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THE CONVERGENCE ESTIMATES FOR GALERKIN-WAVELET SOLUTION OF PERIODIC PSEUDODIFFERENTIAL INITIAL VALUE PROBLEMS

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Using the discrete Fourier transform and Galerkin-Petrov scheme, we get some results on the solutions and the convergence estimates for periodic pseudodifferential initial value problems.

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1. Introduction. In recent years, wavelets have been developing intensively and have become a powerful tool to study mathematics and technology, for example, the theory of the singular integral, singular integro-differential equations, the areas such as sound analysis, image compression, and so on (see [9, 10] and references therein). In this paper, we use a scaling function and a multilevel approach to estimate the error of the problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= a\cdot Au(x,t), \quad x\in\mathcal{T}^n,\ t>0,\ a\in\mathbb{R}, \\ u(x,0) &= [u_0](x), \quad x\in\mathcal{T}^n, \end{aligned} \tag{1.1}$$

where A is a pseudodifferential operator (see [1, 2, 3, 4, 6, 8, 9, 12]) with a symbol $\sigma\in C^\infty(\mathbb{R}^n)$, σ is positively homogeneous of degree $r>0$ such that

$$\left|D^\alpha\sigma(\xi)\right| \le C_\alpha\left(1+|\xi|\right)^{r-|\alpha|}, \quad \text{for all multi-index } \alpha\in\mathbb{N}^n, \tag{1.2}$$

$\mathcal{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, and $[u_0](x)=\sum_{k\in\mathbb{Z}^n}u_0(x+k)$ is a periodic operator.

We discuss only problem (1.1) with the following condition:

$$a\sigma(\xi)\le 0, \quad \forall\xi\in\mathbb{Z}^n. \tag{1.3}$$

2. Preliminaries and notations. The continuous Fourier transform of the function $f\in L_2(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^n}e^{-2\pi ix\xi}f(x)dx, \quad \xi\in\mathbb{R}^n \tag{2.1}$$

with the inverse Fourier formula

$$f(x)=\int_{\mathbb{R}^n} e^{2\pi i x\xi}\hat{f}(\xi)d\xi,\quad \xi\in\mathbb{R}^n \tag{2.2}$$

(see [4, 8, 11]).

The discrete Fourier transform of the function $f\in L_2(\mathcal{J}^n)$ is

$$\mathcal{F}(f)(\xi)=\tilde{f}(\xi):=\int_{[0,1]^n} e^{-2\pi i x\xi}f(x)dx,\quad \xi\in\mathbb{Z}^n, \tag{2.3}$$

and the inverse Fourier transform is

$$f(x):=\sum_{\xi\in\mathbb{Z}^n}\tilde{f}(\xi)e^{2\pi i x\xi} \tag{2.4}$$

(see [6]).

Some simple properties of the discrete Fourier transform are

$$(f,g)_0=\sum_{\xi\in\mathbb{Z}^n}\tilde{f}(\xi)\overline{\tilde{g}(\xi)}, \tag{2.5}$$

where $(\cdot,\cdot)_0$ is the $L_2(\mathcal{J}^n)$-inner product,

$$\|f\|_0^2=\sum_{\xi\in\mathbb{Z}^n}|\tilde{f}(\xi)|^2=||\tilde{f}||_{l_2}^2, \tag{2.6}$$

where $\|\cdot\|_0$ is $L_2(\mathcal{J}^n)$-norm and $\|\cdot\|_{l_2}$ is l_2-norm.

Let $s\in\mathbb{R}$. Denote

$$H^s(\mathcal{J}^n)=\{u\in D'(\mathcal{J}^n)\mid \langle D\rangle^s u\in L_2(\mathcal{J}^n)\}, \tag{2.7}$$

where

$$\langle\xi\rangle=\begin{cases}1 & \text{if } \xi=0,\\ |\xi| & \text{if } \xi\neq 0,\end{cases} \tag{2.8}$$

then $H^s(\mathcal{J}^n)$ is the Sobolev space endowed with the norm

$$\|u\|_s^2=\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{2s}|\tilde{u}(\xi)|^2 \tag{2.9}$$

and the inner product

$$\langle u,v\rangle_s=\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{2s}\tilde{u}(\xi)\overline{\tilde{v}(\xi)}. \tag{2.10}$$

Here, we also define the discrete Sobolev space $H_d^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, of the functions $f\in H^s(\mathbb{R}^n)$ such that the following norm is finite:

$$\|f\|_{s,d}^2=\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{2s}|\hat{f}(\xi)|^2. \tag{2.11}$$

Denote

$$\mathcal{L}_2 = \left\{ f \in L_2(\mathbb{R}^n) : \sum_{\xi \in \mathbb{Z}^n} |f(\cdot - \xi)| \in L_2([0,1]^n) \right\}. \tag{2.12}$$

It is clear that any function $f \in L_2(\mathbb{R}^n)$, which has compact support, or any function, for which $\int_{k+[0,1]^n} |f(x)|^2 dx$ decays exponentially as $|k|$ tends to infinity, belongs to $\mathcal{L}_2$. The periodic operator $[u]$ is totally defined if $u \in \mathcal{L}_2$. Here, we assume that $u_0 \in \mathcal{L}_2$.

Remark 2.1. (1) It follows from (2.1) and (2.3) that if $u \in \mathcal{L}_2$, then $\mathcal{F}([u])(\xi) = \hat{u}(\xi)$, $\xi \in \mathbb{Z}^n$.

(2) It is clear that if $t \le s$, $s,t \in \mathbb{R}$, then $H^t(\mathcal{J}^n) \subset H^s(\mathcal{J}^n)$.

Using the variable separate method and the discrete Fourier transform, the solution of problem (1.1) can be represented as

$$u(x,t) = E(t)[u_0](x) = \sum_{\xi \in \mathbb{Z}^n} \exp\left(a\sigma(\xi)t\right)\mathcal{F}([u_0])(\xi)e^{2\pi i x \xi}, \tag{2.13}$$

where $E(t)$ is a differentiable function and $E(0) = 1$.

We recall that a multiresolution approximation (MRA) of $L_2(\mathbb{R}^n)$ is, as a definition, an increasing sequence V_j, $j \in \mathbb{Z}$, of closed linear subspaces of $L_2(\mathbb{R}^n)$ with the following properties:

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R}^n); \tag{2.14}$$

for all $f \in L_2(\mathbb{R}^n)$ and all $j \in \mathbb{Z}$,

$$f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}; \tag{2.15}$$

for all $f \in L_2(\mathbb{R}^n)$ and $k \in \mathbb{Z}^n$,

$$f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0. \tag{2.16}$$

There exists a function, called the scaling function (SF) $\phi(x) \in V_0$, such that the sequence

$$\{\phi(x-k),\ k \in \mathbb{Z}^n\} \tag{2.17}$$

is a Riesz basic of V_0 (see [5, 9]).

An SF ϕ is called μ-regular ($\mu \in \mathbb{N}$) if, for each $m \in \mathbb{N}$, there exists c_m such that the following condition holds:

$$\left|D^\alpha \phi(x)\right| \le c_m (1+|x|)^{-m}, \quad \forall \alpha,\ |\alpha| \le \mu. \tag{2.18}$$

Remark 2.2. (1) Denote $\phi_{jk}(x) = 2^{nj/2}\phi(2^j x - k)$, $k \in \mathbb{Z}^n$. It follows from (2.14), (2.15), (2.16), and (2.17) that $V_j = \overline{\operatorname{span}}\{\phi_{jk}(x),\ k \in \mathbb{Z}^n\}$, $j \in \mathbb{Z}$.

(2) For each $\mu \in \mathbb{N}$, there exists an SF $\phi(x)$ with compact support, and $\phi(x)$ is μ-regular; so in what follows, we always assume that ϕ has compact support and is μ-regular (see [9]).

Using the periodic operator and an MRA of $L_2(\mathbb{R}^n)$, we can build an MRA of $L_2(\mathcal{J}^n)$ with the SF $[\phi]$ as follows.

Denote

$$\phi_k^j(x) = 2^{nj/2}\sum_{l\in\mathbb{Z}^n}\phi_{jk}(x+l) = 2^{nj/2}\sum_{l\in\mathbb{Z}^n}\phi(2^j(x+l)-k), \quad j \geq 0, \tag{2.19}$$

$$[V_j] = \overline{\operatorname{span}}\{\phi_k^j(x),\ k \in \mathbb{Z}^{nj}\}, \quad j \geq 0, \tag{2.20}$$

where $\mathbb{Z}^{nj} = \mathbb{Z}^n/2^j\mathbb{Z}^n$.

Then, the sequence $[V_j]_{j\geq 0}$ satisfies

$$[V_0] \subset [V_1] \subset \cdots, \qquad \overline{\bigcup_{j\geq 0}[V_j]} = L_2(\mathcal{J}^n). \tag{2.21}$$

It is clear that $\dim[V_j] = 2^{nj}$, and if $(\phi_{jk},\phi_{jl}) = \delta_{kl}$, $k,l \in \mathbb{Z}^n$, then $(\phi_k^j,\phi_l^j) = \delta_{kl}$, $k,l \in \mathbb{Z}^{nj}$ (see [6]).

For each $j \geq 0$, let $P_j : L_2(\mathcal{J}^n) \to [V_j]$ be the orthogonal projection from $L_2(\mathcal{J}^n)$ on $[V_j]$, which has the following property.

Theorem 2.3 (see [6, page 600]). *Let $-\mu - 1 \leq s \leq \mu$, $-\mu \leq q \leq \mu + 1$, and $s \leq q$, then*

$$\|u - P_j u\|_s \leq c2^{j(s-q)}\|u\|_q \tag{2.22}$$

for all $u \in H^q(\mathcal{J}^n)$, where c is independent of j and u.

Denoting $h = 2^{-j}$ and $V_h = [V_j]$, we can write (2.22) as

$$\|v - P_j v\|_s \leq ch^{q-s}\|v\|_q. \tag{2.23}$$

3. The Galerkin-wavelet solution. Fix a distribution with compact support $\eta \in H^{-s'}(\Gamma)$, where $s' \geq 0$ satisfying $AV_h \subset H^{s'}(\mathcal{J}^n)$ and where $\Gamma \subset \mathbb{R}^n$ is some fixed compact domain such as a hypercube. For $f \in H^{s'}(\mathcal{J}^n)$, define

$$\eta_k^j(f) = 2^{-nj/2}\eta(f(2^{-j}(\cdot + k))). \tag{3.1}$$

The space

$$X^j := \operatorname{span}\{\eta_k^j,\ k \in \mathbb{Z}^{nj}\} \tag{3.2}$$

is contained in $(AV_h)'$, which is the dual of AV_h. The corresponding Galerkin-Petrov-wavelet scheme is then given by

$$\eta_k^j\left(\frac{\partial u_h}{\partial t}\right) = a\eta_k^j(Au_h), \quad k \in \mathbb{Z}^{nj}, \tag{3.3}$$

$$u_h(x,0) = R_h[u_0](x), \tag{3.4}$$

where $R_h v$ is a linear approximation of v in V_h and $u_h : [0,\infty) \to V_h$ is a differentiable operator.

Set

$$u_h(x,t) = \sum_{k\in\mathbb{Z}^{nj}} c_k(t)\phi_k^j(x), \tag{3.5}$$

$$R_h[u_0](x) := [u_0]_h(x) := \sum_{k\in\mathbb{Z}^{nj}} c_k(0)\phi_k^i(x). \tag{3.6}$$

Then the scheme (3.3) and (3.4) provides an algebra equation system and the solution can be solved by Fourier series.

Lemma 3.1. *The following formulas hold true:*

$$\begin{aligned} \mathcal{F}(\phi_k^j)(\xi) &= h^{n/2}\hat{\phi}(h\xi)e^{-2\pi ihk\xi}, \\ \mathcal{F}(A\phi_k^j)(\xi) &= h^{n/2}\sigma(\xi)\hat{\phi}(h\xi)e^{-2\pi ihk\xi}. \end{aligned} \tag{3.7}$$

Proof. (a) It follows from (2.3) and (2.19) that

$$\begin{aligned} \mathcal{F}(\phi_k^j)(\xi) &= h^{-n/2}\sum_{l\in\mathbb{Z}^n}\int_{[0,1]^n} e^{-2\pi ix\xi}\phi(2^j(x+l)-k)\,dx \\ &= h^{n/2}\sum_{l\in\mathbb{Z}^n}\int_{2^j(l+[0,1]^n)-k} e^{-2\pi ihx\xi}\phi(x)\,dx e^{-2\pi ikh\xi} \\ &= h^{n/2}\int_{\mathbb{R}^n} e^{-2\pi ihx\xi}\phi(x)\,dx e^{-2\pi ikh\xi} \\ &= h^{n/2}\hat{\phi}(h\xi)e^{-2\pi ihk\xi}. \end{aligned} \tag{3.8}$$

(b) We have

$$\mathcal{F}(Au)(\xi) = \sigma(\xi)\tilde{u}(\xi); \tag{3.9}$$

consequently,

$$\mathcal{F}(A\phi_k^j)(\xi) = \sigma(\xi)\mathcal{F}(\phi_k^j)(\xi) = h^{n/2}\sigma(\xi)\hat{\phi}(h\xi)e^{-2\pi ihk\xi}. \tag{3.10}$$

The proof of the lemma is complete. □

Corollary 3.2. *The following formulas hold true:*

$$\begin{aligned} \eta_k^j(\phi_l^j) &= h^n \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{-2\pi ih(l-k)\xi}, \\ \eta_k^j(A\phi_l^j) &= h^n \sum_{\xi\in\mathbb{Z}^n} \sigma(\xi)\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{-2\pi ih(l-k)\xi}. \end{aligned} \tag{3.11}$$

Proof. (a) Using (2.4), Lemma 3.1, and (3.1), we have

$$\begin{aligned} \eta_k^j(\phi_l^j) &= \eta_k^j\left(\sum_{\xi\in\mathbb{Z}^n} \mathcal{F}(\phi_l^j)(\xi)e^{2\pi ix\xi}\right) \\ &= \eta_k^j\left(\sum_{\xi\in\mathbb{Z}^n} h^{n/2}\hat{\phi}(h\xi)e^{-2\pi ihl\xi}e^{2\pi ix\xi}\right) \\ &= h^n \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(h\xi)e^{-2\pi hl\xi}\eta(e^{2\pi h(x+k)\xi}) \\ &= h^n \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{-2\pi ih(l-k)\xi}. \end{aligned} \tag{3.12}$$

(b) Similarly, we can get the second assertion. □

The following lemma is extracted from [6].

Lemma 3.3. *The following formula holds valid:*

$$\sum_{m\in\mathbb{Z}^{nj}} e^{-2\pi ihm(k-\xi)} = \begin{cases} 2^{nj} & \text{if } \xi = k+2^j\theta,\ \theta\in\mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Set

$$\alpha(k) = \sum_{\xi\in\mathbb{Z}^n} \hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{2\pi ihk\xi}, \tag{3.14}$$

$$\delta(k) = \sum_{\xi\in\mathbb{Z}^n} \sigma(h\xi)\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{2\pi ihk\xi}, \quad k\in\mathbb{Z}^{nj}. \tag{3.15}$$

The series

$$\tilde{\alpha}(\zeta) = h^n \sum_{k\in\mathbb{Z}^{nj}} \alpha(k)e^{-2\pi ihk\zeta}, \tag{3.16}$$

$$\tilde{\delta}(\zeta) = h^n \sum_{k\in\mathbb{Z}^{nj}} \delta(k)e^{-2\pi ihk\zeta}, \tag{3.17}$$

$$\tilde{c}(\zeta,t) = h^n \sum_{k\in\mathbb{Z}^{nj}} c_k(t)e^{-2\pi ihk\zeta}, \quad \zeta\in\mathbb{Z}^n \tag{3.18}$$

are called discrete Fourier series.

It follows from (3.3), (3.5), the positively homogeneous condition, and Corollary 3.2 that

$$\sum_{k\in\mathbb{Z}^{nj}} c_k'(t)\alpha(l-k) = ah^{-r}\sum_{k\in\mathbb{Z}^{nj}} c_k(t)\delta(l-k), \quad l\in\mathbb{Z}^{nj}. \tag{3.19}$$

Thus

$$\tilde{c}_t'(\zeta,t)\tilde{\alpha}(\zeta) = ah^{-r}\tilde{c}(\zeta,t)\tilde{\delta}(\zeta), \tag{3.20}$$

$$\tilde{c}(\zeta,t) = \exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\zeta)}{\tilde{\alpha}(\zeta)}\right)\tilde{c}(\zeta,0). \tag{3.21}$$

For each $\tau = 0,1$, set

$$g_{\phi,\tau}(\zeta) = \sum_{k\in\mathbb{Z}^n}\sigma(h\zeta+k)^\tau\hat{\phi}(h\zeta+k)\overline{\hat{\eta}(h\zeta+k)}. \tag{3.22}$$

Lemma 3.4. *If the series (3.22) converges absolutely, then*

$$\tilde{\alpha}(\zeta) = g_{\phi,0}(\zeta), \qquad \tilde{\delta}(\zeta) = g_{\phi,1}(\zeta). \tag{3.23}$$

Proof. (a) From (3.14) and (3.16), it follows that

$$\tilde{\alpha}(\zeta) = h^n\sum_{k\in\mathbb{Z}^{nj}}\sum_{\xi\in\mathbb{Z}^n}\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}e^{-2\pi ihk(\zeta-\xi)}. \tag{3.24}$$

By the hypothesis of the lemma, we can interchange the summation in the above double sum; then by using the variable change and Lemma 3.3, it is easy to see that

$$\begin{aligned}\tilde{\alpha}(\zeta) &= h^n\sum_{\xi\in\mathbb{Z}^n}\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}\sum_{k\in\mathbb{Z}^{nj}}e^{-2\pi ihk(\zeta-\xi)}\\ &= \sum_{\theta\in\mathbb{Z}^n}\hat{\phi}(h\zeta+\theta)\overline{\hat{\eta}(h\zeta+\theta)} = g_{\phi,0}(\zeta).\end{aligned} \tag{3.25}$$

(b) Similarly, the second assertion of the lemma will be checked. From (3.5), (3.6), and (3.21), it follows that

$$\tilde{u}_h(\xi,t) = \exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right)\mathcal{F}([u_0]_h)(\xi). \tag{3.26}$$

Let $F_h(t)$ be the operator defined by

$$\mathcal{F}(F_h(t)v(\cdot))(\xi) = \exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right)\tilde{v}(\xi), \tag{3.27}$$

then the approximation $u_h(x)$ can be represented by

$$u_h(x) = F_h(t)R_h[u_0](x). \tag{3.28}$$

□

4. The error estimate of approximation solutions. Now to estimate the error, we need some restrictions on the σ, ϕ, and η used above. The triplet (σ,ϕ,η) is called *admissible* if the following properties hold:

(i) there exists $p\in\mathbb{N}$, $p\geq r$, such that the series

$$\sum_{k\in\mathbb{Z}^n}\sigma(h\xi+k)\hat{\phi}(h\xi+k)\overline{\hat{\eta}(h\xi+k)} \tag{4.1}$$

converges absolutely and

$$\sum_{k\in\mathbb{Z}^n}\sigma(h\xi+k)\hat{\phi}(h\xi+k)\overline{\hat{\eta}(h\xi+k)}=\sigma(h\xi)\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}+o(|h\xi|^p) \tag{4.2}$$

as $|h\xi|\to 0$,

(ii) $\hat{\phi}(\xi)\overline{\hat{\eta}(\xi)}\geq 0$, for all $\xi\in\mathbb{R}^n$, $\hat{\phi}(0)\overline{\hat{\eta}(0)}\neq 0$,

(iii) the series

$$\sum_{k\in\mathbb{Z}^n}\hat{\phi}(h\xi+k)\overline{\hat{\eta}(h\xi+k)} \tag{4.3}$$

converges and

$$\sum_{k\in\mathbb{Z}^n}\hat{\phi}(h\xi+k)\overline{\hat{\eta}(h\xi+k)}=\hat{\phi}(h\xi)\overline{\hat{\eta}(h\xi)}+0(|h\xi|^p) \tag{4.4}$$

as $|h\xi|\to 0$.

Remark 4.1. (1) If $\eta=\phi$ and σ is a pseudodifferential operator with symbol $\sigma(\xi)=|\xi|^r$, $0<r\leq\mu$, then the triplet (σ,ϕ,ϕ) is automatically admissible at least for $p=\mu$, where $\mu\in\mathbb{N}$ is used in (2.18) (see [7] for detail).

(2) If $\eta=\phi$ and σ is a pseudodifferential operator with symbol $\sigma(\xi)=\langle\xi\rangle^2$, then the triplet $(\langle\xi\rangle^2,\phi,\phi)$ is admissible for $p=\mu$ (see [6]).

Write

$$u-u_h=\{u-F_h(t)[u_0]\}+F_h(t)\{[u_0]-R_h[u_0]\}. \tag{4.5}$$

We have

$$\begin{aligned}\mathcal{F}(F_h(t)[u_0](\cdot))(\xi)&=\exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right)\mathcal{F}([u_0])(\xi)\\&=\exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right)\hat{u}_0(\xi),\quad \xi\in\mathbb{Z}^n,\end{aligned} \tag{4.6}$$

thus

$$\begin{aligned}&\mathcal{F}(u-F_h(t)[u_0])(\xi)\\&\quad=\left\{\exp\left(at\sigma(\xi)\right)-\exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right)\right\}\hat{u}_0(\xi),\quad \xi\in\mathbb{Z}^n.\end{aligned} \tag{4.7}$$

If the triplet (σ,ϕ,η) is admissible, then it follows from (3.22) and Lemma 3.4 that

$$\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)} = \sigma(h\xi) + 0(|h\xi|^p) \quad \text{as } |h\xi| \longrightarrow 0. \tag{4.8}$$

Theorem 4.2. *Suppose that $r+s' \le s \le p$, $0 \le m \le s$, and it is assumed that the triplet (σ,ϕ,η) is admissible. Then, for $u_0 \in \mathcal{L}_2 \cap H_d^{m+s}(\mathbb{R}^n)$, $0 \le t \le T$, with h small enough, we get*

$$||u - F_h(t)[u_0]||_m \le ch^{s-r}||u_0||_{s+m,d}, \tag{4.9}$$

where c is independent of u, h, and u_0.

Proof. It follows from (4.8) that

$$\left| at\sigma(\xi) - \frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)} \right| \le ch^{p-r}|\xi|^p \quad \text{as } |h\xi| \le 1. \tag{4.10}$$

The equality

$$e^{ta} - e^{tb} = t(a-b)\int_0^1 e^{sta+(1-s)tb}ds, \tag{4.11}$$

(4.10), and (1.3) imply that, for $r \le s \le p$ and $0 \le t \le T$,

$$\left| \exp\left(at\sigma(\xi)\right) - \exp\left(\frac{at}{h^r}\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \right| \le ch^{s-r}|\xi|^s \quad \text{as } |h\xi| \le 1. \tag{4.12}$$

Hence, from (4.7) and (4.12), we obtain

$$\left|\mathcal{F}(u(\cdot,t) - F_h(t)[u_0](\cdot))(\xi)\right| \le ch^{s-r}|\xi|^s\left|\hat{u}_0(\xi)\right| \quad \text{as } |h\xi| \le 1. \tag{4.13}$$

By (1.3) and the admissibility of the triplet (σ,ϕ,η), inequality (4.13) is also valid for all $\xi \in \mathbb{Z}^n$. Hence, for each $0 \le m \le s$, $r+s' \le s \le p$, and $0 \le t \le T$, we get

$$\begin{aligned} ||u - F_h(t)[u_0]||_m^2 &= \sum_{\xi\in\mathbb{Z}^n} \langle\xi\rangle^{2m}\left|\mathcal{F}\{u(\cdot,t) - F_h(t)[u_0](\cdot)\}(\xi)\right|^2 \\ &\le ch^{2(s-r)}\sum_{\xi\in\mathbb{Z}^n}\langle\xi\rangle^{2(m+s)}\left|\hat{u}_0(\xi)\right|^2 \\ &\le ch^{2(s-r)}||u_0||_{m+s,d}^2. \end{aligned} \tag{4.14}$$

The theorem is thus proved. □

From the admissibility of the triplet (σ,ϕ,η) and (1.3), it follows that $F_h(t): H^m(\mathbb{R}^n)\to H^m(\mathbb{R}^n)$, $0\le m\le s$, is a continuous linear operator. Consequently,

$$||F_h(t)([u_0]-R_h[u_0])||_m \le c||[u_0]-R_h[u_0]||_m. \tag{4.15}$$

Therefore, if we assume that

$$||(I-R_h)[u_0]||_m \le ch^s||[u_0]||_{m+s}, \tag{4.16}$$

then

$$||F_h(t)([u_0]-R_h[u_0])||_m \le ch^s||[u_0]||_{m+s}. \tag{4.17}$$

Remark 4.3. It follows from (2.23) that the assumption (4.17) is satisfied, when $R_h=P_j$ for $0\le m$, $m+s\le \mu+1$.

Thus from (4.5), (4.9), and (4.17), we obtain the following theorem.

Theorem 4.4. *If all the hypotheses of Theorem 4.2 and assumption (4.17) are satisfied, then*

$$||u-u_h||_m \le ch^{s-r}||u_0||_{m+s,d}+ch^s||[u_0]||_{m+s}, \tag{4.18}$$

where c is independent of u_0, h.

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