and we have a decomposition $f = f_1 + f_2$ of f as a sum of nonzero idempotents $f_1 = fh$ and $f_2 = f - fh$ such that $f_1 f_2 = 0$. Let $M_1 = f_1 A + M = \{f_1 a + m : a \in A,\ m \in M\}$. It is a regular ideal including M strictly larger than M ($f_1 \in M_1$, $f_1 \notin M$) (also, $ex - x \in M \subset M_1$). This contradicts the maximality of M. Thus, $fh = f$ for each $h \in I$ with $fh \neq 0$. Theorem 3.2 now implies that A is a commutative H^*-algebra. □

4. Some properties of H^*-algebra.

Now, we show that every proper commutative H^*-algebra satisfies assumptions of Theorems 3.1, 3.2, and 3.3. First, note that, in any commutative Banach algebra, an idempotent e is primary if and only if it *cannot* be written as a sum $e = e_1 + e_2$ of two mutually annihilating, $e_1 e_2 = 0$, nonzero idempotents e_1 and e_2.

Indeed, let e be primary and assume that $e = e_1 + e_2$ for some nonzero idempotents e_1 and e_2 such that $e_1 e_2 = 0$. Then, $e_1 e = e_1$ and $e_1 e = e$ since e is primary. This implies $e_2 = 0$, which is a contradiction.

Conversely, assume that $e \neq ef$ for some idempotent f such that $e_1 = ef \neq 0$. Then, $e_2 = f - ef$ is also nonzero idempotent such that $e_1 e_2 = 0$ ($e_2^2 = (e-f)^2 = e - ef - ef + ef = e_2$ and $e_1 e_2 = ef(e-f) = ef - ef = 0$). This means that if e is not primary, then it has a decomposition $e = e_1 + e_2$ into mutually annihilating nonzero idempotents.

In the case of a proper commutative H^*-algebra, the fact that e is primary would also imply that e is selfadjoint: $e^* e = e$ since e^* is also idempotent ($(e^*)^2 = (ee)^* = e^*$). This means that, in this case, e is primary if and only if e is selfadjoint and primitive in the sense of Ambrose [1, Definition 3.3, page 376].

It follows that [1, Corollary 4.1, page 382] implies that, in each proper commutative H^*-algebra, the set A_o (of finite linear combinations of primary idempotents) is dense in A.

The remark in [1, page 369] of (orthogonal complement of any ideal is an ideal of the same kind (in the paragraph above Definition 1.4)) implies that every commutative H^*-algebra satisfies the condition of Theorem 3.3 about the existence of the ideal J^1. But it was shown in the proof of Theorem 3.3 that this fact implies that, for each idempotent e, there is a primary idempotent f such that $ef \neq 0$, which is one of the assumptions of Theorem 3.2.

It remains to show that $xy = 0$ for any x, y in a proper commutative H^*-*algebra* implies that x is orthogonal to y (which implies that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$). We will use the terminology of [1].

So, let A be a commutative proper H^*-algebra. Let P be the set of all primary idempotent. Then, P is a maximal family of doubly orthogonal primitive sa-idempotents (see [1, Definition 3.1]) (it was remarked above that the product ef of any two distinct primary idempotents e, f is zero, $ef = 0$) and so it follows from [1] that $A = \sum_\alpha e_\alpha A$ and each $e_\alpha A$ is isomorphic to the complex field. This means that each $x \in A$ has the form $x = \sum_{e\in P} x(e)e$ for some complex number $x(e)$ for each $e \in P$ and $\sum_{e\in P} |x(e)|^2 \|e\|^2 < \infty$. It is easy to see that the products xy and (x,y) are expressible in terms of this representation by the formulae $xy = \sum_{e\in P} x(e)y(e)e$ and $(x,y) = \sum_{e\in P} x(e)\overline{y}(e)\|e\|^2$. From this, it is easy to show that $xy = 0$ implies $(x,y) = 0$ (note that if the product of any two complex numbers is zero, then either of the numbers (or both) is zero).

5. Some consequences. One of consequences of Theorems 3.1, 3.2, and 3.3 is that each of the above theorems can be used to characterize Hilbert spaces. The reason for that is the fact that each Hilbert space has also a structure of a proper H^*-algebra. To see this, all we have to do is to take any orthonormal base $\{e_\alpha\}\alpha \in \Gamma$ of a Hilbert space H and define multiplication on it by setting $xy = \sum_{\alpha\in\Gamma} x(e_\alpha)y(e_\alpha)e_\alpha$, where $x = \sum_{\alpha\in\Gamma} x(e_\alpha)e_\alpha$ and $y = \sum_{\alpha\in\Gamma} y(e_\alpha)e_\alpha$ are representations of x and y in terms of the orthonormal base $\{e_\alpha\}_{\alpha\in\Gamma}(x(\alpha) = (x, e_\alpha))$ [5]. It is easy to see that H becomes a proper commutative H^*-algebra with respect to the involution $x \to x^*$ with $x^* = \sum_{\alpha\in\Gamma} \overline{x}(e_\alpha)e_\alpha$. Thus, we have the following characterization of a Hilbert space (it is somewhat awkward, yet it is a characterization). It is stated as a corollary of the above theorems.

Corollary 5.1. *Let B be a Banach space. Assume that it is possible to define a multiplication on B with respect to which it is a Banach algebra that has properties stated in either one of the theorems above (Theorems 3.1, 3.2, and 3.3). Then, B is a Hilbert space.*

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