IJMMS 2003:27, 1725–1730
PII. S0161171203106175
http://ijmms.hindawi.com

ON A NONLINEAR COMPACTNESS LEMMA IN $L^p(0,T;B)$

EMMANUEL MAITRE

Received 20 June 2001 and in revised form 2 February 2002

We consider a nonlinear counterpart of a compactness lemma of Simon (1987), which arises naturally in the study of doubly nonlinear equations of elliptic-parabolic type. This paper was motivated by previous results of Simon (1987), recently sharpened by Amann (2000), in the linear setting, and by a nonlinear compactness argument of Alt and Luckhaus (1983).

2000 Mathematics Subject Classification: 46B50, 47H30, 34G20, 35K65.

1. Introduction. Typical applications where the compactness argument stated below is useful are those in which the following kind of doubly nonlinear equations arises:

$$\frac{dB(u)}{dt} + A(u) = f, \tag{1.1}$$

where A is elliptic and B is monotone (not strictly). It is the case, for example, in porous medium, semiconductor equations, and so forth.

In our application, we considered the injection moulding of a thermoplastic with a mold of small thickness with respect to its other dimensions. By averaging Navier-Stokes equations across the thickness of the mold and under an assumption (of Hele-Shaw) stating that the velocity field is proportional to the pressure gradient, the pressure equation can be written as a doubly nonlinear equation [3].

Note that in this context, the equation can degenerate to an elliptic one. In order to get existence of a solution, one usually perform a time discretization, use some result on elliptic operator, and pass to the limit as the time step goes to zero. In nonlinear problems, compactness in time and space is then required. The compactness in space is easily obtained for u from a coerciveness assumption on the elliptic part A, but we have no estimate on $\partial u/\partial t$ since B could degenerate. Theorem 2.1 uses the space compactness of u and some time regularity on $B(u)$ to derive a compactness for $B(u)$, which in turn can be useful to pass to the limit in nonlinear terms of A (provided A has an appropriate structure, e.g., B-pseudomonotone [4]).

2. Main result. We consider two Banach spaces E_1 and E_2. Let $T > 0$, $p \in [1,+\infty]$, and B a (nonlinear) *compact* operator from E_1 to E_2, that is, which maps bounded subsets of E_1 to relatively compact subsets of E_2.

Theorem 2.1. *Let U be a bounded subset of $L^1(0,T;E_1)$ such that $V = B(U)$ is a subset of $L^p(0,T;E_2)$ bounded in $L^r(0,T;E_2)$ with $r > 1$. Assume that*

$$\lim_{h\to 0^+} ||v(\cdot+h)-v||_{L^p(0,T-h;E_2)} = 0 \quad \textit{uniformly for } v \in V. \tag{2.1}$$

Then V is relatively compact in $L^p(0,T;E_2)$ (and in $\mathscr{C}(0,T;E_2)$ if $p = +\infty$).

Remark 2.2. (1) One can easily check that Theorem 2.1 holds if we assume only U bounded in $L^1_{\text{loc}}(0,T;E_1)$ and V bounded in $L^r_{\text{loc}}(0,T;E_2)$.

(2) In the case where B is the canonical injection from E_1 to E_2, the assumption on B corresponds to the compactness of the embedding of E_1 into E_2, and the conclusion falls in the scope of previous results of Simon [5, Theorem 3].

(3) The point in this result is that we do not make any structural assumption on B (e.g., strict monotony which would fall in the scope of results of Visintin [6]) except compactness. Note that in the case of a compact embedding of E_1 into E_2, B needs only to be continuous from E_1 to E_2 for the E_2-topology.

Idea of the proof. A sufficient condition for compactness is to prove that for each couple (t_1,t_2), $\int_{t_1}^{t_2} v(t)dt$ describes a relatively compact subset of E_2 as v describes V. First, the $u(t)$, $u \in U$, are truncated in norm at height $M > 0$ and form a bounded subset of E_1 which B maps to a relatively compact subset $V^M(t)$ of E_2. The key point is that, thanks to equi-integrability assumption, $\int_{t_1}^{t_2} v(t)dt$ can be approximated uniformly in v by Riemann sums involving *truncated elements* of the $V^M(t)$.

Proof. Thanks to the equi-integrability (2.1) of V and results of [5], we only have to prove that for each (t_1,t_2) such that $0 < t_1 < t_2 < T$, the set

$$K = \left\{ \int_{t_1}^{t_2} v(t)dt,\ v \in V \right\} \tag{2.2}$$

is relatively compact in E_2. For that purpose, we introduce for $u \in U$ and $M > 0$ the measurable subset of $[0,T]$ defined by

$$G_u^M = \left\{ t \in [0,T],\ ||u(t)||_{E_1} > M \right\}. \tag{2.3}$$

From our assumptions on U, there exists a constant $C > 0$ such that

$$\forall u \in U, \quad \|u\|_{L^1(0,T;E_1)} \le C, \tag{2.4}$$

and since we have

$$\operatorname{meas}(G_u^M) = \int_{G_u^M} 1\,dt \le \int_{G_u^M} \frac{||u(t)||_{E_1}}{M} dt \le \frac{C}{M}, \tag{2.5}$$

that gives

$$\lim_{M\to+\infty} \operatorname{meas}(G_u^M) = 0, \quad \text{uniformly in } u. \tag{2.6}$$

Introducing the truncated functions

$$u^M(t) = \begin{cases} u(t) & \text{if } t \notin G_u^M, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

we have by construction

$$\forall M > 0,\ \forall u \in U,\ \forall t \in [0,T], \quad \|u^M(t)\|_{E_1} \le M. \tag{2.8}$$

□

Lemma 2.3. *Under condition (2.1), K can be uniformly approximated by Riemann sums involving elements of the form $v^M(t) = B(u^M(t))$ in the following sense: given $\varepsilon > 0$, there exist integers N and M such that for all $v = B(u) \in V$, there exists $s_v^{N,M} \in]0,h[$ such that*

$$\left\| \int_{t_1}^{t_2} v(t)\,dt - \sum_{i=1}^{N} h v^M(\xi_{i-1}^N + s_v^{N,M}) \right\|_{E_2} < \varepsilon, \tag{2.9}$$

where $h = (t_2 - t_1)/N$ and $\xi_i^N = t_1 + ih$.

Proof. We first note that

$$\begin{aligned} &\int_{t_1}^{t_2} v(t)\,dt - \sum_{i=1}^{N} h v^M(\xi_{i-1}^N + s_v^{N,M}) \\ &\quad = \int_{t_1}^{t_2} \left(v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s_v^{N,M}) \chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right) dt. \end{aligned} \tag{2.10}$$

Then we prove the following inequality, where r' stands for the conjugate exponent of r:

$$\begin{aligned} &\frac{1}{h}\int_0^h \int_{t_1}^{t_2} \left\| v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s)\chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right\|_{E_2} dt\,ds \\ &\qquad \le 2T^{1-1/p} \sup_{\sigma\in[-h,h]} \|v(\cdot+\sigma) - v\|_{L^p(0,T-\sigma;E_2)} \\ &\qquad\quad + 2(\operatorname{meas} G_u^M)^{1/r'} \|v - B(0)\|_{L^r(0,T;E_2)}. \end{aligned} \tag{2.11}$$

Denote by I the left-hand side of the stated inequality. Then

$$\begin{aligned} I &= \frac{1}{h}\int_0^h \sum_{i=1}^N \int_{\xi_{i-1}^N}^{\xi_i^N} \|v(t)-v^M(\xi_{i-1}^N+s)\|_{E_2}\,dt\,ds \\ &= \frac{1}{h}\sum_{i=1}^N \int_{\xi_{i-1}^N}^{\xi_i^N}\int_{\xi_{i-1}^N}^{\xi_i^N} \|v(t)-v^M(s)\|_{E_2}\,dt\,ds. \end{aligned} \tag{2.12}$$

Using Fubini's theorem and setting $\sigma = s-t$, we get

$$I = \frac{1}{h}\sum_{i=1}^N \int_{\xi_{i-1}^N}^{\xi_i^N}\int_{\xi_{i-1}^N-t}^{\xi_i^N-t} \|v(t)-v^M(t+\sigma)\|_{E_2}\,d\sigma\,dt, \tag{2.13}$$

which gives, thanks to a new application of Fubini's theorem,

$$\begin{aligned} I &= \frac{1}{h}\int_{-h}^h \sum_{i=1}^N \int_{\max(\xi_{i-1}^N,\xi_{i-1}^N-\sigma)}^{\min(\xi_i^N,\xi_i^N-\sigma)} \|v(t)-v^M(t+\sigma)\|_{E_2}\,dt\,d\sigma \\ &\le \frac{1}{h}\int_{-h}^h \int_{\max(t_1,t_1-\sigma)}^{\min(t_2,t_2-\sigma)} \|v(t)-v^M(t+\sigma)\|_{E_2}\,dt\,d\sigma. \end{aligned} \tag{2.14}$$

From the definition of v^M, we thus have

$$\begin{aligned} I \le{}& \frac{1}{h}\int_{-h}^h \int_{\max(t_1,t_1-\sigma)}^{\min(t_2,t_2-\sigma)} \|v(t)-v(t+\sigma)\|_{E_2}\,dt\,d\sigma \\ &+ \frac{1}{h}\int_{-h}^h \int_{\max(t_1,t_1-\sigma)}^{\min(t_2,t_2-\sigma)} \chi_{G_u^M}(t+\sigma)\|v(t)-B(0)\|_{E_2}\,dt\,d\sigma. \end{aligned} \tag{2.15}$$

As V is a bounded subset of $L^r(0,T;E_2)$, one has the second term bounded by

$$\begin{aligned} &\frac{1}{h}\int_{-h}^h \left(\int_{\max(t_1,t_1-\sigma)}^{\min(t_2,t_2-\sigma)} \chi_{G_u^M}(t+\sigma)\,dt\right)^{1/r'} \left(\int_{t_1}^{t_2} \|v(t)-B(0)\|_{E_2}^r\,dt\right)^{1/r} d\sigma \\ &\qquad \le 2\left(\operatorname{meas} G_u^M\right)^{1/r'} \|v-B(0)\|_{L^r(0,T;E_2)}, \end{aligned} \tag{2.16}$$

and the Hölder inequality gives the announced estimation (2.11).

Using (2.1) and (2.6), and as v belongs to a bounded subset V of $L^r(0,T;E_2)$, we conclude from (2.11) that

$$\frac{1}{h}\int_0^h\int_{t_1}^{t_2} \left\| v(t) - \sum_{i=1}^N v^M(\xi_{i-1}^N+s)\chi_{]\xi_{i-1}^N,\xi_i^N]}(t)\right\|_{E_2} dt\,ds \longrightarrow 0, \tag{2.17}$$

when M,N go to infinity, uniformly in v. We claim that there exists at least one $s = s_v^{N,M} \in [0,h]$ such that

$$\int_{t_1}^{t_2} \left\| v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s_v^{N,M}) \chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right\|_{E_2} dt \longrightarrow 0, \tag{2.18}$$

when M, N go to infinity, uniformly in v. Indeed, we set, for the sake of readability,

$$f_{N,M}^v(s) = \int_{t_1}^{t_2} \left\| v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s) \chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right\|_{E_2} dt \tag{2.19}$$

so that the uniform convergence (2.17) reads

$$\frac{1}{h} \int_0^h f_{N,M}^v(s)\,ds \longrightarrow 0, \tag{2.20}$$

when $M,N = 1/h$ go to infinity, uniformly in v. Then for fixed v, N, and M, there exists at least one $s = s_v^{N,M} \in [0,h]$ such that

$$f_{N,M}^v(s_v^{N,M}) \leq \frac{1}{h} \int_0^h f_{N,M}^v(s)\,ds. \tag{2.21}$$

If not, we would have the reverse strict inequality for all $s \in [0,h]$ which by averaging on $[0,h]$ would lead to a contradiction. Then as $f_{N,M}^v$ is positive, the uniform convergence (2.20) implies

$$f_{N,M}^v(s_v^{N,M}) \longrightarrow 0, \tag{2.22}$$

when $M,N = 1/h$ go to infinity, uniformly in v, which is exactly (2.18).

A fortiori, (2.9) holds thanks to (2.10) and since

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} \left(v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s_v^{N,M}) \chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right) dt \right\|_{E_2} \\ &\qquad \leq \int_{t_1}^{t_2} \left\| v(t) - \sum_{i=1}^{N} v^M(\xi_{i-1}^N + s_v^{N,M}) \chi_{]\xi_{i-1}^N,\xi_i^N]}(t) \right\|_{E_2} dt. \end{aligned} \tag{2.23}$$

This proves Lemma 2.3. □

To conclude the proof of Theorem 2.1, note that Lemma 2.3 means that $K \subset \varepsilon B_{E_2} + K_{M,N}$, where B_{E_2} is the unit open ball of E_2 and

$$K_{M,N} = \left\{ \sum_{i=1}^{N} h v^M (\xi_{i-1}^N + s_v^{N,M}),\ v^M = B(u^M),\ u \in U \right\}. \tag{2.24}$$

For fixed M, N and from (2.8), we note that $u^M(\xi_{i-1}^N + s_v^{N,M})$ is bounded in E_1 uniformly in $u \in U$. As B is compact, $K_{M,N}$ is thus a relatively compact subset of E_2. Thus, K is also relatively compact in E_2.

Corollary 2.4. *Let U be a bounded subset of $L^1(0,T;E_1)$ such that $V = B(U)$ is bounded in $L^r(0,T;E_2)$ with $r > 1$. Assume that*

$$\frac{\partial V}{\partial t} = \left\{ \frac{\partial v}{\partial t},\ v \in V \right\} \tag{2.25}$$

is bounded in $L^1(0,T;E_2)$. Then V is relatively compact in $L^p(0,T;E_2)$ for any $p < +\infty$.

Proof. Condition (2.1) is satisfied (see [5, Lemma 4]). □

Acknowledgment. The author would like to thank the referee for careful reading of this paper, which led to an improved formulation of the results.

References

[1] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z. **183** (1983), no. 3, 311–341.

[2] H. Amann, *Compact embeddings of vector-valued Sobolev and Besov spaces*, Glas. Mat. Ser. III **35(55)** (2000), no. 1, 161–177.

[3] E. Maitre, *Sur une classe d'équations à double non linéarité : application à la simulation de l'écoulement d'un fluide visqueux compressible*, Ph.D. thesis, University of Grenoble I, France, 1997.

[4] E. Maitre and P. Witomski, *A pseudo-monotonicity adapted to doubly nonlinear elliptic-parabolic equations*, Nonlinear Anal. **50** (2002), no. 2, Ser. A: Theory Methods, 223–250.

[5] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

[6] A. Visintin, *Strong convergence results related to strict convexity*, Comm. Partial Differential Equations **9** (1984), no. 5, 439–466.

Emmanuel Maitre: Laboratoire de Modélisation et Calcul, Université Joseph Fourier, BP 53, 38041 Grenoble Cedex 9, France

E-mail address: Emmanuel.Maitre@imag.fr