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COMMON PERIODIC POINTS FOR A CLASS OF CONTINUOUS COMMUTING MAPPINGS ON AN INTERVAL

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The existence of common periodic points for a family of continuous commuting self-mappings on an interval is proved and two illustrative examples are given in support of our theorem and definition.

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1. Introduction and preliminaries. All mappings considered here are assumed to be continuous from the interval $I = [u,v]$ to itself. Let $F(f)$ and $P(f)$ be the set of fixed and periodic points of f, respectively, and let $\overline{P(f)}$ be the closure of $P(f)$. Denote $L(x,f)$ by the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. By Schwartz's theorem [4], it is easy to show that $L(x,f) \cap P(f) \neq \emptyset$ for each x in I. Obviously, $F(f)$ is a closed set and $\emptyset \neq F(f) \subset P(f)$. Define the classes of mappings

$$\begin{aligned} A &= \{f : I \longrightarrow I \mid F(f) = [a_f, b_f],\ a_f \leq b_f\}, \\ B &= \{f : I \longrightarrow I \mid P(f) = F(f)\}, \\ D &= \{f : I \longrightarrow I \mid \overline{P(f)} = P(f)\}. \end{aligned} \tag{1.1}$$

The following definition was introduced by Cano [2].

Definition 1.1. A class of mappings T is said to be an *H-class* if $T = T' \cup \{h\}$, where T' is any subset of $A \cup B$ composed of commuting mappings and h is any mapping which commutes with the elements of T'.

Boyce [1] and Huneke [3] showed that if f and g are two commuting self-mappings of I, then f and g need not have a common fixed point in I. Cano [2] proved the following theorem.

Theorem 1.2. *There is a common fixed point for every H-class in I.*

In this note, we consider a larger class of mappings which has the common periodic point property and properly contains the class H considered by Cano. Two illustrative examples are given in support of our theorem and definition.

We first introduce the following definition.

Definition 1.3. A class of mappings T is said to be a *C-class* if $T = T' \cup \{h\}$ and T is a commuting family of mappings, where T' is any subset of $A \cup D$ and h is any mapping.

Obviously, $B \subset D$. The following example proves that B is a proper subset of D.

Example 1.4. Let $I = [0,1]$ and $f(x) = 1-x$. It is easy to show that $F(f) = \{1/2\} \neq [0,1] = P(f) = \overline{P(f)}$, that is, $f \in D$ and $f \overline{\in} B$.

Remark 1.5. Clearly, H-class is C-class, but the converse is not true.

2. Main results.

Our main result is as follows.

Theorem 2.1. *There is a common periodic point for every C-class in I.*

Proof. Let T be a C-class and T_1 a finite subset of T. We can write T_1 as

$$T_1 = \{f_1, f_2, \dots, f_n\} \cup \{h\} \cup \{g_1, g_2, \dots, g_m\}, \tag{2.1}$$

where $f_1 \in A$, $i = 1,2,\dots,n$, and h is a possible arbitrary mapping that commutes with the elements of T, $g_j \in D$, $j = 1,2,\dots,m$. Suppose that there are different $i,k \in \{1,2,\dots,n\}$ such that $F(f_i) \cap F(f_k)$ is not an interval, that is, $F(f_i) \cap F(f_k) = \varnothing$. Let $F(f_i) = [a_i, b_i]$ and $F(f_k) = [a_k, b_k]$. Clearly, $\max\{a_i, a_k\} > \min\{a_i, a_k\}$. Without loss of generality, we can assume $a_k > a_i$. Since f_i and f_k commute and a_i, $b_i \in F(f_i)$, then $f_i(f_k(a_i)) = f_k(f_i(a_i)) = f_k(a_i)$, that is, $f_k(a_i) \in F(f_i)$. Hence, $f_k(a_i) > a_i$. Similarly, we can show that $f_k(b_i) < b_i$. Let $w(x) = f(x) - x$ for $x \in F(f_i)$. Since $w(a_i) > 0$ and $w(b_i) < 0$, there is $c \in (a_i, b_i)$ such that $w(c) = 0$, that is, $f_k(c) = c$. Therefore,

$$c \in (a_i, b_i) \cap F(f_k) \subset F(f_i) \cap F(f_k) \neq \varnothing, \tag{2.2}$$

a contradiction. Thus, $F(f_i) \cap F(f_k)$ is an interval for any two distinct $i,k \in \{1,2,\dots,n\}$. It is easy to show that $\cap_{i=1}^{n} F(f_i)$ is an interval. Let $\cap_{i=1}^{n} F(f_i) = [a,b]$. By the commutativity of h with the f_i's, h takes $[a,b]$ into $[a,b]$, and so, it must have a fixed point $z \in [a,b]$. Now, $\{g_1^n(z)\}_{n=0}^{\infty}$ has a limit point $z_1 \in P(g_1)$ because $P(g_1)$ is a closed set. Clearly, there exists a subsequence $\{g_1^{n_k}(z)\}_{k=1}^{\infty}$ of $\{g_1^n(z)\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} g_1^{n_k}(z) = z_1 = g_1^r(z_1) \in P(g_1). \tag{2.3}$$

Since $z \in (\cap_{i=1}^{n} F(f_i)) \cap F(h)$, by (2.3), we have

$$\begin{gathered} f_i(g_1^{n_k}(z)) = g_1^{n_k}(f_i(z)) = g_1^{n_k}(z) \longrightarrow z_1, \quad k \longrightarrow \infty, \\ f_i(g_1^{n_k}(z)) \longrightarrow f_i(z_1), \quad k \longrightarrow \infty. \end{gathered} \tag{2.4}$$

From (2.4), we have $f_i(z_1) \in F(f_i)$. Using the same method, we can show that $z_1 \in F(h)$. So,

$$z_1 \in \left(\cap_{i=1}^{n} F(f_i)\right) \cap F(h) \cap P(g_1). \tag{2.5}$$

Similarly, $\{g_j^n(z_{j-1})\}_{n=0}^{\infty}$, $j = 2,3,\ldots,m$, has a limit point

$$z_j \in \left(\cap_{i=1}^{n} F(f_i)\right) \cap F(h) \cap \left(\cap_{i=1}^{j} P(g_i)\right). \tag{2.6}$$

Thus,

$$\varnothing \neq \left(\cap_{i=1}^{n} F(f_i)\right) \cap F(h) \cap \left(\cap_{j=1}^{m} P(g_j)\right) \tag{2.7}$$

which implies that

$$\varnothing \neq \left(\cap_{f\in T\cap A} F(f)\right) \cap F(h) \cap \left(\cap_{f\in T\cap D} P(f)\right) \subset \cap_{f\in T} P(f) \tag{2.8}$$

by the compactness of I. When T contains no such h, $T \cap A = \varnothing$, or $T \cap D = \varnothing$, we have the same result from the above proof. This completes the proof. □

We at last give an example in which Theorem 2.1 holds but Theorem 1.2 is not applicable.

Example 2.2. Let $I = [-1,1]$,

$$f(x) = \begin{cases} 1+x & \text{if } x \in [-1,0], \\ 1-x & \text{if } x \in (0,1], \end{cases} \qquad g(x) = \begin{cases} -x & \text{if } x \in [-1,0], \\ x & \text{if } x \in (0,1]. \end{cases} \tag{2.9}$$

Let h be a continuous mapping and commute with f and g. It is easy to see that

$$F(f) = \left\{\frac{1}{2}\right\}, \qquad P(f) = \overline{P(f)} = [0,1], \qquad F(g) = [0,1]; \tag{2.10}$$

that is, $f \in D$, $f \,\overline{\in}\, B$, and $g \in A$. Clearly, f and g are continuous and

$$f(g(x)) = g(f(x)) = \begin{cases} 1+x & \text{if } x \in [-1,0], \\ 1-x & \text{if } x \in (0,1]. \end{cases} \tag{2.11}$$

Thus, $\{f,g,h\}$ is a C-class but $\{f,g,h\}$ is not an H-class. Hence, Theorem 2.1 holds, that is, f, g, and h have a common periodic point. But Theorem 1.2 is not applicable.

Remark 2.3. Example 2.2 and Remark 1.5 prove the greater generality of Theorem 2.1 over Theorem 1.2.

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References

[1] W. M. Boyce, *Commuting functions with no common fixed point*, Trans. Amer. Math. Soc. **137** (1969), 77–92.

[2] J. Cano, *Common fixed points for a class of commuting mappings on an interval*, Proc. Amer. Math. Soc. **86** (1982), no. 2, 336–338.

[3] J. P. Huneke, *On common fixed points of commuting continuous functions on an interval*, Trans. Amer. Math. Soc. **139** (1969), 371–381.

[4] A. J. Schwartz, *Common periodic points of commuting functions*, Michigan Math. J. **12** (1965), 353–355.

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