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REMARKS ON EMBEDDABLE SEMIGROUPS IN GROUPS AND A GENERALIZATION OF SOME CUTHBERT'S RESULTS

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Let $(U(t))_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a Banach space X. In this paper, we establish that if, for some $t_0>0$, $U(t_0)$ is a Fredholm (resp., semi-Fredholm) operator, then $(U(t))_{t\geq 0}$ is a Fredholm (resp., semi-Fredholm) semigroup. Moreover, we give a necessary and sufficient condition guaranteeing that $(U(t))_{t\geq 0}$ can be embedded in a C_0-group on X. Also we study semigroups which are near the identity in the sense that there exists $t_0>0$ such that $U(t_0)-I\in\mathcal{J}(X)$, where $\mathcal{J}(X)$ is an arbitrary closed two-sided ideal contained in the set of Fredholm perturbations. We close this paper by discussing the case where $\mathcal{J}(X)$ is replaced by some subsets of the set of polynomially compact perturbations.

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1. Introduction. Let X be a Banach space over the complex field and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators on X. The subset of all compact operators of $\mathcal{L}(X)$ is designated by $\mathcal{K}(X)$. For $A\in\mathcal{L}(X)$, we let $\sigma(A)$, $\rho(A)$, $R(\lambda,A)$, $N(A)$, and $R(A)$ denote the spectrum, the resolvent set, the resolvent operator, the null space, and the range of A, respectively. The nullity of A, $\alpha(A)$, is defined as the dimension $N(A)$ and the deficiency of A, $\beta(A)$, is defined as the codimension of $R(A)$ in X.

Write

$$\begin{aligned}\Phi_+(X)&=\{A\in\mathcal{L}(X):\alpha(A)<\infty,\ R(A)\text{ is closed in }X\},\\ \Phi_-(X)&=\{A\in\mathcal{L}(X):\beta(A)<\infty\ (\text{then }R(A)\text{ is closed in }X)\}.\end{aligned}\tag{1.1}$$

By $\Phi_\pm(X):=\Phi_+(X)\cup\Phi_-(X)$ we denote the set of semi-Fredholm operators in $\mathcal{L}(X)$, while $\Phi(X):=\Phi_+(X)\cap\Phi_-(X)$ is the set of Fredholm operators in $\mathcal{L}(X)$. If $A\in\Phi_\pm(X)$, the number $i(A)=\alpha(A)-\beta(A)$, a finite or infinite integer is the index of A. Let X^* denotes the dual space of X and A^* the dual operator of A.

Let $(U(t))_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on X. We say that $(U(t))_{t\geq 0}$ is a Fredholm (resp., semi-Fredholm) semigroup if $U(t)$ is in $\Phi(X)$ (resp., $\Phi_\pm(X)$) for all $t>0$.

In [7, Theorem 16.3.6], it is proved that a C_0-semigroup of bounded linear operators $(U(t))_{t\geq 0}$ can be embedded in a C_0-group if and only if there exists $t_0 > 0$ such that $0 \in \rho(U(t_0))$. The main goal of Section 2 is to give a generalization of this result to Fredholm semigroup. Our approach consists in relaxing the requirement *there exists* $t_0 > 0$ *such that* $0 \in \rho(U(t_0))$ and replacing it by the weaker one *there exists* $t_0 > 0$ *such that* $U(t_0) \in \Phi(X)$. In fact, we prove under this hypothesis that $(U(t))_{t\geq 0}$ is a Fredholm semigroup, that is, $U(t) \in \Phi(X)$ for all $t \geq 0$. In particular, we show that if there exists $t_0 > 0$ such that $U(t_0) \in \Phi_\pm(X)$, then $(U(t))_{t\geq 0}$ is a semi-Fredholm semigroup, that is, $U(t) \in \Phi_\pm(X)$ for all $t \geq 0$.

In Section 3, we extend some results owing to Cuthbert [2] which deal with C_0-semigroups having the property of being near the identity, in the sense that, for some value of t, $U(t) - I \in \mathcal{K}(X)$. We show that Cuthbert's results remain valid if, for some $t_0 > 0$, $U(t_0) - I \in \mathcal{J}(X)$ where $\mathcal{J}(X)$ is an arbitrary closed two-sided ideal of $\mathcal{L}(X)$ contained in the ideal of Fredholm perturbations $\mathcal{F}(X)$. In the last section, some generalizations of the results obtained in Section 3 to polynomially compact perturbations are also given.

2. Embeddable C_0-semigroups in C_0-groups.

Let X be a Banach space and let $(U(t))_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on X.

Theorem 2.1. *A C_0-semigroup $(U(t))_{t\geq 0}$ can be embedded in a C_0-group on X if and only if there exists $t_0 > 0$ such that $U(t_0) \in \Phi(X)$.*

To prove Theorem 2.1, the following proposition is required.

Proposition 2.2. *Let $t_0 > 0$ and let $(U(t))_{t\geq 0}$ be a C_0-semigroup on X.*
(i) *If $U(t_0) \in \Phi_+(X)$, then $U(t) \in \Phi_+(X)$ and $\alpha(U(t)) = 0$ for all $t \geq 0$.*
(ii) *If $U(t_0) \in \Phi_-(X)$, then $U(t) \in \Phi_-(X)$ and $\beta(U(t)) = 0$ for all $t \geq 0$.*
(iii) *If $U(t_0) \in \Phi(X)$, then $U(t) \in \Phi(X)$ and $i(U(t)) = 0$ for all $t \geq 0$.*

Obviously, Proposition 2.2 shows that if, for some $t_0 > 0$, $U(t_0) \in \Phi_\pm(X)$, then $(U(t))_{t\geq 0}$ is a semi-Fredholm semigroup. In the case where $U(t_0) \in \Phi(X)$, $(U(t))_{t\geq 0}$ is a Fredholm semigroup and $i(U(t)) = 0$ for all $t \geq 0$.

Proof of Proposition 2.2. (i) We first show that $U(t_0)$ is injective. Since $\alpha(U(t_0)) < \infty$, then 0 is an eigenvalue with finite multiplicity of $U(t_0)$. Let $x \neq 0$ be an eigenvector associated to 0. Putting $t_1 = t_0/2$, then $U(t_0)x = U(t_1)U(t_1)x = 0$, hence 0 is an eigenvalue of $U(t_1)$. Proceeding by induction, we define a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $U(t_n)$, $\forall n \in \mathbb{N}$. For $n \geq 0$, we define the sets

$$\Lambda_n = N(U(t_n)) \bigcap \{x \in X : \|x\| = 1\}. \tag{2.1}$$

Clearly, the inclusion $N(U(s)) \subseteq N(U(t))$, for $s \leq t$, and the compactness of Λ_0 imply that $(\Lambda_n)_n$ is a decreasing sequence (in the sense of the inclusion) of nonempty compact subsets of X. Thus $\bigcap_{n=0}^{\infty}\Lambda_n \neq \emptyset$. If $x \in \bigcap_{n=0}^{\infty}\Lambda_n$, then

$$||U(t_n)x - x|| = \|x\| = 1 \quad \forall n \geq 1. \tag{2.2}$$

Since $t_n \to 0$ as $n \to \infty$, (2.2) contradicts the strong continuity of $(U(t))_{t\geq 0}$. This shows that $N(U(t_0)) = \{0\}$, that is, $\alpha(U(t_0)) = 0$.

Let $0 \leq t \leq t_0$. The inclusion $N(U(t)) \subseteq N(U(t_0))$ implies that $\alpha(U(t)) = 0$. Assume now that $t > t_0$ and $x \in N(U(t))$, then there exists an integer n such that $nt_0 > t$ and therefore $U(nt_0)x = U(nt_0 - t)U(t)x = 0$. Hence, we have $x = 0$ and consequently $N(U(t)) = \{0\}$ for all $t > t_0$ which ends the proof of (i).

(ii) To prove this item, we will proceed by duality. Let $(U^*(t))_{t\geq 0}$ be the dual semigroup of $(U(t))_{t\geq 0}$. Since $\beta(U(t)) = \alpha(U^*(t))$, then it suffices to show that $\alpha(U^*(t)) = 0$ for all $t \geq 0$. By hypothesis, we have $\alpha(U^*(t_0)) < \infty$. Let x^* be an element of $N(U^*(t_0))$. Arguing as above, we construct a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to 0$ as $n \to \infty$ such that 0 is an eigenvalue of $U^*(t_n)$, for all $n \in \mathbb{N}$ a decreasing sequence

$$\Sigma_n = N(U^*(t_n)) \bigcap \{x^* \in X^* : ||x^*|| = 1\} \tag{2.3}$$

of nonempty compact subsets of X^*. We infer that $\bigcap_{n=0}^{\infty}\Sigma_n \neq \emptyset$. Let $x^* \in \bigcap_{n=0}^{\infty}\Sigma_n$, then for all $n \in \mathbb{N}$

$$|\langle U^*(t_n)x^* - x^*, x\rangle| = |\langle x^*, x\rangle| \quad \forall x \in X. \tag{2.4}$$

Using the fact that $(U^*(t))_{t\geq 0}$ is continuous in the weak* topology at $t = 0$, we conclude that

$$\lim_{t\to 0} |\langle U^*(t_n)x^* - x^*, x\rangle| = 0 \quad \forall x \in X. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain $\langle x^*, x\rangle = 0$ for all $x \in X$. This shows that $x^* = 0$ and therefore $\alpha(U^*(t_0)) = 0$. Arguing as above, we show that $\alpha(U^*(t)) = 0$ for all $t \geq 0$.

(iii) This follows from (i) and (ii).

To complete the proof of (i) it suffices to show that $R(U(t))$ is closed in X for all $t \geq 0$. Assume that $U(t_0) \in \Phi_+(X)$, then $\alpha(U(t_0)) < \infty$ and $\beta(U(t_0)) = \infty$ (if $\beta(U(t_0)) < \infty$ the proof is contained in (ii) see below). Let $U^*(t_o)$ be the dual operator of $U(t_0)$. Obviously, $U^*(t_0) \in \Phi_-(X)$ and consequently $\beta(U^*(t_0)) < \infty$. Hence $\beta(U^*(t)) < \infty$ for all $t \geq 0$. Now applying Kato's lemma [8, Lemma 332] we infer that $R(U^*(t))$ is closed in X^* for all $t \geq 0$. This together with the closed graph theorem of Banach [15, page 205] implies that $R(U(t))$ is closed in X for all $t \geq 0$.

Assume now that $U(t_0) \in \Phi_-(X)$, then $\beta(U(t_0)) < \infty$ and $\alpha(U(t_0)) = \infty$ (if $\alpha(U(t_0)) < \infty$ the proof is contained in (i)). It follows from the first part of the statement (ii) that $\beta(U(t)) < \infty$ for all $t \geq 0$. Again using Kato's lemma

[8, Lemma 332] we see that $R(U(t))$ is closed in X for all $t \geq 0$ which completes the proof of (ii).

Now if $U(t_0) \in \Phi(X)$, then $\alpha(U(t_0)) < \infty$ and $\beta(U(t_0)) < \infty$. It follows from the discussion above that $R(U(t))$ is closed in X for all $t \geq 0$. This ends the proof of Proposition 2.2. □

Proof of Theorem 2.1. The proof follows immediately from Proposition 2.2 and [7, Theorem 16.3.6]. □

3. An extension of some results by Cuthbert. Throughout this section X denotes a Banach space and $(U(t))_{t\geq 0}$ designates a strongly continuous semigroup with infinitesimal generator A.

As mentioned in the introduction, this section is motivated by Cuthbert's work [2] dealing with C_0-semigroups which have the property of being near the identity, in the sense that, for some positive value of $t > 0$, $U(t)-I \in \mathcal{K}(X)$. We discuss the possibility of extending Cuthbert's results to other operator ideals of $\mathcal{L}(X)$. To this purpose, we introduce the concept of Fredholm perturbations (see [1, 4, 12]).

Definition 3.1. We say that an operator $F \in \mathcal{L}(X)$ is a Fredholm perturbation if $A+F \in \Phi(X)$ whenever $A \in \Phi(X)$. The operator F is called an upper (resp., lower) semi-Fredholm perturbation if $F+A \in \Phi_+(X)$ (resp., $F+A \in \Phi_-(X)$) whenever $A \in \Phi_+(X)$ (resp., $A \in \Phi_-(X)$).

The sets of Fredholm, upper semi-Fredholm, and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, and $\mathcal{F}_-(X)$, respectively. These sets of operators were introduced and investigated in [4] (see also [12]). In particular, it is proved that $\mathcal{F}_+(X)$ and $\mathcal{F}(X)$ are closed two-sided ideals of $\mathcal{L}(X)$ while $\mathcal{F}_-(X)$ is a closed subset of $\mathcal{L}(X)$.

Our main objective here is to show that Cuthbert's results remain valid if we replace $\mathcal{K}(X)$ by any closed two-sided ideal contained in $\mathcal{F}(X)$.

In the following, $\mathcal{J}(X)$ denotes an arbitrary *nonzero closed two-sided ideal* of $\mathcal{L}(X)$ satisfying

$$\mathcal{J}(X) \subseteq \mathcal{F}(X). \tag{3.1}$$

Remark 3.2. (1) It is worth noticing that, in general, the structure ideal of $\mathcal{L}(X)$ is extremely complicated. Most of the results on ideal structure deal with the well-known closed ideals which have arisen from applied work with operators. We can quote, for example, compact operators, weakly compact operators, strictly singular operators, strictly cosingular operators, upper semi-Fredholm perturbations, and Fredholm perturbations. In general, we have

$$\begin{aligned} &\mathcal{K}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{F}_+(X) \subseteq \mathcal{F}(X), \\ &\mathcal{K}(X) \subseteq C\mathcal{S}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{F}(X), \end{aligned} \tag{3.2}$$

where $\mathcal{S}(X)$ and $C\mathcal{S}(X)$ denote, respectively, the ideals of $\mathcal{L}(X)$ consisting of strictly singular and strictly cosingular operators on X. The inclusion $\mathcal{S}(X) \subseteq \mathcal{F}_+(X)$ is due to Kato (cf. [8]) while $C\mathcal{S}(X) \subseteq \mathcal{F}_-(X)$ was proved by Vladimirskiĭ [13].

(2) If X is isomorphic to an L_p space with $1 \le p \le \infty$ or to $C(\Xi)$ where Ξ is a compact Hausdorff space, then we have

$$\mathcal{K}(X) \subseteq \mathcal{S}(X) = \mathcal{F}_+(X) = C\mathcal{S}(X) = \mathcal{F}_-(X) = \mathcal{F}(X) \tag{3.3}$$

(cf. [9, equations (2.9) and (2.10)]).

A Banach space X is said to be an h-space if each closed infinite-dimensional subspace of X contains a complemented subspace isomorphic to X [14]. Any Banach space isomorphic to an h-space; c, c_0 and l_p $(1 \le p < \infty)$ are h-spaces. In [14, Theorem 6.2], Whitley proved that, if X is an h-space, then $\mathcal{S}(X)$ is the greatest proper ideal of $\mathcal{L}(X)$. This, together with (3.2), implies that

$$\mathcal{K}(X) \subseteq \mathcal{F}_+(X) = \mathcal{S}(X) = \mathcal{F}(X), \qquad \mathcal{K}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{S}(X) = \mathcal{F}(X). \tag{3.4}$$

We denote by $\mathbb{O}$ the set

$$\mathbb{O} = \{t > 0 \text{ such that } U(t) - I \in \mathcal{J}(X)\}. \tag{3.5}$$

It should be noted that for a given C_0-semigroup, the set $\mathbb{O}$ can be empty.

Remark 3.3. Note that, under assumption (3.1), if $\mathbb{O} \neq \emptyset$, then the C_0-semigroup $(U(t))_{t\ge 0}$ can be embedded in a C_0-group on X. (It suffices to write $U(t_0) = I + [U(t_0) - I]$ for some $t_o \in \mathbb{O}$ and to apply Theorem 2.1.) This statement improves [2, Theorem 1].

Observe that the relation

$$(U(t)-I)(U(s)-I) = (U(t+s)-I) - (U(s)-I) - (U(t)-I), \tag{3.6}$$

implies that

$$s \in \mathbb{O},\ t \in \mathbb{O} \Longrightarrow s+t \in \mathbb{O}, \qquad s \in \mathbb{O},\ t \notin \mathbb{O} \Longrightarrow s+t \notin \mathbb{O}. \tag{3.7}$$

It follows from these relations that $\mathbb{O}$ is the intersection of an additive subgroup of real number with the positive real line. Therefore, $\mathbb{O}$ may be in one of the following forms:

(i) $\mathbb{O} =]0, \infty[$;
(ii) $\mathbb{O} = \{nx$, for some $x > 0$; and $n = 1, 2, \dots\}$;
(iii) $\mathbb{O}$ is a dense subset of $]0, \infty[$ with empty interior.

The following examples taken from [2] show that all the three types of sets may occur, the above classification of $\mathbb{O}$-sets is not empty; and sets of type (ii) can arise from semigroups having bounded or unbounded infinitesimal generators.

Examples 3.4. Take $X = l_1$, the Banach space of absolutely convergent sequences. As mentioned above (see Remark 3.2(1)), $\mathcal{K}(X)$ is the sole closed two-sided proper ideal of $\mathcal{L}(X)$, that is, $\mathcal{K}(X) = \mathcal{F}(X)$.

(1) Let $(U(t))_{t\geq 0}$ be the C_0-semigroup given by $U(t) = I$ for all $t \geq 0$. Clearly, for all $t > 0$, $U(t) - I \in \mathcal{K}(X)$. Accordingly, $\mathbb{O} =]0,\infty[$ and $A = 0$.

(2) (a) Assume that $U(t) = \operatorname{diag}\{e^{it}, e^{-it}, e^{it}, e^{-it}, \ldots\}$ for all $t \geq 0$. In this case, we have $\mathbb{O} = \{2n\pi,\ n = 1,2,3,\ldots\}$ and $A = \operatorname{diag}\{i,-i,i,-i,\ldots\}$, the infinitesimal generator of $(U(t))_{t\geq 0}$, is bounded.

(b) Suppose now that $U(t) = \operatorname{diag}\{e^{it}, e^{2it}, e^{3it}, e^{4it}, \ldots\}$ for all $t \geq 0$. Here, we have also $\mathbb{O} = \{2n\pi,\ n = 1,2,3,\ldots\}$ but $A = \operatorname{diag}\{i, 2i, 3i, 4i, \ldots\}$, the infinitesimal generator of $(U(t))_{t\geq 0}$, is unbounded.

(3) The C_0-semigroup $(U(t))_{t\geq 0}$ with $U(t) = \operatorname{diag}\{e^{it}, e^{2!it}, e^{3!it}, \ldots, e^{n!it}, \ldots\}$ provides an example of $\mathbb{O}$-set of type (iii).

In the next theorem, we derive some relationships between the type of $\mathbb{O}$-sets and the structure of the semigroup. In particular, we show that $\mathbb{O}$ has the first form if and only if A is a Fredholm perturbation. If $\mathbb{O}$ takes the third form, then A is necessarily unbounded.

Theorem 3.5. *Assume that condition (3.1) is satisfied. Then the following statements are equivalent:*

(i) $\mathbb{O} =]0,+\infty[$;

(ii) *A is a Fredholm perturbation;*

(iii) *$\lambda R(\lambda, A) - I$ is a Fredholm perturbation for some (in fact for all) $\lambda > \omega$.*

This result extends [2, Theorem 2] to large classes of operators which contain properly the set of compact operators.

Proof of Theorem 3.5. (i)⇒(ii). The first step in the proof of this implication consists in showing that (i) implies that A is bounded. The proof of this implication is similar to that of [2, Theorem 2]. Details are omitted.

Next, since A is bounded, then $U(t)$ is uniformly continuous for $t \geq 0$ (see [7]). Hence, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|U(t) - I\| < \varepsilon \quad \text{for } t < \delta. \tag{3.8}$$

Accordingly, for any $t < \delta$, we have

$$\left\|\frac{1}{t}\int_0^t U(s)\,ds - I\right\| = \left\|\frac{1}{t}\int_0^t (U(s) - I)\,ds\right\| \leq \frac{1}{t}\int_0^t \|U(s) - I\|\,ds < \varepsilon. \tag{3.9}$$

Hence, for ε small enough, $\int_0^t U(s)ds$ is invertible for all $t < \delta$. Moreover, using the identity

$$U(t) - I = A \int_0^t U(s)ds, \tag{3.10}$$

together with the fact that A and $U(t)$ commute, we infer that

$$A = \left[\int_0^t U(s)ds \right]^{-1} (U(t) - I). \tag{3.11}$$

Since $\mathcal{J}(X)$ is an ideal, we infer that $A \in \mathcal{J}(X)$.

(ii)⇒(i). Assume that $A \in \mathcal{J}(X)$. Using again identity (3.10) and the ideal structure of $\mathcal{J}(X)$ we see that $U(t) - I \in \mathcal{J}(X)$ for all $t \geq 0$.

(ii)⇒(iii). This follows from the identity $\lambda R(\lambda, A) - I = AR(\lambda, A)$ and the ideal structure of $\mathcal{J}(X)$.

(iii)⇒(ii). Assume that $\lambda R(\lambda, A) - I \in \mathcal{J}(X)$ for all $\lambda > \omega$. Note that the identity $\lambda R(\lambda, A) - I = AR(\lambda, A)$ and (3.11) lead to

$$R(\lambda, A)(U(t) - I) = (\lambda R(\lambda, A) - I) \int_0^t U(s)ds \quad \forall t \geq 0. \tag{3.12}$$

Writing (3.12) in the form

$$\begin{aligned} &\lambda R(\lambda, A)(U(t) - I) - (U(t) - I) + (U(t) - I) \\ &\quad = \lambda(\lambda R(\lambda, A) - I) \int_0^t U(s)ds \quad \forall t \geq 0, \end{aligned} \tag{3.13}$$

we infer that

$$U(t) - I = (\lambda R(\lambda, A) - I)\left[U(t) - I + \lambda \int_0^t U(s)ds \right]. \tag{3.14}$$

Next, using the fact that $[U(t) - I + \lambda \int_0^t U(s)ds] \in \mathcal{L}(X)$, we get that $U(t) - I \in \mathcal{J}(X)$ for all $t \geq 0$, that is, $\mathbb{O} =]0, \infty[$. This achieves the proof. □

The next result asserts that if the $\mathbb{O}$-set is in the form (iii), then the infinitesimal generator of $(U(t))_{t\geq 0}$ is necessarily unbounded. It generalizes [2, Theorem 3].

Proposition 3.6. *Assume that condition (3.1) holds true. If $\mathbb{O}$ is a dense subset of $]0, \infty[$ with no interior points, then A is unbounded.*

Proof. Assume, for contradiction, that A is bounded. Then, proceeding as in the proof of the implication (i)⇒(ii) in Theorem 3.5 we see that if $t < \delta$ and $t \in \mathbb{O}$, then $A \in \mathcal{J}(X)$. So, by Theorem 3.5, we get $\mathbb{O} =]0, \infty[$. This contradicts the hypothesis. □

Remark 3.7. (1) Notice that if $\mathcal{J}(X)$ is a nonzero closed two-sided ideal of $\mathcal{L}(X)$ satisfying (3.1), then it follows from [4, Proposition 4, page 70] that

$$\overline{\mathcal{F}_0(X)} \subseteq \mathcal{J}(X) \subseteq \mathcal{F}(X), \tag{3.15}$$

where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators on X. This shows that $\overline{\mathcal{F}_0(X)}$ is the minimal ideal (in the sense of the inclusion) in $\mathcal{L}(X)$ for which the results of this section are valid. Evidently, if X has the approximation property, then we have $\overline{\mathcal{F}_0(X)} = \mathcal{K}(X)$.

(2) Even though the description of the ideal structure of $\mathcal{L}(X)$ is a complex task, there exist some Banach spaces X for which $\mathcal{L}(X)$ has only one proper nonzero closed two-sided ideal. The first result in this direction was established by Calkin (cf. [4]). He proved that if X is a separable Hilbert space, then $\mathcal{K}(X)$ is the unique proper nonzero closed two-sided ideal of $\mathcal{L}(X)$. An extension of this result was obtained by Gohberg et al. [4]. They proved the same result for $X = l_p$, $1 \le p < \infty$, and $X = c_0$. In [6], Herman establishes the same result for a large class of Banach spaces, namely Banach spaces which have perfectly homogeneous block bases and satisfy $(+)$ (for the definition and the meaning of the symbol $(+)$ we refer to [6]). (Evidently, the spaces l_p, $1 \le p < \infty$, and c_0 belong to this class.) Thus, if X has perfectly homogeneous block bases which satisfy $(+)$, then

$$\mathcal{K}(X) = \mathcal{F}_+(X) = \mathcal{F}_-(X) = \mathcal{F}(X). \tag{3.16}$$

Consequently, for this class of spaces the results of this section use the ideal of compact operators and coincide with those obtained in [2]. Hence, for such spaces the Cuthbert results are optimal.

4. Further extensions. Let X be a Banach space. An operator $R \in \mathcal{L}(X)$ is called a Riesz operator if $\lambda - R \in \Phi(X)$ for all scalars $\lambda \neq 0$. Let $\mathcal{R}(X)$ denote the class of all Riesz operators. For further discussions concerning this family of operators, we refer to [1, 12] and the references therein. For our purpose, we recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators, $\mathcal{R}(X)$ is not an ideal of $\mathcal{L}(X)$ [1], and $\mathcal{F}(X)$ is the largest ideal contained in $\mathcal{R}(X)$ [12]. Hence the sets $\mathcal{K}(X)$, $\mathcal{S}(X)$, $C\mathcal{S}(X)$, $\mathcal{F}_+(X)$, and $\mathcal{F}_-(X)$ are also contained in $\mathcal{R}(X)$.

Let $A \in \mathcal{L}(X)$. The Fredholm region of A is defined as $\{\lambda \in \mathbb{C};\ \lambda - A \in \Phi(X)\}$ and denoted by Φ_A. Next, let $\Phi_A^0 := \{\lambda \in \Phi_A : i(\lambda - A) = 0\}$ and define the set

$$\sigma_b(A) := \mathbb{C} \backslash \rho_b(A), \tag{4.1}$$

where

$$\rho_b(A) := \{\lambda \in \Phi_A^0 \text{ such that all scalars near } \lambda \text{ are in } \rho(A)\}. \tag{4.2}$$

Following [5, 11], $\sigma_b(\cdot)$ is called the Browder essential spectrum.

We say that an operator $F \in \mathcal{L}(X)$ is polynomially compact (see [3]) if there is a nonzero complex polynomial $p(z)$ such that the operator $p(F)$ is compact. We designate by $P\mathcal{K}(X)$ the set of polynomially compact operators on X. Let $F \in P\mathcal{K}(X)$, the nonzero polynomial $p(z)$ of least degree and leading coefficient 1 such that $p(F)$ is compact will be called the minimal polynomial of F. We denote by $\Xi(X)$ the subset of $P\mathcal{K}(X)$ defined by

$$\Xi(X) := \Big\{ F \in P\mathcal{K}(X) \text{ such that the minimal polynomial of } F \quad p(z) = \sum_{r=0}^{p} a_r z^r \text{ satisfies } p(-1) \neq 0 \Big\}. \tag{4.3}$$

We first prove the following lemma which is required in the sequel.

Lemma 4.1. *If $F \in \Xi(X)$, then $I+F \in \Phi(X)$ and $i(I+F) = 0$.*

Proof. Since $p(F) \in \Xi(X)$ ($p(\cdot)$ denotes the minimal polynomial of F), then $\sigma_b(p(F)) = \{0\}$. By hypothesis $p(-1) \neq 0$, then $p(-1) \notin \sigma_b(p(F))$. Next, making use of the spectral mapping theorem for the Browder essential spectrum [5, Theorem 4] we conclude that $-1 \notin \sigma_b(F)$, that is, $-1 \in \rho_b(F)$. This ends the proof. □

The developments below are mainly suggested by the fact that, in general, the sets $\mathcal{F}(X)$ and $\Xi(X)$ do not coincide. Indeed, if $p(z) = (z-\lambda_1)^{n_1} \cdots (z-\lambda_k)^{n_k}$ is the minimal polynomial of $F \in \Xi(X)$, then, by the structure theorem of Gilfeather [3, Theorem 1], the spectrum of F consists of countably many points with $\{\lambda_1,\ldots,\lambda_k\}$ as only possible limit points and such that all but possibly $\{\lambda_1,\ldots,\lambda_k\}$ are eigenvalues with finite-dimensional generalized eigenspaces. This, together with the fact that the operators belonging to $\mathcal{F}(X)$ satisfy the Riesz-Schauder theory of compact operators (see above), implies that $\mathcal{F}(X) \neq \Xi(X)$. Thus the next result improves Proposition 3.6.

Proposition 4.2. *Let $(U(t))_{t\geq 0}$ be a C^0-semigroup on X. If*

$$\{t > 0 \text{ such that } U(t) - I \in \Xi(X)\} \neq \varnothing, \tag{4.4}$$

then $(U(t))_{t\geq 0}$ can be embedded in a C_0-group on X.

Proof. By hypothesis, there exists t_0 such that $U(t_0) - I \in \Xi(X)$. Since $U(t_0) = I + [U(t_0) - I]$, the use of Lemma 4.1 implies that $U(t_0) \in \Phi(X)$. Now, the result follows from Theorem 2.1. □

Due to some technical difficulties, we do not know whether or not Theorem 3.5 is valid for perturbations belonging to $\Xi(X)$. So, we discuss this result for a subset of $\Xi(X)$ consisting of power compact operators, that is,

$$\mathcal{P}(X) := \{F \in \mathcal{L}(X) \text{ such that } F^n \in \mathcal{K}(X) \text{ for some integer } n \geq 1\}. \tag{4.5}$$

Our principal motivation here rely on the fact that, for some classes of Banach spaces, we have $\mathcal{F}(X) \subseteq \mathcal{P}(X)$. In particular, if X is isomorphic to an L_p space with $1 \le p \le \infty$ or to $C(\Omega)$ where Ω is a metric compact Hausdorff space, then $\mathcal{S}(X) = \mathcal{F}(X)$ (cf. (3.3)). Moreover, by [10, Theorem 1], we have $\mathcal{S}(X)\mathcal{S}(X) \subseteq \mathcal{K}(X)$. These conclusions are also valid if X is an l_p space with $1 \le p < \infty$ and c_0 [6]. Note also that if X has the Dunford-Pettis property (a Banach space X is said to have the Dunford-Pettis property if for every Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into relatively norm compact sets of Y), then $\mathcal{W}(X)\mathcal{W}(X) \subseteq \mathcal{K}(X)$ where $\mathcal{W}(X)$ stands for the set of weakly compact operators. However, although the inclusion $\mathcal{P}(X) \subseteq \mathcal{R}(X)$ is valid for arbitrary Banach spaces (use the Ruston characterization of Riesz operators [1]), in general, we have $\mathcal{P}(X) \neq \mathcal{F}(X)$. In the light of these observations, we project to extend Theorem 3.5 to semigroups $(U(t))_{t\ge 0}$ for which there exists $t_0 > 0$ such that $U(t_0) - I \in \mathcal{P}(X)$. Evidently, since $\mathcal{P}(X) \subseteq \Xi(X)$, Proposition 4.2 holds also true for power compact perturbations. More precisely, we have the following theorem.

Theorem 4.3. *Let $(U(t))_{t\ge 0}$ be a C_0-semigroup on X with type ω and let A denote its infinitesimal generator. Define the set $\mathbb{O}$ by*

$$\mathbb{O} = \{t \ge 0 \text{ such that } U(t) - I \in \mathcal{P}(X)\}. \tag{4.6}$$

Then, the following items are equivalent:

(i) $\mathbb{O} =]0, +\infty[$;

(ii) $A \in \mathcal{P}(X)$;

(iii) $[\lambda R(\lambda, A) - I] \in \mathcal{P}(X)$ *for some (in fact for all)* $\lambda > \omega$.

Proof. We try to imitate the procedure in the proof of Theorem 3.5. Let us first observe that if $U(t) - I \in \mathcal{P}(X)$, then there exists $m \ge 1$ such that $(U(t) - I)^m \in \mathcal{K}(X)$. Using the spectral mapping theorem (see, e.g., [15, page 227]), one sees that that spectrum of $U(t) - I$ is either finite or a countable set accumulating only at zero. Moreover,

$$\sigma(U(t) - I) = \sigma(U(t)) - 1. \tag{4.7}$$

This means that, apart possibly from the point 1, $\sigma(U(t)) = \{e^{\eta t} : \eta \in P\sigma(A)\}$ ($P\sigma(A)$ stands for the point spectrum of A) and, for any $\varepsilon > 0$, the set $\{\lambda \in \sigma(U(t)) : |\lambda - 1| > \varepsilon\}$ is finite for all $t > 0$. Then arguing as in the proof of [2, Theorem 2], we conclude that (i) implies that $A \in \mathcal{L}(X)$. Furthermore, similar arguments as in the proof of Theorem 3.5 [(i)⇒(ii)] imply that

$$A = \left[\int_0^t U(s)\,ds\right]^{-1}(U(t) - I) = (U(t) - I)\left[\int_0^t U(s)\,ds\right]^{-1} \tag{4.8}$$

which leads to $A \in \mathcal{P}(X)$.

The remainder of the proof is verbatim that of Theorem 3.5. It suffices to use the fact that $U(t)-I$ and $[\int_0^t U(s)ds]^{-1}$ (resp., A and $R(\lambda, A)$) commute. □

We close this section by noticing that Proposition 3.6 is also valid for power compact perturbations. The proof uses Theorem 4.3.

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