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ON CONFORMAL DILATATION IN SPACE

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We study the conformality problems associated with quasiregular mappings in space. Our approach is based on the concept of the infinitesimal space and some new Grötzsch-Teichmüller type modulus estimates that are expressed in terms of the mean value of the dilatation coefficients.

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1. Introduction. Let G be an open set in $\mathbb{R}^n$. A continuous mapping $f : G \to \mathbb{R}^n$ is called K-quasiregular, $K \geq 1$, if $f \in W^{1,n}_{\text{loc}}(G)$ and if $\|f'(x)\|^n \leq K J_f(x)$ a.e., where $J_f(x)$ stands for the Jacobian determinant of $f'(x)$ and $\|f'(x)\| = \sup |f'(x)h|$, where the supremum is taken over all unit vectors $h \in \mathbb{R}^n$. A homeomorphic K-quasiregular mapping is called K-quasiconformal. We will employ the following distortion coefficients:

$$K_f(x) = \frac{\|f'(x)\|^n}{J_f(x)}, \qquad L_f(x) = \frac{J_f(x)}{\ell(f'(x))^n}, \qquad H_f(x) = \frac{\|f'(x)\|}{\ell(f'(x))}, \tag{1.1}$$

that are called the outer, inner, and linear dilatation of f at x, respectively. Here, $\ell(f'(x)) = \inf |f'(x)h|$. These dilatation coefficients are well defined at regular points of f and, by convention, we let $K_f(x) = L_f(x) = H_f(x) = 1$ at the nonregular points and for a constant mapping.

It is well known that if $n \geq 3$ and one of the dilatation coefficients of a quasiregular mapping f, say $L_f(x)$, is close to 1, then f is close to a Möbius transformation. In spite of this Liouville's phenomenon, the pointwise condition $L_f(x) \to 1$ as $x \to y$, $y \in G$, implies neither conformality for f at y nor the properties typical for the conformal mappings. The mapping

$$f(x) = x(1 - \log|x|), \qquad f(0) = 0, \tag{1.2}$$

shows that $|f(x)|/|x| \to \infty$ as $x \to 0$ although $L_f(x) = (1 - 1/\log|x|)^{n-1} \to 1$. Nevertheless, the conformal behavior of f at a point can be studied in terms of some other measures of closeness of the distortion coefficient to 1. The first such result is due to Teichmüller [29] and Wittich [31]. They proved that if f is a quasiconformal homeomorphism of the unit disk $|z| < 1$ in the complex plane $\mathbb{C}$ onto itself normalized by $f(0) = 0$ and such that

$$\int_{|z|<1} \frac{L_f(z) - 1}{|z|^2} dx\, dy < \infty, \quad z = x + iy, \tag{1.3}$$

then $|f(z)|/|z| = C(1 + \eta(|z|))$ with some $C > 0$ and $\eta(t) \to 0$ as $t \to 0$. In what follows, we call such C the conformal dilatation coefficient of f at 0. Belinskii [3] derived the conformal differentiability of f at 0 from condition (1.3). The complete treatment of the classical Teichmüller-Wittich-Belinskii conformality theorem for quasiconformal mappings in plane is given in [18, Chapter V, Section 6]. Similar problems have been studied by Shabat [27], Lehto [17], Reich and Walczak [23], and Brakalova and Jenkins [6]. Another approach to the investigation of the pointwise behavior of the quasiconformal mappings based on the Beltrami equation is due to Bojarskiĭ [5] (see also [15, 26]).

Consider the class of space radial mappings $f : B \to B$ defined on the unit ball B in $\mathbb{R}^n$ centered at the origin as follows: fix an arbitrary locally integrable function g on $[0,1]$ with $g(t) \geq 1$ for a.e. t, and let

$$f(x) = xe^{-\alpha(|x|)}, \qquad \alpha(|x|) = \int_{|x|}^{1} \frac{g(t) - 1}{t} dt, \qquad f(0) = 0. \tag{1.4}$$

It follows from (1.4) that $g(|x|) = J_f(x)/\ell(f'(x))^n$ a.e., and therefore $g(|x|)$ agrees with the inner dilatation coefficient of f at x. A simple observation shows that f is conformally differentiable at the origin if and only if the integral $\alpha(0)$ in (1.4) converges. For an arbitrary quasiregular mapping $f : B \to B$, $f(0) = 0$, we may consider a condition similar to (1.3), namely

$$\int_{\mathcal{U}} \frac{L_f(x) - 1}{|x|^n} dx < \infty \tag{1.5}$$

for some neighborhood $\mathcal{U}$ of zero, and one can expect that condition (1.5) is sufficient for $f(x)$ to be conformal at $x = 0$. In this direction, we know the only two following statements. Suominen [28] proved that the condition

$$\int_{\mathcal{U}} \frac{(D_f(x))^{1/n} - 1}{|x|^n} dx < \infty, \tag{1.6}$$

equivalent to (1.5), implies that $|f(x)| \sim C|x|$ as $x \to 0$ for K-quasiconformal mappings in Riemannian manifolds. Reshetnyak [25, page 204] showed that the stronger Dini requirement

$$\int_0^1 \frac{\delta_f(t)}{t} dt < \infty, \tag{1.7}$$

where $\delta_f(t) = \operatorname{ess\,sup}_{|x|<t}(K_f(x) - 1)$, guarantees the conformal differentiability of f at 0.

In this paper, we give a direct generalization to nonconstant quasiregular mappings in $\mathbb{R}^n$, $n \geq 2$, of the classical theorem of Teichmüller and Wittich, replacing assumption (1.3) by (1.5), and give bounds for the conformal dilatation coefficient $C = \lim_{x\to 0} |f(x)|/|x|$, see Theorem 3.1. The proof is based on the concept of the infinitesimal space developed in [13] and new Grötzsch-type modulus estimates for quasiregular mappings in $\mathbb{R}^n$, $n \geq 2$, where integrals similar to (1.5) control the distortion. A uniform version of the theorem as well as several consequences concerning, in particular, the study of some rectifiability problems for quasispheres, see [1, 2, 8], are also given. The conformal differentiability under condition (1.5) remains an open problem.

For convenience, we will prove the main statements only for the inner dilatation coefficient $L_f(x)$ because, for the other dilatations, the corresponding results follow from the well-known relations (see, e.g., [30, page 44])

$$L_f(x) \leq K_f^{n-1}(x), \qquad K_f(x) \leq L_f^{n-1}(x), \qquad H_f^n(x) = K_f(x)L_f(x) \tag{1.8}$$

that hold for every $n \geq 2$.

The following standard notations are used in this paper. The norm of a vector $x \in \mathbb{R}^n$ is written as $|x| = \langle x,x\rangle^{1/2} = (x_1^2+\cdots+x_n^2)^{1/2}$, where $x_1,\dots,x_n$ are the coordinates of x and $\langle x,y\rangle$ denotes the usual inner product of vectors x and y in $\mathbb{R}^n$. If $0 < a < b < \infty$, the domain $R(a,b) = B(0,b)\setminus \overline{B}(0,a)$ is called a spherical annulus, where $B(x_0,r)$ is the ball $\{x \in \mathbb{R}^n \mid |x-x_0| < r\}$. A space ring is a domain D such that the boundary ∂D consists of two nonempty connected sets A_1 and A_2 in the compactified space $\overline{\mathbb{R}}^n$.

2. Modulus estimates. Let $\mathscr{C}$ be a family of arcs or curves in space $\mathbb{R}^n$. A nonnegative and Borel measurable function ρ defined in $\mathbb{R}^n$ is called admissible for the family $\mathscr{C}$ if the relation

$$\int_\gamma \rho\, ds \geq 1 \tag{2.1}$$

holds for every locally rectifiable $\gamma \in \mathscr{C}$. The quantity

$$\mathbf{M}(\mathscr{C}) = \inf_\rho \int_{\mathbb{R}^n} \rho^n dx, \tag{2.2}$$

where the infimum is taken over all ρ admissible with respect to the family $\mathscr{C}$ is called the modulus of the family $\mathscr{C}$ (see, e.g, [30, page 16] and [10]). This quantity is a conformal invariant and possesses the monotonicity property which says, in particular, that if $\mathscr{C}_1 < \mathscr{C}_2$, that is, every $\gamma \in \mathscr{C}_2$ has a subcurve which belongs to $\mathscr{C}_1$, then (see, e.g., [30, page 16])

$$\mathbf{M}(\mathscr{C}_1) \geq \mathbf{M}(\mathscr{C}_2). \tag{2.3}$$

Let $\mathcal{R}$ be a space ring whose complement consists of two components C_0 and C_1. A curve γ is said to join the boundary components in $\mathcal{R}$ if γ lies in $\mathcal{R}$, except for its endpoints that lie in different boundary components of $\mathcal{R}$.

In these terms, the modulus of a space ring has the representation (see, e.g., [10, 14])

$$\operatorname{mod}\mathcal{R} = \left(\frac{\omega_{n-1}}{\mathbf{M}(\Gamma)}\right)^{1/(n-1)}, \tag{2.4}$$

where Γ is the family of curves joining the boundary components in $\mathcal{R}$ and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$ (see, e.g., [10, 32]). Note also that the modulus $\mathbf{M}(\Gamma)$ coincides with the conformal capacity of the space ring $\mathcal{R}$ by a result of Loewner [19] (see, e.g., [10]).

In the sequel, we employ only the following two families of curves, lying in the spherical annulus $R(a,b)$, and its images under quasiconformal mappings. The first one that we denote by $\Gamma_{R(a,b)}$ consists of all locally rectifiable curves γ that join the boundary components in $R(a,b)$. The second family $\Gamma^{\nu}_{R(a,b)}$, with $\nu \in S^{n-1}$ fixed, consists of all locally rectifiable curves γ that join in $R(a,b)$ the two components of $L \cap R(a,b)$, where $L = \{t\nu : t \in R\}$ is the line through 0 and ν.

In order to derive the desired estimates, we need the following two statements.

Lemma 2.1. *Let $f : G \to G'$ be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then, for each curve family Γ in G,*

$$\mathbf{M}(f(\Gamma)) \le \int_G \rho^n L_f(x)\,dx \tag{2.5}$$

for every admissible ρ for Γ.

Proof. To prove (2.5), we first recall the inequality due to Väisälä (see [30, page 95])

$$\mathbf{M}(\Gamma) \le \int_G \rho^{*n}(f(x))\|f'(x)\|^n\,dx \tag{2.6}$$

that holds for every curve family Γ in G and every admissible ρ^* for $\Gamma' = f(\Gamma)$.

We give a short proof for (2.6). Let Γ_0 denote the family of all locally rectifiable curves $\gamma \in \Gamma$ such that f is absolutely continuous on every closed subcurve of γ. Since f is ACL^n, it follows from Fuglede's theorem (see, e.g., [30, page 95]) that $\mathbf{M}(\Gamma \setminus \Gamma_0) = 0$. Hence, $\mathbf{M}(\Gamma) = \mathbf{M}(\Gamma_0)$.

Next, let ρ^* be admissible for Γ'. Define $\rho : \mathbb{R}^n \to [0,\infty]$ by $\rho(x) = \rho^*(f(x))L(x,f)$ for $x \in G$ and $\rho(x) = 0$ for $x \notin G$, where

$$L(x,f) = \limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}. \tag{2.7}$$

Then, ρ is a Borel function, and for $\gamma \in \Gamma_0$,

$$\int_\gamma \rho\, ds \geq \int_{f\circ\gamma} \rho^*\, ds \geq 1. \tag{2.8}$$

Thus, ρ is admissible for Γ_0, and therefore

$$\begin{aligned}\mathbf{M}(\Gamma) = \mathbf{M}(\Gamma_0) &\leq \int_G \rho^n dx = \int_G \rho^{*n}(f(x))L^n(x,f)dx \\ &= \int_G \rho^{*n}(f(x))\|f'(x)\|^n dx\end{aligned} \tag{2.9}$$

since f is differentiable a.e. in G and $L(x,f) = \|f'(x)\|$ at every point of differentiability.

Applying formula (2.6) to the inverse of f, we obtain

$$\mathbf{M}(f(\Gamma)) \leq \int_G \rho^n L_f(x)dx \tag{2.10}$$

for every admissible ρ for Γ. □

Lemma 2.2. *Let $\mathcal{R}$ be a space ring that contains the spherical annulus $R(a,b)$, and let E_1 and E_2 be two disjoint subsets of $\mathcal{R}$ such that each sphere $S^{n-1}(t)$, $a<t<b$, meets both E_1 and E_2. If $\mathscr{E}$ is the family of all curves joining E_1 and E_2 in $\mathcal{R}\setminus\{E_1\cup E_2\}$, then*

$$\mathbf{M}(\mathscr{E}) \geq c_n \log\frac{b}{a}, \tag{2.11}$$

where

$$c_n = \frac{1}{2}\omega_{n-2}\left(\int_0^\infty t^{(2-n)/(n-1)}(1+t^2)^{1/(1-n)}\right)^{1-n}. \tag{2.12}$$

If $\mathcal{R} = R(a,b)$ and E_1 and E_2 are the components of $L\cap R(a,b)$, where L is a line through the origin in the direction of a unit vector ν, then

$$\mathbf{M}(\mathscr{E}) = c_n \log\frac{b}{a}. \tag{2.13}$$

This useful result, the proof of which is based on the combination of the space moduli technique and Hardy-Littlewood-Polya's symmetrization principle, is due to Gehring [11] (see also [30, page 31], [7, page 58], and [24, page 108]).

Let $f:\mathbb{R}^n \to \mathbb{R}^n$, $f(0)=0$, $n\geq 2$, be a quasiconformal mapping. We will use the following standard notations:

$$M_f(r)=\max_{|x|=r}|f(x)|, \qquad m_f(r)=\min_{|x|=r}|f(x)|. \tag{2.14}$$

Theorem 2.3. *Let $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(0)=0$, $n\geq 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then, for every spherical annulus $R(a,b)$,*

$$\begin{aligned}\log\frac{b}{a}&-\operatorname{mod} f(R(a,b))\\ &\leq \frac{\operatorname{mod}^n f(R(a,b))}{\sum_{k=1}^{n-1}(\log(b/a))^{n-k}\operatorname{mod}^k f(R(a,b))}\frac{1}{\omega_{n-1}}\int_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx.\end{aligned} \tag{2.15}$$

Proof. Let $R(a,b)$ be a spherical annulus in $\mathbb{R}^n$ and let $\Gamma_{R(a,b)}$ be the family of curves which join the boundary components of $R(a,b)$. Then, (2.5) yields

$$\mathbf{M}(f(\Gamma_{R(a,b)}))\leq\int_{R(a,b)}\rho^n L_f(x)dx \tag{2.16}$$

for every admissible ρ with respect to a family $\Gamma_{R(a,b)}$. Using formula (2.4), we obtain from (2.16)

$$(\operatorname{mod} f(R(a,b)))^{1-n}\leq\frac{1}{\omega_{n-1}}\int_{R(a,b)}\rho^n L_f(x)dx. \tag{2.17}$$

On the other hand, the function

$$\rho_0(x)=\frac{1}{|x|\log(b/a)} \tag{2.18}$$

is admissible with respect to $\Gamma_{R(a,b)}$ since for every curve $\gamma\in\Gamma_{R(a,b)}$,

$$\int_\gamma\rho_0 ds\geq\int_a^b\frac{1}{r\log(b/a)}dr=1. \tag{2.19}$$

Substituting ρ_0 in (2.17) and noting that

$$\frac{1}{\omega_{n-1}}\int_{R(a,b)}\rho_0^n(x)dx=\left(\log\frac{b}{a}\right)^{1-n}, \tag{2.20}$$

we arrive at the inequality

$$\begin{aligned}
&(\operatorname{mod} f(R(a,b)))^{1-n} - \left(\log \frac{b}{a}\right)^{1-n} \\
&\quad \le \frac{1}{\omega_{n-1}(\log(b/a))^n} \int_{R(a,b)} \frac{L_f(x)-1}{|x|^n} dx
\end{aligned} \tag{2.21}$$

that can be rewritten in the form of (2.15). □

Corollary 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \ge 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then, for every spherical annulus $R(a,b)$,*

$$\log \frac{b}{a} - \operatorname{mod} f(R(a,b)) \le \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x)-1}{|x|^n} dx. \tag{2.22}$$

Proof. If $\log(b/a) \le \operatorname{mod} f(R(a,b))$, then inequality (2.22) is trivial. If $\log(b/a) > \operatorname{mod} f(R(a,b))$, then (2.21) can be rewritten as

$$\left(\frac{\beta}{\alpha}\right)^{n-1} - 1 \le \frac{M}{\beta}, \tag{2.23}$$

where $\beta = \log(b/a)$, $\alpha = \operatorname{mod} f(R(a,b))$, and M is the right-hand side of (2.22). Now,

$$\frac{\beta}{\alpha} - 1 \le \left(\frac{\beta}{\alpha}\right)^{n-1} - 1 \le \frac{M}{\beta} \le \frac{M}{\alpha} \tag{2.24}$$

and this gives (2.22). □

Corollary 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \ge 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then, for every spherical annulus $R(a,b)$,*

$$\log \frac{b}{a} - \log \frac{M_f(b)}{m_f(a)} \le \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x)-1}{|x|^n} dx. \tag{2.25}$$

Proof. Since the space ring $f(R(a,b))$ is contained in the spherical annulus $R(m_f(a), M_f(b))$, the monotonicity principle for the modulus yields

$$\operatorname{mod} f(R(a,b)) \le \operatorname{mod} R(m_f(a), M_f(b)) = \log \frac{M_f(b)}{m_f(a)} \tag{2.26}$$

because for every annulus $R(a,b)$,

$$\operatorname{mod} R(a,b) = \log \frac{b}{a}. \tag{2.27}$$

□

Theorem 2.6. *Let $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(0)=0$, $n\geq 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then, for every spherical annulus $R(a,b)$ and each $\nu\in S^{n-1}$,*

$$\mathbf{M}(f(\Gamma^{\nu}_{R(a,b)}))-c_n\log\frac{b}{a}\leq\int_{R(a,b)}\rho_0^n(x,\nu)\frac{L_f(x)-1}{|x|^n}dx,\tag{2.28}$$

where

$$\rho_0(x,y)=\left(\frac{c_n}{\omega_{n-2}}\right)^{1/(n-1)}\left(1-\left\langle\frac{x}{|x|},\frac{y}{|y|}\right\rangle^2\right)^{(2-n)/2(n-1)}\tag{2.29}$$

and c_n is the constant defined by (2.12).

Proof. Fix a unit vector $\nu=y/|y|\in\mathbb{R}^n$ and consider the family $\Gamma^{\nu}_{R(a,b)}$ of curves which join $\{t\nu:-b<t<-a\}$ to $\{t\nu:a<t<b\}$ in $R(a,b)$. By Lemma 2.1,

$$\mathbf{M}(f(\Gamma^{\nu}_{R(a,b)}))\leq\int_{R(a,b)}\rho^nL_f(x)dx\tag{2.30}$$

for each admissible ρ with respect to $\Gamma^{\nu}_{R(a,b)}$.

Now, we show that the function $\rho_\nu(x)=\rho_0(x,y)/|x|$ is admissible for the family $\Gamma^{\nu}_{R(a,b)}$.

Indeed, let γ be a rectifiable curve in $\Gamma^{\nu}_{R(a,b)}$ and let $\varphi(x)=x/|x|$. Then, $\varphi\circ\gamma$ is a curve on S^{n-1} and γ joins the antipodal points $\pm y/|y|$. Since $\|\varphi'(x)\|=1/|x|$, then using the arc length parametrization of γ, we see that

$$\begin{aligned}\int_\gamma\rho_\nu(x)ds&=\int_0^{\ell(\gamma)}\rho_0(\gamma(s),y)\frac{ds}{|\gamma(s)|}\\&=\int_0^{\ell(\gamma)}\rho_0(\gamma(s),y)\|\varphi'(\gamma(s))\|ds\\&\geq\int_0^{\ell(\gamma)}\rho_0(\gamma(s),y)|\varphi'(\gamma(s))\gamma'(s)|ds\\&=\int_0^{\ell(\gamma)}\rho_0(\varphi(\gamma(s)),y)|\varphi'(\gamma(s))\gamma'(s)|ds\\&=\int_{\varphi\circ\gamma}\rho_0(x,y)ds.\end{aligned}\tag{2.31}$$

In order to continue the estimation of the above integral, we rewrite $\rho_0(x,y)$ as

$$\rho_0(x,y)=p_n^{-1}\left(\frac{(1-\langle x/|x|,y/|y|\rangle^2)^{1/2}}{2}\right)^{(2-n)/(n-1)}\tag{2.32}$$

with

$$p_n = 2\int_0^\infty r^{(2-n)/(n-1)}\left(1+r^2\right)^{1/(1-n)} dr \tag{2.33}$$

and introduce a certain coordinate system on the sphere S^{n-1}.

Denote by $\mathbb{V}^{n-1}$ a hyperplane passing through the origin and orthogonal to the vector $y/|y|$. Let $t = P(x) : S^{n-1} \to \mathbb{V}^{n-1}$ be the stereographic projection with the pole at the point $y/|y|$ and $F(t)$ be the inverse mapping. Provide the sphere S^{n-1} with the spherical coordinates $\alpha_1,\dots,\alpha_{n-1}$ in such a way that α_1 stands for the angle between the radius vectors going from the origin to the points x and $-y/|y|$ of the unit sphere. In these terms, $|t| = \tan(\alpha_1/2)$, and therefore $\sin\alpha_1 = 2|t|/(1+|t|^2)$. On the other hand, $1-\langle x/|x|, y/|y|\rangle^2 = \sin^2\alpha_1$, so

$$\rho_0(\cdot,y)\circ F(t) = p_n^{-1}\left(\frac{1+|t|^2}{|t|}\right)^{(n-2)/(n-1)}. \tag{2.34}$$

Since $x = F(t)$ is conformal and $\|F'(t)\| = 2/(1+|t|^2)$, we get

$$\begin{aligned}\int_{\varphi\circ y}\rho_0 ds &= \int_{P\circ\varphi\circ y}\rho_0\circ F\|\|F'(t)\|\||dt| \\ &\geq 2p_n^{-1}\int_0^\infty |t|^{(2-n)/(n-1)}\left(1+|t|^2\right)^{1/(1-n)} d|t| = 1,\end{aligned} \tag{2.35}$$

and hence by (2.31), ρ_ν is admissible for $\Gamma^\nu_{R(a,b)}$.

Since

$$\begin{aligned}\int_{R(a,b)}\rho_\nu^n dx &= \int_a^b\left(\int_{S^{n-1}(r)}\rho_\nu^n dm_{n-1}\right)dr \\ &= \int_a^b\left[\int_{S^{n-1}}\rho_\nu^n(ru)r^{n-1}dm_{n-1}(u)\right]dr \\ &= \int_a^b\frac{dr}{r}\int_{S^{n-1}}\rho_0^n dm_{n-1}(x) = c_n\log\frac{b}{a},\end{aligned} \tag{2.36}$$

we obtain

$$\int_{S^{n-1}}\rho_0^n dm_{n-1}(x) = c_n. \tag{2.37}$$

Inequality (2.30) with $\rho = \rho_\nu$ yields

$$\mathbf{M}\left(f\left(\Gamma^\nu_{R(a,b)}\right)\right) - c_n\log\frac{b}{a} \leq \int_{R(a,b)}\rho_0^n(x,y)\frac{L_f(x)-1}{|x|^n}dx \tag{2.38}$$

and we arrive at the stated conclusion. As for (2.36),

$$\begin{aligned}
&\int_{S^{n-1}} \rho_0^n \, dm_{n-1}(x) Z \\
&\quad = 2^{n-1} \int_{\mathbb{V}^{n-1}} \frac{\rho_0^n(F(t),y)}{(1+|t|^2)^{n-1}} dm_{n-1}(t) \\
&\quad = p_n^{-n} 2^{n-1} \int_{\mathbb{V}^{n-1}} |t|^{n(2-n)/(n-1)} (1+|t|^2)^{n(n-2)/(n-1)} (1+|t|^2)^{1-n} dm_{n-1}(t) \\
&\quad = \omega_{n-2} p_n^{-n} 2^{n-1} \frac{p_n}{2} = \omega_{n-2} p_n^{1-n} 2^{n-2} = c_n.
\end{aligned} \tag{2.39}$$

□

Corollary 2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a K-quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then,*

$$\int_{S^{n-1}} \mathbf{M}(f(\Gamma^{\nu}_{R(a,b)})) dm_{n-1}(\nu) \leq c_n \int_{R(a,b)} \frac{L_f(x) dx}{|x|^n}. \tag{2.40}$$

Proof. The function $\rho_0(x,y)$ is symmetric in the sense that $\rho_0(x,y) = \rho_0(y,x)$, $x, y \in S^{n-1}$, and therefore

$$\int_{S^{n-1}} \rho_0^n(x,y) dm_{n-1}(y) = \int_{S^{n-1}} \rho_0^n(x,y) dm_{n-1}(x) = c_n. \tag{2.41}$$

If we integrate inequality (2.28) with respect to the parameter y over the sphere S^{n-1}, then by Fubini's theorem and relation (2.41), we get the required inequality (2.40). □

Corollary 2.8. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a K-quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then,*

$$\log \frac{m_f(b)}{M_f(a)} - \log \frac{b}{a} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx. \tag{2.42}$$

Proof. If $m_f(b) \leq M_f(a)$, then inequality (2.42) is trivial. Assume that $m_f(b) > M_f(a)$. Then, the space ring $\mathcal{R} = f(R(a,b))$ contains the spherical annulus $R(M_f(a), m_f(b))$. The curve family $\mathcal{E} = f(\Gamma^{\nu}_{R(a,b)})$ satisfies all the assumptions of Lemma 2.2 with respect to $R(M_f(a), m_f(b))$. Therefore, (2.11) and (2.13) imply that

$$\mathbf{M}\Big(f\Big(\Gamma^{\nu}_{R(a,b)}\Big)\Big) \geq \mathbf{M}\Big(\Gamma^{\nu}_{R(M_f(a), m_f(b))}\Big) = c_n \log \frac{m_f(b)}{M_f(a)}. \tag{2.43}$$

This, together with (2.40), yields (2.42). □

The following statements may be of independent interest.

Theorem 2.9. *Let f be a K-quasiconformal mapping of a spherical annulus $R(a,b)$ onto another spherical annulus $R(c,d)$ with the inner dilatation coefficient $L_f(x)$. Then,*

$$\begin{aligned}-\frac{1}{\omega_{n-1}}\int_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx &\le \log\frac{b}{a}-\log\frac{d}{c}\\ &\le \frac{(\log(d/c))^n}{\sum_{k=1}^{n-1}(\log(b/a))^{n-k}\cdot(\log(d/c))^k}\cdot\frac{1}{\omega_{n-1}}\int_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx.\end{aligned} \tag{2.44}$$

Proof. The first inequality follows from Corollary 2.8 and the second one is a consequence of Theorem 2.3.

If f is a K-quasiconformal mapping in the plane, then (2.44) yields

$$\left(\frac{b}{a}\right)^{1/K} \le \frac{d}{c} \le \left(\frac{b}{a}\right)^{K} \tag{2.45}$$

and we recognize the classical Grötzsch inequality for annuli (see, e.g., [18, page 38]). □

Corollary 2.10. *Let f be a K-quasiconformal mapping of a spherical annulus $R(a,b)$ onto another spherical annulus $R(c,d)$ with the inner dilatation coefficient $L_f(x)$. Then,*

$$\left|\log\frac{d}{c}-\log\frac{b}{a}\right| \le \frac{1}{\omega_{n-1}}\int_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx. \tag{2.46}$$

Indeed, if $\log(d/c) > \log(b/a)$, then (2.46) follows from inequality (2.42). If $\log(d/c) < \log(b/a)$, then (2.46) follows from inequality (2.25).

For $n = 2$, we arrive at the modulus estimations under quasiconformal mappings in the plane with the variable dilatation coefficient established by Belinskii [3].

Note that all the inequalities proved in this section remain valid also for ACL^n homeomorphisms in $\mathbb{R}^n$ with locally integrable dilatation coefficients. Moreover, estimates (2.44) and (2.46) are sharp. For instance, the radial mappings of type (1.4) provide the equality in (2.46).

3. Conformal dilatation coefficient.

We apply estimates proved in Section 2 to establish a space version of the regularity problem studied by Teichmüller [29] and Wittich [31].

Theorem 3.1. *Let $f:\mathbb{R}^n\to\mathbb{R}^n$, $n\ge 3$, $f(0)=0$, be a nonconstant K-quasiregular mapping with the inner dilatation coefficient $L_f(x)$ and*

$$I(r) = \frac{1}{\omega_{n-1}}\int_{\mathcal{U}}\frac{L_f(x)-1}{|x|^n}dx < \infty \tag{3.1}$$

for some neighborhood $\mathcal{U}$ *of* 0*. Then, the radius of injectivity of* f *at* 0*,* $R_f(0)$*, satisfies* $R_f(0) > 0$ *and there exists a constant* C *with*

$$\min_{|x|=R} |f(x)| \frac{e^{-I(R)}}{R} \le C \le \max_{|x|=R} |f(x)| \frac{e^{I(R)}}{R}, \quad 0 < R \le R_f(0), \tag{3.2}$$

$$\frac{|f(x)|}{|x|} \longrightarrow C \quad \text{as } x \to 0. \tag{3.3}$$

Remark 3.2. The statement of Theorem 3.1 is also valid if $n = 2$ and f is a homeomorphism.

Corollary 3.3. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$*,* $f(0) = 0$*,* $n \ge 2$*, be a* K*-quasiconformal mapping satisfying (3.1). Then, (3.3) holds and (3.2) can be replaced by estimates*

$$\min_{|x|=1} |f(x)| e^{-I(1)} \le C \le \max_{|x|=1} |f(x)| e^{I(1)}. \tag{3.4}$$

In the case $n = 2$, we arrive at the Teichmüller-Wittich result for K-quasiconformal mappings in the plane (see also [18, Lemma 6.1]). For $n \ge 3$, the asymptotic behavior of f described in Corollary 3.3 has been proved by Suominen [28] for K-quasiconformal mapping in Riemannian manifolds.

It is well known that a sense-preserving locally L-bilipschitz mapping $f : G \to \mathbb{R}^n$ is $L^{2(n-1)}$-quasiregular; a locally L-bilipschitz mapping f satisfies, for each $L' > L$, $x \in G$, and for some $\delta > 0$, the double inequality

$$\frac{1}{L'} \le \frac{|f(y) - f(z)|}{|y - z|} \le L', \tag{3.5}$$

whenever $y, z \in B(x, \delta)$. A more general class than sense-preserving locally bilipschitz mappings is provided by the class of mappings of bounded length distortion (BLD), see [21]. These mappings also form a subclass of quasiregular mappings.

Corollary 3.4. *Let* $f : G \to \mathbb{R}^n$ *be a bilipschitz mapping and*

$$\int_{\mathcal{U}} \frac{L_f(x) - 1}{|x - a|^n} dx < \infty \tag{3.6}$$

for some neighborhood $\mathcal{U}$ *of* $a \in G$*. Then, there is a constant* $C > 0$ *such that*

$$\frac{|f(x) - f(a)|}{|x - a|} \longrightarrow C \quad \text{as } x \longrightarrow a. \tag{3.7}$$

This statement was also proved in [16].

Remark 3.5. If we replace (3.1) by the following stronger requirement:

$$\int_0^1 \frac{\delta_f(t)}{t}dt < \infty, \tag{3.8}$$

where

$$\delta_f(t) = \operatorname*{ess\,sup}_{|x|<t} (K_f(x) - 1), \tag{3.9}$$

then, by the well-known theorem of Reshetnyak (see [25, page 204]), $f(x)$ will be conformally differentiable at the origin.

The well-known theorem of Liouville states that if the dilatation coefficient of a quasiregular mapping is close to 1, then f is close to a Möbius transformation. The next lemma that gives a weak integral condition for this phenomenon will be used for the proof of Theorem 3.1. We recall some basic notions from the space infinitesimal geometry studied in [13].

Let $f : G \to \mathbb{R}^n$, $n \ge 2$, be a nonconstant K-quasiregular mapping, $y \in G$, $t_0 = \operatorname{dist}(y, \partial G)$, and $R(t) = t_0/t$, $t > 0$. For $x \in B(0, R(t))$, we set

$$F_t(x) = \frac{f(tx+y) - f(y)}{\tau(y,f,t)}, \tag{3.10}$$

where

$$\tau(y,f,t) = \left(\frac{\operatorname{meas} f(B(y,t))}{\Omega_n}\right)^{1/n}. \tag{3.11}$$

Here, Ω_n denotes the volume of the unit ball B in $\mathbb{R}^n$. Let $T(y,f)$ be a class of all the limit functions for the family of the mappings F_t as $t \to 0$, where the limit is taken in terms of the locally uniform convergence. The set $T(y,f)$ is called the infinitesimal space for the mapping f at the point y. The elements of $T(y,f)$ are called infinitesimal mappings, and the family (3.10) is called an approximating family for f at y. The family $T(y,f)$ is not empty and consists only of nonconstant K-quasiregular mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ for which $F(0) = 0$, $F(\infty) = \infty$, and $\operatorname{meas} F(B) = \Omega_n$, see [13, Theorem 2.7].

Lemma 3.6. *Let $f : G \to \mathbb{R}^n$, $n \ge 2$, be a nonconstant K-quasiregular mapping with the inner dilatation coefficient $L_f(x)$, and let E be a compact subset of G. If*

$$\frac{1}{\Omega_n t^n}\int_{|x-y|<t} (L_f(x) - 1)\,dx \longrightarrow 0 \quad as\ t \longrightarrow 0 \tag{3.12}$$

uniformly in $y \in E$, then

(i) *for $n \ge 3$, the infinitesimal space $T(y,f)$ consists of linear isometric mappings only;*

(ii) *for $n \ge 3$, the mapping f is locally homeomorphic in E;*

(iii) *the mapping f preserves infinitesimal spheres and spherical annuli centered at y in the sense that*

$$\frac{\max_{|x-y|=r}|f(x)-f(y)|}{\min_{|x-y|=r}|f(x)-f(y)|}\longrightarrow 1 \quad \text{as } r\longrightarrow 0, \tag{3.13}$$

and for each $c \geq 1$, $c^{-1} \leq |x|/|z| \leq c$,

$$\frac{|f(x+y)-f(y)|}{|f(z+y)-f(y)|}-\frac{|x|}{|z|}\longrightarrow 0 \tag{3.14}$$

as $x, z \to 0$ uniformly in $y \in E$.

Proof of Lemma 3.6. (i) Let F_t be the approximating family for f at y. Assume that $t_j \to 0$ as $j \to \infty$ and $F_{t_j}(x) \to F(x)$ locally uniformly as $j \to \infty$. By formula (3.10), we get that

$$K_{F_{t_j}}(x) = K_f(t_j x + y) \quad \text{a.e.}, \tag{3.15}$$

and hence (3.12) can be written as

$$\int_{|x|<R}\Big(K_{F_{t_j}}(x)-1\Big)dx \longrightarrow 0 \quad \text{as } j\longrightarrow\infty \tag{3.16}$$

for every positive constant R. The latter limit implies that $K_{F_{t_j}}(x) \to 1$ as $j \to \infty$ in measure in $\mathbb{R}^n$. Without loss of generality, we may assume that $K_{F_{t_j}}(x) \to 1$ a.e. This can be achieved by passing to a subsequence. By [12, Theorem 3.1], the limit mapping F is a nonconstant 1-quasiregular mapping. Applying Liouville's theorem, we see that F is a Möbius mapping. Since $F(0) = 0$, $F(\infty) = \infty$, and $\operatorname{meas} F(B) = \Omega_n$, we come to the conclusion that F is a linear isometry.

(ii) By [20, Lemma 4.5], we see that

$$\limsup_{j\to\infty} i_{F_{t_j}}(0) \leq i_F(0) = 1, \tag{3.17}$$

where $i_f(x)$ denotes the local topological index of f at x. Thus, all the mappings $F_{t_j}(x)$ are locally injective at 0 for $j > j_0$. By (3.10), we deduce that f is locally injective at y, too.

(iii) Assume the converse. Then, there exist $c \geq 1$, sequences $y_j \in E$, and $x_j, z_j \to 0$ as $j \to \infty$ satisfying the condition $c^{-1} \leq |x_j|/|z_j| \leq c$, such that

$$\left|\frac{|f(x_j+y_j)-f(y_j)|}{|f(y_j+y_j)-f(y_j)|}-\frac{|x_j|}{|y_j|}\right| \geq \varepsilon > 0. \tag{3.18}$$

Consider the following auxiliary family of nonconstant K-quasiregular mappings:

$$F_j(x) = \frac{f(|x_j|x + y_j) - f(y_j)}{\tau(y_j, f, |x_j|)} \tag{3.19}$$

with the distortion coefficients $K_{F_j}(x) = K_f(|x_j|x + y_j)$. Then, the convergence

$$\frac{1}{\Omega_n t^n} \int_{|x-y_j|<t} (L_f(x) - 1)\,dx \longrightarrow 0 \quad \text{as } t \longrightarrow 0, \tag{3.20}$$

is uniform in $y \in E$ with $t = |x_j|R$, $R > 0$, and hence

$$\int_{|x|<R} \left(L_{F_j}(x) - 1\right) dx \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{3.21}$$

for every positive R. Since E is a compact subset of G, then we can repeat the corresponding sequential arguments to show that every limit function for the family of the mappings F_j, as $j \to \infty$, is a linear isometry F. Without loss of generality, we may assume that $F_j \to F$ as $j \to \infty$.

Set $\zeta_j = x_j/|x_j|$ and $w_j = z_j/|x_j|$. We may assume that $\zeta_j \to \zeta_0$, $|\zeta_0| = 1$, and $w_j \to w_0$, $c^{-1} \le |w_0| \le c$, as $j \to \infty$. Otherwise, we can pass to some appropriate subsequences. Since $F_j(\zeta_j) = (f(x_j + y_j) - f(y_j))/\tau(y_j, f, |x_j|) \to F(\zeta_0)$, $F_j(w_j) = (f(z_j + y_j) - f(y_j))/\tau(y_j, f, |x_j|) \to F(w_0)$, and F is linear isometry, it follows that

$$\begin{aligned} 0 &= \frac{|F(\zeta_0)|}{|F(w_0)|} - \frac{|\zeta_0|}{|w_0|} \\ &= \lim_{j\to\infty} \left| \frac{|F_j(\zeta_j)|}{|F_j(w_j)|} - \frac{|x_j|}{|z_j|} \right| \\ &= \lim_{j\to\infty} \left| \frac{|f(x_j + y_j) - f(y_j)|}{|f(z_j + y_j) - f(y_j)|} - \frac{|x_j|}{|z_j|} \right|. \end{aligned} \tag{3.22}$$

Formula (3.22) provides a contradiction to inequality (3.18). Now, (3.13) is a consequence of (3.14). □

Proof of Theorem 3.1. Let $f : G \to \mathbb{R}^n$, $n \ge 3$, be a nonconstant K-quasiregular mapping. For every such mapping $f(x)$ and every $y \in G$, we define the radius of injectivity $R_f(y)$ of f at y as a supremum over all $\rho > 0$ such that $f(x_1) \ne f(x_2)$ for $x_1 \ne x_2$ in the ball $|x - y| < \rho$ in G, see [20].

Assume that the integral

$$I(r) = \frac{1}{\omega_{n-1}} \int_{|x|<r} \frac{L_f(x) - 1}{|x|^n}\,dx \tag{3.23}$$

converges for some $r > 0$. Now,

$$\frac{1}{r^n}\int_{|x|<r}(L_f(x)-1)\,dx \le \int_{|x|<r}\frac{L_f(x)-1}{|x|^n}\,dx, \tag{3.24}$$

and hence

$$\frac{1}{\Omega_n r^n}\int_{|x|<r}(L_f(x)-1)\,dx \longrightarrow 0, \quad \text{as } r \longrightarrow 0, \tag{3.25}$$

and we make use of the weak conformality result stated in Lemma 3.6. In particular, the mapping f is locally homeomorphic at the origin $R_f(0) > 0$, and that

$$\lim_{r\to 0}\log\frac{M_f(r)}{m_f(r)} = 0. \tag{3.26}$$

Hence, in order to deduce (3.3), it suffices to show that

$$\lim_{r\to 0}\log\frac{M_f(r)}{r} = a \tag{3.27}$$

and we do this by showing that the Cauchy criterion

$$-\varepsilon < \log\frac{M_f(r_2)}{M_f(r_1)} - \log\frac{r_2}{r_1} < \varepsilon \tag{3.28}$$

holds for $0 < r_1 < r_2 < \delta$.

Fix a positive number R, $0 < R < R_f(0)$ and first prove the first inequality in (3.28).

The convergence of the integral (3.23) implies that given $\varepsilon > 0$, there exists $\delta > 0$ such that $I(\delta) < \varepsilon/2$. Therefore, for every $0 < r_1 < r_2 < \delta$ by Corollary 2.5,

$$\log\frac{r_2}{r_1} - \log\frac{M_f(r_2)}{m_f(r_1)} \le I(\delta) < \frac{\varepsilon}{2}. \tag{3.29}$$

By (3.26), we can assume as well that

$$\log\frac{M_f(r_2)}{m_f(r_1)} = \log\frac{M_f(r_2)}{M_f(r_1)} + \log\frac{M_f(r_1)}{m_f(r_1)} \le \log\frac{M_f(r_2)}{M_f(r_1)} + \frac{\varepsilon}{2}. \tag{3.30}$$

From (3.29) and (3.30), we derive the first inequality in (3.28).

For the second inequality in (3.28), we may assume that, by Corollary 2.8,

$$\log \frac{m_f(r_2)}{M_f(r_1)} - \log \frac{r_2}{r_1} \le I(\delta) < \frac{\varepsilon}{2}. \tag{3.31}$$

From (3.26), we see that

$$\log \frac{M_f(r_2)}{M_f(r_1)} - \log \frac{m_f(r_2)}{M_f(r_1)} = \log \frac{M_f(r_2)}{m_f(r_2)} < \frac{\varepsilon}{2}. \tag{3.32}$$

Combining (3.31) with (3.32), we obtain the second inequality of (3.28) and, therefore, the aforementioned Cauchy criterion.

In order to prove inequalities (3.2), first note that by Corollary 2.5,

$$\log \frac{R}{r} - \log \frac{M_f(R)}{m_f(r)} < I(R) \tag{3.33}$$

for every $0 < r \le R$. Using relation (3.26), we deduce that

$$\log \frac{M_f(r)}{r} < \log \frac{M_f(R)}{R} + I(R) + \varepsilon(r), \tag{3.34}$$

where $\varepsilon(r) \to 0$ as $t \to 0$. Thus,

$$\lim_{r\to 0} \log \frac{M_f(r)}{r} \le \log \frac{M_f(R)}{R} + I(R). \tag{3.35}$$

Next, by Corollary 2.8,

$$\log \frac{m_f(R)}{M_f(r)} - \log \frac{R}{r} < I(R). \tag{3.36}$$

Since (3.36) implies that

$$\lim_{r\to 0} \log \frac{M_f(r)}{r} \ge \log \frac{m_f(R)}{R} - I(R), \tag{3.37}$$

the proof is complete. □

The following statement is a stronger version of Theorem 3.1.

Theorem 3.7. *Let $f : G \to \mathbb{R}^n$, $n \ge 3$, be a nonconstant K-quasiregular mapping, and let E be a compact set in G. If the improper integral*

$$I(y, \mathcal{U}) = \frac{1}{\omega_{n-1}} \int_{\mathcal{U}} \frac{L_f(x) - 1}{|x - y|^n} dx \tag{3.38}$$

converges uniformly in $y \in E$ for some neighborhood $\mathcal{U}$ of E, then there exists a positive continuous function $C(y)$, $y \in E$, such that

$$\frac{|f(x) - f(y)|}{|x - y|} \longrightarrow C(y) \quad \text{as } x \longrightarrow y \tag{3.39}$$

uniformly in $y \in E$, and for $0 < R < R_f(y)$,

$$\begin{aligned} &\min_{|x-y|=R} |f(x) - f(y)| \frac{e^{-I(y,B(y,R))}}{R} \\ &\quad \le C(y) \le \max_{|x-y|=R} |f(x) - f(y)| \frac{e^{I(y,B(y,R))}}{R}. \end{aligned} \tag{3.40}$$

Here, $R_f(y)$ stands for the radius of injectivity of f at y.

Proof. For each fixed $y \in E$, we consider the following auxiliary K-quasiregular mappings:

$$F(x) = f(x + y) - f(y) \tag{3.41}$$

defined for $|x - y| < \operatorname{dist}(y, \partial G)$. Denoting by $L_F(x, y)$ the inner dilatation coefficient for F, we see that $L_F(x, y) = L_f(x + y)$ a.e. in a neighborhood of the point $y \in E$. Then, the convergence of $I(y, \mathcal{U})$ in E implies that for every fixed $y \in E$, there exists an $r > 0$ such that

$$\int_{B(r)} \frac{L_F(x, y) - 1}{|x|^n} dx < \infty. \tag{3.42}$$

So, the mapping F satisfies all the conditions of Theorem 3.1, and hence

$$\frac{|F(x)|}{|x|} = \frac{|f(x + y) - f(y)|}{|x|} \longrightarrow C(y) \quad \text{as } x \longrightarrow 0 \tag{3.43}$$

for every fixed $y \in E$.

In order to show that the limit (3.43) is uniform with respect to $y \in E$, we have to analyze the proof of Theorem 3.1 and to make use of the uniform convergence of $I(y, \mathcal{U})$ in $y \in E$. Recall that its proof is based on the following two distortion estimates of Corollaries 2.8 and 2.5:

$$\log \frac{r_2}{r_1} - \log \frac{M_F(r_2)}{m_F(r_1)} \le \frac{1}{\omega_{n-1}} \int_{R(r_1, r_2)} \frac{L_F(x, y) - 1}{|x|^n} dx, \tag{3.44}$$

$$\log \frac{m_F(r_2)}{M_F(r_1)} - \log \frac{r_2}{r_1} \le \frac{1}{\omega_{n-1}} \int_{R(r_1, r_2)} \frac{L_F(x, y) - 1}{|x|^n} dx, \tag{3.45}$$

and the weak conformality result

$$\log \frac{M_F(r)}{m_F(r)} \longrightarrow 0 \quad \text{as } r \longrightarrow 0, \tag{3.46}$$

provided by Lemma 3.6. Now the uniform convergence of $I(y,\mathcal{U})$ with respect to $y \in E$ and Lemma 3.6 imply the uniform convergence in (3.46). Hence, from (3.44) and (3.46) and the uniform convergence of $I(y,\mathcal{U})$, we obtain that for every $\varepsilon > 0$, there is $\delta > 0$ such that $0 < r_1 < r_2 < \delta$ implies that

$$\left| \log \frac{M_F(r_2)}{M_F(r_1)} - \log \frac{r_2}{r_1} \right| < \varepsilon \tag{3.47}$$

for every $y \in E$, where

$$M_F(r) = \max_{|x|=r} |F(x)| = \max_{|x|=r} |f(x+y) - f(y)|. \tag{3.48}$$

Thus, we have arrived at the Cauchy criterion for the function $M_F(r)/r$ to converge to a nonzero limit uniformly in $y \in E$. The proof is complete. □

Corollary 3.8. *Let $f : G \to \mathbb{R}^n$ be a locally bilipschitz mapping, let E be a compact set in G, and let the integral*

$$\int_{\mathcal{U}} \frac{L(x)-1}{|x-y|^n} dx \tag{3.49}$$

converge uniformly in $y \in E$ for some neighborhood $\mathcal{U}$ of E. Then, there exists a positive continuous function $C(y)$, $y \in E$, such that

$$\frac{|f(x)-f(y)|}{|x-y|} \longrightarrow C(y) \quad \text{as } x \longrightarrow y \tag{3.50}$$

uniformly in $y \in E$.

This statement follows immediately from Theorem 3.7 if we recall that every locally L-bilipschitz mapping in G is K-quasiregular with $K \le L^{2(n-1)}$.

Corollary 3.9. *Let $f : G \to \mathbb{R}^n$, $n \ge 2$, be a K-quasiconformal mapping, and let E be a compact subset of G. If the improper integral*

$$\int_{\mathcal{U}} \frac{L_f(x)-1}{|x-y|^n} dx \tag{3.51}$$

converges uniformly in $y \in E$ for some neighborhood $\mathcal{U}$ of E, then there exists a positive constant L such that

$$\frac{1}{L}|x-z| \le |f(x)-f(z)| \le L|x-z|, \tag{3.52}$$

whenever $x, z \in E$.

Proof. We first show that

$$M = \sup_{x,z \in E, x \neq z} \frac{|f(x) - f(z)|}{|x - z|} < \infty. \tag{3.53}$$

Assume the converse. Then, there exist sequences $x_j, z_j \in E$ such that

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = \infty. \tag{3.54}$$

Without loss of generality, we may assume that $x_j \to x_0$ and $z_j \to z_0$. Since E is a compact set, then $x_0, z_0 \in E$. If $x_0 \neq z_0$, then

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = \frac{|f(x_0) - f(z_0)|}{|x_0 - z_0|} \neq \infty. \tag{3.55}$$

If $x_0 = z_0 = y$, then

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = C(y) \tag{3.56}$$

by Theorem 3.7. Since $C(y) < \infty$, then (3.56) provides a contradiction to (3.54).

Repeating the preceding arguments and taking into account both the injectivity of f in G and the inequality $C(y) > 0$, $y \in E$, we get that

$$N = \inf_{x,z \in E, x \neq z} \frac{|f(x) - f(z)|}{|x - z|} > 0. \tag{3.57}$$

Inequalities (3.53) and (3.57) imply the existence of a positive constant L such that (3.52) holds whenever $x, z \in E$. □

It is well known that a quasiconformal mapping $f : G \to \mathbb{R}^n$ being an ACL^n homeomorphism need not preserve the Hausdorff dimension of some subsets E of G of a smaller dimension than n, and the image $f(\gamma)$ of a rectifiable curve $\gamma \subset G$ under quasiconformal mapping f may fail to be rectifiable. The following statement, a consequence of (3.52), provides a sufficient condition for the rectifiability of $f(\gamma)$.

Corollary 3.10. *Let $f : G \to \mathbb{R}^n$, $n \geq 2$, be a K-quasiconformal mapping, and let γ be a compact rectifiable curve in G. If the improper integral*

$$\int_{\mathcal{U}} \frac{L_f(x) - 1}{|x - y|^n} dx \tag{3.58}$$

converges uniformly in $y \in \gamma$ *for some neighborhood* $\mathcal{U}$ *of* γ*, then* $\Gamma = f(\gamma)$ *is rectifiable.*

Formula (3.52) provides the following double inequality:

$$\frac{1}{L} \leq \frac{\text{length } f(\gamma)}{\text{length } \gamma} \leq L, \tag{3.59}$$

and the constant L can be also estimated by means of formula (3.40).

Theorem 3.7 provides a bilipschitz condition for f on compact subsets E of G and hence the rectifiability of $f(E)$ and the absolute continuity properties of f on such sets E can be derived from Theorem 3.7 as in Corollary 3.10. In particular, rectifiability properties of quasispheres, that is, images of S^{n-1} under quasiconformal mappings can be derived from Theorem 3.7. We first recall some definitions and previous results.

For a set $E \subset \mathbb{R}^n$ and for $\delta > 0$, let

$$\Lambda_\alpha^\delta(E) = \gamma_{n,\alpha} \inf_{\{B_j\}} \sum_j d(B_j)^\alpha, \tag{3.60}$$

where the infimum is taken over all countable coverings $\{B_j\}$ of E with $d(B_j) < \delta$. Here, the B_j are balls of $\mathbb{R}^n$ and $d(B_j)$ is the diameter of B_j (see [9, page 7]). The constant $\gamma_{n,\alpha}$ in (3.60) is the normalizing constant. The quantity

$$\Lambda_\alpha(E) = \lim_{\delta \to 0} \Lambda_\alpha^\delta(E), \tag{3.61}$$

finite or infinite, is called the α-dimensional normalized Hausdorff measure of the set E.

Mattila and Vuorinen [22] proved that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal, $K(t) = K(f \mid (B(x,1+t) \setminus B(x,1-t)))$, $0 < t < 1$, where $K(f)$ denotes the maximal dilatation of the mapping f, then the Dini condition

$$\int_0^1 \frac{K(t)^{1/(n-1)} - 1}{t} dt < \infty \tag{3.62}$$

implies that $\Lambda_{n-1}(f(S^{n-1})) < \infty$.

This result can be extended. First, the well-known theorem of Reshetnyak states that the Dini condition (3.62) implies the uniform conformal differentiability of the mapping f in S^{n-1} (see [25, page 204]). Hence, (3.62) gives a sufficient condition for the quasisphere $f(S^{n-1})$ to be smooth. On the other hand, the following statement provides a condition weaker than (3.62) for the rectifiability of $f(S^{n-1})$.

Corollary 3.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, be a K-quasiconformal mapping and suppose that the improper integral*

$$\int_{\mathcal{U}} \frac{L_f(x) - 1}{|x - y|^n} dx \tag{3.63}$$

converges uniformly in $y \in S^{n-1}$ for some neighborhood $\mathcal{U}$ of S^{n-1}. Then,

$$\Lambda_{n-1}(f(S^{n-1})) \leq L^{n-1} \omega_{n-1}, \tag{3.64}$$

where

$$L = \sup_{x, y \in S^{n-1}, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}. \tag{3.65}$$

Finally, we note one interesting aspect of Theorems 3.1 and 3.7. These statements can give new results for quasiconformal mappings in the plane by first extending them to higher dimension. For example, consider a quasiconformal mapping f of the plane to itself which conjugates the actions of two Kleinian groups. The dilatation of such a mapping may be uniformly bounded away from 1 a.e., and hence the two-dimensional versions of the results due to Teichmüller, Wittich, and Belinskii tells us nothing. However, such a mapping can be extended to three dimensions in a conformally natural way, and in some cases, one can show that the extension satisfies (3.1) a.e. with respect to the Patterson-Sullivan measure on the limit set. This particular example is described in detail in [4].

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