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TIME ESTIMATES FOR THE CAUCHY PROBLEM FOR A THIRD-ORDER HYPERBOLIC EQUATION

VLADIMIR VARLAMOV

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A classical solution is considered for the Cauchy problem: $(u_{tt} - \Delta u)_t + u_{tt} - \alpha\Delta u = f(x,t)$, $x \in \mathbb{R}^3$, $t > 0$; $u(x,0) = f_0(x)$, $u_t(x,0) = f_1(x)$, and $u_{tt}(x) = f_2(x)$, $x \in \mathbb{R}^3$, where $\alpha = \text{const}$, $0 < \alpha < 1$. The above equation governs the propagation of time-dependent acoustic waves in a relaxing medium. A classical solution of this problem is obtained in the form of convolutions of the right-hand side and the initial data with the fundamental solution of the equation. Sharp time estimates are deduced for the solution in question which show polynomial growth for small times and exponential decay for large time when $f = 0$. They also show the time evolution of the solution when $f \neq 0$.

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1. Introduction. In recent years, interest has grown in theoretical investigation of wave motion in media with dispersion and absorption. This is explained not only by the practical needs of fluid dynamics, theory of viscoelasticity, and geophysics, but also by the particular nature of equations arising here. In the present paper, we are concerned with studying a third-order hyperbolic equation in the three-dimensional case

$$\tau\left(u_{tt} - c_f^2\Delta u\right)_t + u_{tt} - c_e^2\Delta u = f(x,t), \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $u(x,t)$ is the dynamic pressure (perturbation of the average pressure) and Δ is the Laplace operator. The positive constant coefficients τ, c_e, and c_f are the relaxation time and the limiting phase speeds of sound, respectively. In order to understand their meaning, one has to describe briefly the corresponding physical situation.

In a relaxing medium, the propagation of an acoustic wave perturbs the state of a thermodynamical equilibrium. Having been disturbed, such a medium tends towards a state of equilibrium, but with new values of parameters. If the relaxation time is much smaller than the period of oscillations, the propagation of sound occurs with the same speed c_e as in the absence of relaxation. If the inverse relation holds, the relaxation processes are "frozen" (not fast enough to follow the oscillations) and the sound propagates with the "frozen" sound speed $c_f > c_e$. In fact, for the majority of relaxing media (mixtures of gases, chemically reacting fluids, water with bubbles, etc.), the ratio c_e^2/c_f^2 is rather close to one [1, 3, 7]. Similar processes occur in standard viscoelastic materials [4] and in cracked and porous media [7], where wave propagation disturbs the state of a mechanical equilibrium.

In [5, 6], Renno obtained an integral representation of the fundamental solution of (1.1) involving modified Bessel functions and constructed the classical solution of the Cauchy problem for (1.1) using spherical means [2]. In [8, 9], the author of the present paper constructed the fundamental solution of (1.1) in the form of a contour integral in the complex plane and applied the method of retarded potentials for solving the initial-value problem for the homogeneous equation (1.1). In the present paper, we would like to continue this study, deduce some properties of the fundamental solution, and, with their help, obtain sharp time estimates of the classical solution of the Cauchy problem for (1.1).

2. Posing of the problem and auxiliary results. We denote by $C^n(\mathbb{R}^3)$ the space of functions having continuous derivatives through order n in $\mathbb{R}^3$ and by $C_0^n(\mathbb{R}^3)$ the subspace of $C^n(\mathbb{R}^3)$ of functions whose derivatives through order n have compact support in $\mathbb{R}^3$.

We denote by $B(x,t)$ a ball of radius t with the centre at the point x. In the sequel, we will need functions defined on such a ball, that is, $h(x,t)$, $x \in B(x,t)$, $t > 0$. Making a change of variable $x = t\xi$, $0 < |\xi| < 1$, we obtain the function $h(t\xi,t)$ which we denote by $\tilde{h}$. Analogous notations will be used for its derivatives, for example, $\tilde{h}_t = \partial_t \tilde{h}(t\xi,t)$, where $t > 0$, $|\xi| < 1$.

To simplify the notation in the rest of the paper, we introduce the nondimensional variables $\bar{x}_i = x_i/(c_f\tau)$, $\bar{t} = t/\tau$ and set $\alpha = c_f^2/c_e^2$. Note that $0 < \alpha < 1$. We are interested in studying a Cauchy problem for (1.1) which in the scaled variables can be posed as follows:

$$(u_{tt} - \Delta u)_t + u_{tt} - \alpha \Delta u = f(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \tag{2.1}$$

$$u(x,0) = f_0(x), \qquad u_t(x,0) = f_1(x), \qquad u_{tt}(x) = f_2(x), \quad x \in \mathbb{R}^3. \tag{2.2}$$

In [9], a contour integral representation was obtained for the fundamental solution $E(x,t)$ of (2.1) which can be written as

$$E(x,t) = \frac{e(x,t)}{4\pi|x|} H(t - |x|), \tag{2.3}$$

where

$$\begin{gathered} e(x,t) = \frac{1}{2\pi i} \int_{C^+} \frac{\exp\left[\xi(t - \gamma(\xi)|x|)\right]}{\xi + \alpha} d\xi, \\ \gamma(\xi) = \sqrt{\frac{1+\xi}{\alpha+\xi}}, \qquad |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}. \end{gathered} \tag{2.4}$$

Here,

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases} \tag{2.5}$$

is the Heaviside function and C^+ is the positively oriented circle with the center in $\xi = -\alpha$ and radius $1-\alpha$, that is, $\{\xi : |\xi + \alpha| = 1-\alpha\}$. The branch of the function $\gamma(\xi)$ is determined in the following way. The cut is made along the interval $[-1,-\alpha]$ and the branch of $\gamma(\xi)$ satisfying the condition $\gamma(0) = 1$ is chosen. Note that $e = e(|x|,t)$.

In [5], Renno has obtained another representation for the kernel of the fundamental solution

$$e(r,t) = \exp(\alpha r - bt)F(r,t), \tag{2.6}$$

where

$$\begin{aligned} F(r,t) = \Big\{ I_0(\omega(r,t)) + \int_0^1 \exp\left(\eta(r,t)y^2\right)\big[4\eta(r,t)yI_0(\psi(r,t)y) \\ + \psi(r,t)I_1(\psi(r,t)y)\big] \\ \times I_0\Big(\omega(r,t)\sqrt{1-y^2}\Big)\,dy \Big\}, \end{aligned} \tag{2.7}$$

$$\omega(r,t) = a\sqrt{t^2 - r^2}, \qquad \eta(r,t) = a(t-r), \qquad \psi(r,t) = 2\sqrt{\alpha a r(t-r)},$$

$$a = \frac{1-\alpha}{2}, \qquad b = \frac{1+\alpha}{2}, \qquad r = |x|.$$

It follows from (2.7) that $e(r,t) > 0$. Since representation (2.7) is rather complicated, only a rough estimate was deduced in [5] for the large time behavior of solutions of (2.1) with $f = f_0 = f_1 = 0$ and $f_2 \in C_0^2(\mathbb{R}^3)$; namely,

$$|u| \leq \frac{C}{t}, \quad \text{as } t \longrightarrow \infty. \tag{2.8}$$

In [6], without the use of (2.7), Renno has proved that the solution of problem (2.1) with $f = 0$ decayed exponentially in time for sufficiently large t. In the present paper, sharp time estimates uniform in space will be obtained for all $t > 0$. They show dependence on the initial data, polynomial growth of solutions for small times, and exponential decay at infinity in time. They also show evolution of the wave process described by (2.1) due to the influence of the source term of the equation. In order to obtain these results, we use both formula (2.7) and the contour integral representation obtained in [9].

Modifying the main theorem of [9] in such a way that it covers the contribution of the source term of the equation, we can obtain the following statement.

Theorem 2.1. *If $f(x,t) \in C^3(\mathbb{R}^3 \times \mathbb{R}^+)$ and $f_i(x) \in C^{5-i}(\mathbb{R}^3)$, $i = 0,1,2$, then there exists the unique classical solution of problem (2.1) represented in* *the form*

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_f(x,t), \tag{2.9}$$

where

$$\begin{aligned}
u_0(x,t) &= \frac{1}{4\pi}\int_{B(x,t)} \frac{e(x-y,t)}{|x-y|}\left[f_2(y) + \frac{2}{3}\left(f_1(y) - \frac{1}{3}f_0(y) - \Delta f_0(y)\right)\right]dy,\\
u_1(x,t) &= \frac{1}{4\pi}\partial_t \int_{B(x,t)} \frac{e(x-y,t)}{|x-y|}\left[f_1(y) + \frac{2}{3}f_0(y)\right]dy,\\
u_2(x,t) &= \frac{1}{4\pi}\partial_t^2 \int_{B(x,t)} \frac{e(x-y,t)}{|x-y|} f_0(y)\,dy,\\
u_f(x,t) &= \frac{1}{4\pi}\int_0^t d\tau \int_{B(x,\tau)} \frac{e(x-y,\tau)}{|x-y|} f(y,t-\tau)\,dy.
\end{aligned} \tag{2.10}$$

Set

$$\beta(\xi,t) = bt - \alpha|\xi| \tag{2.11}$$

and observe that

$$0 < a < \beta(\xi,t) < b. \tag{2.12}$$

We can write that

$$\tilde{e} = e(t\xi,t) = \exp[-\beta(\xi,t)]F(t\xi,t) = \exp[-\beta(\xi,t)]\tilde{F}. \tag{2.13}$$

Since $I_0(x), I_1(x) > 0$ for $x > 0$ (see [10]), it follows from (2.7) that $\tilde{F} > 0$, and therefore $\tilde{e} > 0$. The following statement provides some more properties of the function $\tilde{e}$.

Lemma 2.2. *The function $\tilde{e}$ satisfies the following inequalities:*

$$\tilde{e}_t + b\tilde{e} > 0, \tag{2.14}$$

$$\tilde{e}_{tt} + 2b\tilde{e}_t + b^2\tilde{e} > 0. \tag{2.15}$$

Proof. Differentiating (2.13) with respect to t, we get

$$\tilde{e}_t = -b\exp[-\beta(\xi,t)]\tilde{F} + \exp[-\beta(\xi,t)]\tilde{F}_t, \tag{2.16}$$

$$\tilde{e}_{tt} = b^2\exp[-\beta(\xi,t)]\tilde{F} - 2b\exp[-\beta(\xi,t)]\tilde{F}_t + \exp[-\beta(\xi,t)]\tilde{F}_{tt}. \tag{2.17}$$

According to (2.7), we have

$$\begin{aligned}
\tilde{\omega}_t &= a\sqrt{1-|\xi|^2} > 0, & \tilde{\eta}_t &= a(1-|\xi|) > 0,\\
\tilde{\psi}_t &= 2\sqrt{\alpha a(1-|\xi|)} > 0, & \tilde{\omega}_{tt} &= \tilde{\eta}_{tt} = \tilde{\psi}_{tt} = 0.
\end{aligned} \tag{2.18}$$

Next, we use the properties of the modified Bessel functions involved in (2.7) and their derivatives through the second order. These properties are based on the formulas (see Watson [10, page 79])

$$I_0'(x) = I_1(x),$$
$$I_\nu'(x) = \frac{1}{2}\left[I_{\nu-1}(x) + I_{\nu+1}(x)\right], \quad \nu \geq 1, \tag{2.19}$$

and the fact that $I_\nu(x) > 0$ for $\nu \geq 0$, $x > 0$. Therefore, we conclude that $\tilde{F}_t > 0$ and $\tilde{F}_{tt} > 0$. Then, (2.14) follows immediately from (2.16). Excluding the second term in the right-hand side of (2.17) by means of (2.16), we obtain (2.15). □

Lemma 2.3. *For all $t > 0$ and integers $n \geq 0$, the following relations are true*

$$\partial_t^n \int_{B(0,1)} \frac{e(t\xi,t)}{|\xi|} d\xi = 2\pi(-1)^n a^n e^{-at}, \tag{2.20}$$

$$\int_0^t \tau^2 d\tau \int_{B(0,1)} \frac{e(\tau\xi,\tau)}{|\xi|} d\xi = \frac{2 - e^{-at}\left[(1+at)^2 + 1\right]}{a^3}. \tag{2.21}$$

Proof. Making use of formula (2.4), setting $k = \xi\gamma(\xi)$ and changing the order of integration, we get

$$\begin{aligned}\int_{B(0,1)} \frac{e(t\xi,t)}{|\xi|} d\xi &= 4\pi \int_0^1 e(t\rho,t)\, d\rho \\ &= \frac{2}{i} \int_{C^+} \frac{e^{\xi t}}{\xi + \alpha} d\xi \int_0^1 \exp\left(-k(\xi)t\rho\right)\rho\, d\rho \\ &= \frac{2}{i} \int_{C^+} \frac{\exp\left\{\left[\xi - k(\xi)\right]t\right\}}{\xi + \alpha} \varphi(\xi,t)\, d\xi,\end{aligned} \tag{2.22}$$

where

$$\varphi(\xi,t) = \frac{\exp\left(k(\xi)t\right) - \left[1 + k(\xi)t\right]}{\left[k(\xi)t\right]^2}. \tag{2.23}$$

Setting

$$z = \frac{1+\xi}{\alpha+\xi}, \tag{2.24}$$

we map the cut $[-1,-\alpha]$ onto the negative semi-axis $(-\infty,0]$ and transform the last integral to the form

$$\frac{2}{i} \int_{\Gamma^+} \exp\left(-\frac{1-\alpha z}{\sqrt{z}+1} t\right) \frac{\Phi(z,t)}{z-1} dz, \tag{2.25}$$

where

$$\Phi(z,t) = \frac{\exp\left[K(z)t\right] - \left[1 + K(z)t\right]}{\left[K(z)t\right]^2}, \quad K(z) = \frac{(1-\alpha z)\sqrt{z}}{z-1} \tag{2.26}$$

and Γ^+ is a positively oriented circle $\{z : |z-1| = 1\}$. Note that the only singular point of the integrand lying inside the contour of integration is the essential singularity at $z = 1$. Calculating the residue at this point, we evaluate the integral (2.25) and obtain (2.20) for the case where $n = 0$.

In order to prove (2.20) with $n \geq 1$, we remark that representation (2.7) implies that $e(r,t) \in C^\infty\{(r,t) : 0 < r < t,\ t > 0\}$. Therefore, for all integers $n \geq 1$,

$$\partial_t^n \int_{B(0,1)} \frac{e(t\xi,t)}{|\xi|} d\xi = \int_{B(0,1)} \partial_t^n \frac{e(t\xi,t)}{|\xi|} d\xi \tag{2.27}$$

whence (2.20) follows immediately.

Relation (2.21) is deduced by integrating (2.20) with $n = 0$ with respect to the temporal variable from 0 to t. □

Having completed the preliminary considerations, we now pass to the main results.

3. The main theorem.

Since the following statement deals with the long-time behavior of solutions of the problem in question, in addition to the assumptions of Theorem 2.1, it is required that the initial data have compact support in $\mathbb{R}^3$.

Theorem 3.1. *If $f, f_0 \in C_0^3(\mathbb{R}^3)$, $f_1 \in C_0^4(\mathbb{R}^3)$, and $f_2 \in C_0^5(\mathbb{R}^3)$, then for all $t > 0$ and uniformly with respect to $x \in \mathbb{R}^3$, the following estimates hold:*

$$||u_0||_{L^\infty} \leq C_1 t^2 e^{-at}, \tag{3.1}$$

$$||u_1||_{L^\infty} \leq (C_2 t + C_3 t^2) e^{-at}, \tag{3.2}$$

$$||u_2||_{L^\infty} \leq (C_4 + C_5 t + C_6 t^2) e^{-at}, \tag{3.3}$$

$$||u_f||_{L^\infty} \leq C_7 \{2 - e^{-at}[(1+at)^2 + 1]\} a^{-3}, \tag{3.4}$$

where $C_i = \text{const} > 0$ depend on α and the initial data.

Proof. Setting $x - y = t\xi$, $t > 0$, $0 < |\xi| < 1$, we can rewrite the term $u_0(x,t)$ in (2.9) as

$$u_0(x,t) = \frac{t^2}{4\pi} \int_{B_1} \frac{e(t\xi,t)}{|\xi|} F_0(x - t\xi) d\xi, \tag{3.5}$$

where

$$F_0(y) = f_2(y) + \frac{2}{3}\left[f_1(y) - \frac{1}{3} f_0(y) - \Delta f_0(y) \right]. \tag{3.6}$$

Since $e(t\xi,t) > 0$, applying the mean-value theorem to the last integral and taking into consideration (2.20) with $n = 0$, we conclude that

$$|u_0(x,t)| \le C_1 t^2 e^{-at}, \quad t > 0,\ x \in \mathbb{R}^3, \tag{3.7}$$

where

$$C_1 = \frac{1}{2} \sup_{y\in\mathbb{R}^3} |F_0(y)|. \tag{3.8}$$

Whence, (3.1) follows.

For the term u_1, we have

$$\begin{aligned} u_1(x,t) = {} & \frac{t}{2\pi}\int_{B(0,1)} \frac{e(t\xi,t)}{|\xi|} F_1(x-t\xi)\,d\xi \\ & + \frac{t^2}{4\pi}\int_{B(0,1)} \frac{\partial_t e(t\xi,t)}{|\xi|} F_1(x-t\xi)\,d\xi \\ & - \frac{t^2}{4\pi}\int_{B(0,1)} \frac{e(t\xi,t)}{|\xi|} \xi\cdot\nabla F_1(x-t\xi)\,d\xi, \end{aligned} \tag{3.9}$$

where

$$F_1(y) = f_1(y) + \frac{2}{3} f_0(y), \tag{3.10}$$

$x\cdot y$ is the scalar product, and $\nabla F(x-t\xi)$ is the gradient with respect to the coordinates $x - t\xi$. Next, we apply the mean-value theorem to the first and the third integrals in the right-hand side of the last formula. In order to have positive kernels in the integrand, we rewrite the second integral in the last formula as

$$\int_{B(0,1)} \frac{\tilde{e}_t + b\tilde{e}}{|\xi|} F_1(x-t\xi)\,d\xi - b\int_{B(0,1)} \frac{\tilde{e}}{|\xi|} F_1(x-t\xi)\,d\xi. \tag{3.11}$$

Applying the mean-value theorem and Lemma 2.2, we obtain that for all $t > 0$ and uniformly with respect to $x \in \mathbb{R}^3$

$$|u_1(x,t)| \le (C_2 t + C_3 t^2) e^{-at}, \tag{3.12}$$

where

$$\begin{aligned} C_2 &= \frac{1}{2}\sup_{y\in\mathbb{R}^3} |F_1(y)|, \\ C_3 &= \frac{1+3\alpha}{4}\sup_{y\in\mathbb{R}^3} |F_1(y)| + \frac{1}{2}\sup_{y\in\mathbb{R}^3} |\nabla F_1(y)|. \end{aligned} \tag{3.13}$$

This implies (3.2).

Similarly, setting $x-y=t\xi$ in the representation of u_2 and differentiating the result with respect to t, we find that

$$\begin{aligned}
&u_2(x,t)\\
&\quad=\frac{1}{2\pi}\int_{B(0,1)}\frac{e(t\xi,t)}{|\xi|}f_0(x-t\xi)d\xi\\
&\qquad+\frac{t}{2\pi}\int_{B(0,1)}\left[\frac{\partial_t e(t\xi,t)}{|\xi|}f_0(x-t\xi)-\frac{e(t\xi,t)}{|\xi|}\xi\cdot f_0(x-t\xi)\right]d\xi\\
&\qquad+\frac{t^2}{2\pi}\int_{B(0,1)}\left\{\frac{\partial_t^2 e(t\xi,t)}{|\xi|}f_0(x-t\xi)-\frac{2\partial_t e(t\xi,t)}{|\xi|}\xi\cdot\nabla f_0(x-t\xi)\right.\\
&\qquad\qquad\left.+\frac{e(t\xi,t)}{|\xi|}\left[\xi\cdot((\xi\cdot\nabla)\nabla f_0(x-t\xi))+\xi\cdot\nabla f_0(x-t\xi)\right]\right\}d\xi.
\end{aligned}\tag{3.14}$$

For the second derivative in the direction of ξ appearing in the last expression, the following estimate holds

$$|\xi\cdot((\xi\cdot\nabla)\nabla f_0(y))|\le\sqrt{3\sum_{i,j=1}^{3}|\partial^2_{y_iy_j}f(y)|^2}.\tag{3.15}$$

We rewrite the integral containing $\tilde{e}_{tt}$ as

$$\begin{aligned}
&\int_{B(0,1)}\frac{\tilde{e}_{tt}+2b\tilde{e}_t+b^2\tilde{e}}{|\xi|}f_0(x-t\xi)d\xi-2b\int_{B(0,1)}\frac{\tilde{e}_t+2b\tilde{e}}{|\xi|}f_0(x-t\xi)d\xi\\
&\quad-b^2\int_{B(0,1)}\frac{\tilde{e}}{|\xi|}f_0(x-t\xi)d\xi.
\end{aligned}\tag{3.16}$$

Applying the mean-value theorem and Lemmas 2.2 and 2.3, we obtain the estimate of (3.16). Finally, we get, for all $t>0$ and uniformly with respect to $x\in\mathbb{R}^3$,

$$|u_2(x,t)|\le(C_4+C_5t+C_6t^2)e^{-at},\tag{3.17}$$

where

$$\begin{aligned}
C_4&=\sup_{y\in\mathbb{R}^3}|f_0(y)|,\\
C_5&=\frac{1+3\alpha}{4}\sup_{y\in\mathbb{R}^3}|f_0(y)|+\frac{1}{2}\sup_{y\in\mathbb{R}^3}|\nabla f_0(y)|,\\
C_6&=\frac{1}{2}\left[\left(\frac{1-\alpha}{2}\right)^2+2(1+\alpha)\alpha+\left(\frac{1+\alpha}{2}\right)^2\right]\sup_{y\in\mathbb{R}^3}|f_0(y)|\\
&\quad+\frac{1+3\alpha}{2}\sup_{y\in\mathbb{R}^3}|\nabla f_0(y)|+\frac{1}{2}\sup_{y\in\mathbb{R}^3}\sqrt{3\sum_{i,j=1}^{3}|\partial^2_{y_iy_j}f(y)|^2}.
\end{aligned}\tag{3.18}$$

From the last inequality, (3.3) follows.

It is easy to see that

$$|u_f(x,t)| \le \sup_{\bar{\Gamma}(x,t)} |f(y,\tau)| \int_0^t \tau^2 d\tau \int_{B(0,1)} \frac{e(\tau\xi,\tau)}{|\xi|} d\xi, \tag{3.19}$$

where $\Gamma(x,t) = \{(y,\tau) \mid -(t-\tau) > |y-x|,\ 0 < \tau < t\}$ is an open backward light cone with the base $B(x,t)$. Using (2.21), we arrive at (3.4). The proof is completed. □

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Vladimir Varlamov: Department of Mathematics, University of Texas - Pan American, Edinburg, TX 78539-2999, USA

E-mail address: varlamov@panam.edu