IJMMS 2003:3, 133–152
PII. S0161171203204038
http://ijmms.hindawi.com


# VECTOR FIELDS ON NONORIENTABLE SURFACES

**ILIE BARZA and DORIN GHISA**

Received 4 April 2002

A one-to-one correspondence is established between the germs of functions and tangent vectors on a NOS **X** and the bi-germs of functions, respectively, elementary fields of tangent vectors (EFTV) on the orientable double cover of **X**. Some representation theorems for the algebra of germs of functions, the tangent space at an arbitrary point of **X**, and the space of vector fields on **X** are proved by using a symmetrisation process. An example related to the normal derivative on the border of the Möbius strip supports the nontriviality of the concepts introduced in this paper.

2000 Mathematics Subject Classification: 30F15, 30F50.

**1. Introduction.** It was Felix Klein who first had the idea that, in order to do more than topology on NOS, they should be endowed with dianalytic structures. This was exploited much later by Schiffer and Spencer [13], who undertook, for the first time, the analysis of NOS. In the monograph [2], the category of Klein surfaces was introduced, and, in [3], Andreian Cazacu clarified completely the concept of morphism of Klein surfaces. She proved that the interior transformations of Stoïlow [14], on which he had based his topological principles of analytic functions, are essential in the definition of the morphisms of Klein surfaces. The category of Riemann surfaces appears as a subcategory of the category of Klein surfaces. Moreover, since the last one contains border free, as well as bordered surfaces, it also contains the subcategory of bordered Riemann surfaces. For this reason, if the contrary is not explicitly mentioned, by a surface, Riemann surface, or Klein surface, we will understand in this paper a bordered or a bordered free surface, respectively Riemann surface or Klein surface.

In this paper, the methods used in [4, 5, 6, 7, 8] are extended to the study of vector fields on NOS. The extension required new concepts and techniques.

The first three sections present the general frame of the problems we are dealing with and which are based on a result of Felix Klein (Theorem 2.1).

In section 4, the concept of bi-germ of functions on a symmetric Riemann surface is defined.

In Section 5, the operators of symmetrisation/antisymmetrisation of the bi-germs are introduced. Both operators play a basic role in analysis on nonorientable surfaces.

Section 6 deals with the study of vector fields on symmetric Riemann surfaces and on NOS. A symmetrisation process, dual to the symmetrisation of bi-germs, is defined. The connection between the vector fields on a nonorientable Klein surface and the vector fields on its orientable double covering is given (Theorem 6.2).

In Section 7, we give the detailed construction of the unit normal vector field to the border of the Möbius strip.

**2. Prerequisites.** Let $\mathbf{X}$ be a surface. If $p \in \mathbf{X}$, a chart at $p$ is a couple $(U,\varphi)$ consisting of an open neighborhood $U$ of $p$ and a homeomorphism $\varphi : U \to V$ where $V$ is a relatively open subset of the closed upper half-plane

$$\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}. \tag{2.1}$$

A *dianalytic atlas* on $\mathbf{X}$ is a family of charts $\mathcal{T} = \{(U_\alpha,\varphi_\alpha) \mid \alpha \in I\}$ such that $\mathbf{X} = \bigcup_{\alpha\in I} U_\alpha$ and, for every two charts $(U_\alpha,\varphi_\alpha)$ and $(U_\beta,\varphi_\beta) \in \mathcal{T}$, the transfer function $\varphi_\beta \circ \varphi_\alpha^{-1}$ is either conformal or anticonformal on each connected component of $\varphi_\alpha(U_\alpha \cap U_\beta)$.

The couple $(\mathbf{X},\mathcal{T})$ is called a *Klein surface* if $\mathcal{T}$ is a *maximal dianalytic atlas* on $\mathbf{X}$.

We notice that a Klein surface can be an orientable as well as a nonorientable surface. However, the universal covering $\hat{\mathbf{X}}$ of any Klein surface $\mathbf{X}$, being simply connected, is necessarily orientable. Consequently, there is a conformal structure on $\hat{\mathbf{X}}$ that makes the canonical projection

$$p : \hat{\mathbf{X}} \longrightarrow \mathbf{X} \tag{2.2}$$

locally analytic or antianalytic in terms of each chart; we call such a map *dianalytic*.

Let $\mathcal{G}$ be the group of covering transformations of $\hat{\mathbf{X}}$ over $\mathbf{X}$. Some elements of $\mathcal{G}$ might be conformal and some anticonformal mappings of $\hat{\mathbf{X}}$ on itself. If $\mathbf{X}$ is a nonorientable Klein surface, then $\mathcal{G}$ contains necessarily anticonformal mappings. Suppose that this is the case and let $\mathcal{G}_1$ be the subgroup of $\mathcal{G}$ formed with all its conformal elements. Then, the orbit space

$$\mathbb{O}_2 := \mathbb{O}_2(\mathbf{X}) := \hat{\mathbf{X}}/\mathcal{G}_1 \tag{2.3}$$

is an orientable surface. Moreover, $\mathbb{O}_2$ has a unique analytic structure, which makes the canonical projection

$$\pi : \hat{\mathbf{X}} \longrightarrow \mathbb{O}_2 \tag{2.4}$$

an analytic mapping. If $g \in \mathcal{G} \backslash \mathcal{G}_1$, then $\mathcal{G} = \mathcal{G}_1 \cup g\mathcal{G}_1$ and $\mathcal{G}_1 \cap g\mathcal{G}_1 = \emptyset$. Therefore, we can define

$$\mathbf{h} : \mathbb{O}_2 \longrightarrow \mathbb{O}_2 \tag{2.5}$$

by $\mathbf{h}(\hat{x}) := \widehat{gx}$, where $\hat{y}$ is the fiber of $y \in \hat{\mathbf{X}}$ with respect to $\pi : \hat{\mathbf{X}} \to \mathbb{O}_2$, that is, $\hat{y} = \{gy \mid g \in \mathcal{G}_1\}$.

Obviously, $\mathbf{h}$ is an *antianalytic involution* since $g \in \mathcal{G} \backslash \mathcal{G}_1$ and $g^2 \in \mathcal{G}_1$. It is *fixed point free* since $\mathbf{h}(\hat{x}) = \hat{x}$ would imply $g \in \mathcal{G}_1$, contrary to the hypothesis.

Klein called *symmetry* an involution of a Riemann surface of the type previously described. There are lots of other types of symmetries (see, e.g., [1, 2, 10, 11], and so on); however, in this paper, we will only use the concept of symmetry in the sense of Klein.

The following theorem has its origins in Klein's works.

**Theorem 2.1.** *If* $(\mathbb{O}_2, \mathbf{h})$ *is a symmetric Riemann surface and if* $\langle \mathbf{h} \rangle$ *is the two-element group generated by* $\mathbf{h}$*, then the covering projection*

$$\mathbf{q} : \mathbb{O}_2 \longrightarrow \mathbb{O}_2 / \langle \mathbf{h} \rangle \tag{2.6}$$

*induces a structure of nonorientable Klein surface on* $\mathbb{O}_2/\langle \mathbf{h} \rangle$ *with respect to which* $\mathbf{q}$ *is a morphism of Klein surfaces.*

*Conversely, if* $\mathbf{X}$ *is a nonorientable Klein surface, there exists a symmetric Riemann surface* $(\mathbb{O}_2, \mathbf{h})$ *such that* $\mathbf{X}$ *is dianalytically equivalent to* $\mathbb{O}_2/\langle \mathbf{h} \rangle$*.*

We call $\mathbb{O}_2 := \mathbb{O}_2(\mathbf{X})$ the orientable double cover of $\mathbf{X}$, and we notice that this concept is different from those defined by other types of symmetries.

**3. Vector fields.** Let $\mathbf{X}$ be a Klein surface. For $P \in \mathbf{X}$, we denote by $\mathcal{C}^\infty(P)$ the algebra of germs of complex functions defined on neighborhoods of $P$ whose real and imaginary parts are functions of class $\mathcal{C}^\infty$. To every differentiable curve $\alpha : ]-\varepsilon, \varepsilon[ \to \mathbf{X}$ ($\varepsilon > 0$ and $\alpha(0) = P$), we associate the operator

$$\vec{X}_{P,\alpha} = \vec{X}_P : \mathcal{C}^\infty(P) \longrightarrow \mathbb{C} \tag{3.1}$$

defined by

$$\vec{X}_P(f) := \frac{d(f \circ \alpha)}{dt}(0), \tag{3.2}$$

for every $f \in \mathcal{C}^\infty(P)$. The map $\vec{X}_P$ is a linear operator and satisfies Leibniz rule for the derivation of the product of functions. This operator $\vec{X}_P$ is called *a tangent vector* at $P$ to $\mathbf{X}$. The vector space $\mathbf{T}_P\mathbf{X}$ of all tangent vectors at $P$ to $\mathbf{X}$ is called *the tangent space* at $P$ to $\mathbf{X}$.

If $(U,\varphi)$ is a chart at $P$ and if $\varphi(Q)=(x,y)$ are the local coordinates in $U$, $\varphi(P)=(0,0)$, then $x\to\varphi^{-1}(x,0)$ and $y\to\varphi^{-1}(0,y)$, the so-called coordinate curves at $P$, determine a basis for $\mathbf{T}_P\mathbf{X}$

$$\mathcal{B}=\mathcal{B}_{(U,\varphi)}:=\left\{\left(\frac{\partial}{\partial x}\right)_P,\left(\frac{\partial}{\partial y}\right)_P\right\} \tag{3.3}$$

defined by

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)_P(f)&:=\frac{\partial(f\circ\varphi^{-1})}{\partial x}(\varphi(P)),\\ \left(\frac{\partial}{\partial y}\right)_P(f)&:=\frac{\partial(f\circ\varphi^{-1})}{\partial y}(\varphi(P)),\end{aligned} \tag{3.4}$$

where $\partial/\partial x$ and $\partial/\partial y$ are the usual partial derivatives in the complex plane $\mathbb{C}$. (See, e.g., [9, 11, 12].)

Every $\vec{X}_P\in\mathbf{T}_P\mathbf{X}$ has the form

$$\vec{X}_P=X^1(P)\left(\frac{\partial}{\partial x}\right)_P+X^2(P)\left(\frac{\partial}{\partial y}\right)_P, \tag{3.5}$$

where $X^1(P)$ and $X^2(P)$ are complex coefficients.

The *tangent bundle* of $\mathbf{X}$ is the set

$$\mathbf{TX}:=\bigcup_{P\in\mathbf{X}}\mathbf{T}_P\mathbf{X}. \tag{3.6}$$

It is well known that $\mathbf{TX}$ has a canonical structure of differential manifold of real dimension 4.

If $A$ is a subset of $\mathbf{X}$, a *vector field* on $A$ is any function

$$\vec{X}:A\longrightarrow\mathbf{TX} \tag{3.7}$$

such that $\vec{X}(P):=\vec{X}_P\in\mathbf{T}_P\mathbf{X}$ for every $P\in A$.

In particular, for a chart $(U,\varphi)$, we have the vector fields $\partial/\partial x$ and $\partial/\partial y$

$$\frac{\partial}{\partial x},\frac{\partial}{\partial y}:U\longrightarrow\mathbf{TX}, \tag{3.8}$$

$P\to(\partial/\partial x)(P):=(\partial/\partial x)_P$, respectively, $P\to(\partial/\partial y)(P):=(\partial/\partial y)_P$ as defined by (3.4).

If $\vec{X}:A\to\mathbf{TX}$ is a vector field of class $\mathscr{C}^\infty$, then the restriction of $\vec{X}$ to $U$ has the form

$$\vec{X}_U=X^1\frac{\partial}{\partial x}+X^2\frac{\partial}{\partial y}, \tag{3.9}$$

where $X^1,X^2:U\to\mathbb{C}$ are functions of class $\mathscr{C}^\infty$ on $U$.

We will study in more detail the vector fields on $\mathbf{X}$ and $\mathbb{O}_2 := \mathbb{O}_2(\mathbf{X})$.

**4. Bi-germs of functions on $\mathbb{O}_2$.** In all what follows, $\mathbf{X}$ will be a nonorientable Klein surface, $\mathbb{O}_2 := \mathbb{O}_2(\mathbf{X})$ will be the corresponding double cover of $\mathbf{X}$, $\mathbf{h}$ will be the symmetry of $\mathbb{O}_2$ as in Theorem 2.1, and $\mathbf{X}$ will be identified with the orbit space $\mathbb{O}_2/\langle\mathbf{h}\rangle$. The map $\mathbf{q} : \mathbb{O}_2 \to \mathbb{O}_2/\langle\mathbf{h}\rangle$ will be the canonical projection.

Different parametric disks or half-disks, if centered on the border $\mathcal{B}(\mathbf{X})$, will be *evenly* covered by $\mathbf{q}$. Then, the inverse image by $\mathbf{q}$ of such a parametric disk (half-disk) $\widetilde{D}$ consists of a pair $D$ and $\mathbf{h}D$ of symmetric disks (half-disks) on $\mathbb{O}_2$

$$\mathbf{q}^{-1}(\widetilde{D}) = D \cup \mathbf{h}D, \qquad D \cap \mathbf{h}D = \varnothing. \tag{4.1}$$

This brings us to the necessity of considering germs of $\mathcal{C}^\infty$-functions on symmetric subsets of $\mathbb{O}_2$ as defined next.

For $P \in \mathbb{O}_2$, we denote by $\{P;\mathbf{h}P\}$ the orbit of $P$ with respect to $\langle\mathbf{h}\rangle$ and by $\widetilde{P} = \widetilde{\mathbf{h}P}$ the image by $\mathbf{q}$ of $P$ and $\mathbf{h}P$ on $\mathbf{X} := \mathbb{O}_2/\langle\mathbf{h}\rangle$.

Since $\mathbf{q}$ is a dianalytic map, if $F : \mathbf{X} \to \mathbb{C}$ is a function of class $\mathcal{C}^\infty$ or if $F$ is harmonic or dianalytic, then so is $F \circ \mathbf{q}$ and vice versa.

A subset $\Sigma$ of $\mathbb{O}_2$ will be called *symmetric* if it is $\mathbf{h}$-invariant, that is, $\mathbf{h}\Sigma = \Sigma$. Obviously, the restriction of $\mathbf{h}$ to any symmetric set $\Sigma$ continues to be an involution.

In the study of local properties of functions at points $\widetilde{P} \in \mathbf{X}$, we only need symmetric neighborhoods of $\{P;\mathbf{h}P\}$. Such a neighborhood is, by definition, a subset $\Sigma$ such that there is a parametric disk (half disk) $D$ centered at $P$, $D \subset \Sigma$. Implicitly, $\mathbf{h}D$ is a parametric disk (half disk) centered at $\mathbf{h}P$ and $D \cup \mathbf{h}D \subseteq \Sigma$.

If $\mathcal{D}$ is the family of parametric disks (half-disks) centered at $P$, the family $\{D \cup \mathbf{h}D \mid D \in \mathcal{D}\}$ is a fundamental system of symmetric neighborhoods of $\{P;\mathbf{h}P\}$. If $D \cap \mathbf{h}D = \varnothing$, then

$$\widetilde{D} := \{\widetilde{P} \mid P \in D\} \tag{4.2}$$

is a parametric disk (half-disk) on $\mathbf{X}$, and it is evenly covered by $\mathbf{q}$. Let $\mathcal{C}^\infty(\widetilde{P})$ be the algebra of germs of functions of class $\mathcal{C}^\infty$ at $\widetilde{P}$, and let $F_{\widetilde{P}} \in \mathcal{C}^\infty(\widetilde{P})$.

We will denote by $F$ any class representative of the germ $F_{\widetilde{P}}$, and we will consider that the domain of $F$ is a disk or a half-disk as previously defined (i.e., evenly covered by $\mathbf{q}$). The functions $f_P := F \circ \mathbf{q}_D$ and $f_{\mathbf{h}P} := F \circ \mathbf{q}_{\mathbf{h}D}$ ($\mathbf{q}_D$ and $\mathbf{q}_{\mathbf{h}D}$ being the restrictions of $\mathbf{q}$ to $D$ and, respectively, to $\mathbf{h}D$) are functions of class $\mathcal{C}^\infty$. Having in mind the conventional notations, we can write

$$f_P \in \mathcal{C}^\infty(P), \qquad f_{\mathbf{h}P} \in \mathcal{C}^\infty(\mathbf{h}P). \tag{4.3}$$

Thus, the germ $F_{\tilde{P}}$ induces on $\mathbf{X}$ the two-element set of germs $\{f_P; f_{\mathbf{h}P}\}$. This set is a *nonordered* pair of elements of the set $\mathscr{C}^\infty(P) \cup \mathscr{C}^\infty(\mathbf{h}P)$, with the pair having an element in each member of this union. With this preparation in mind, we can give the following definition.

**Definition 4.1.** A bi-germ of class $\mathscr{C}^\infty$ at $\{P; \mathbf{h}P\}$ is a function

$$\chi : \{P; \mathbf{h}P\} \longrightarrow \mathscr{C}^\infty(P) \cup \mathscr{C}^\infty(\mathbf{h}P) \tag{4.4}$$

such that $\chi(Q) \in \mathscr{C}^\infty(Q)$ for every $Q \in \{P; \mathbf{h}P\}$.

We denote by $\mathscr{C}^\infty(P; \mathbf{h}P)$ the set of bi-germs of class $\mathscr{C}^\infty$ at $\{P; \mathbf{h}P\}$. The algebraic operations in $\mathscr{C}^\infty(P; \mathbf{h}P)$ are the natural operations induced by the algebra $\mathscr{C}^\infty(Q)$, that is, for arbitrary $\chi, \lambda \in \mathscr{C}^\infty(P; \mathbf{h}P)$, $Q \in \{P; \mathbf{h}P\}$, and $a \in \mathbb{C}$,

$$\begin{aligned} (\chi + \lambda)(Q) &:= \chi(Q) + \lambda(Q), \\ (\chi\lambda)(Q) &:= \chi(Q)\lambda(Q), \\ (a\chi)(Q) &= a\chi(Q). \end{aligned} \tag{4.5}$$

Obviously, $\mathscr{C}^\infty(P; \mathbf{h}P)$ with these operations is a commutative algebra with unit. We call it *the algebra of bi-germs of class* $\mathscr{C}^\infty$ *at* $\{P; \mathbf{h}P\}$.

**Remark 4.2.** The algebra $\mathscr{C}^\infty(P; \mathbf{h}P)$ is not the product algebra $\mathscr{C}^\infty(P) \times \mathscr{C}^\infty(\mathbf{h}P)$ since the last one consists of ordered pairs of germs.

It is convenient to denote the bi-germ $\chi \in \mathscr{C}^\infty(P; \mathbf{h}P)$ by $\{\chi_P; \chi_{\mathbf{h}P}\}$. The two germs $\chi_P \in \mathscr{C}^\infty(P)$ and $\chi_{\mathbf{h}P} \in \mathscr{C}^\infty(\mathbf{h}P)$ are different, and, generally, there is no connection between them. With these notations, the algebraic operations in $\mathscr{C}^\infty(P; \mathbf{h}P)$ become

$$\{\chi_P; \chi_{\mathbf{h}P}\} + \{\lambda_P; \lambda_{\mathbf{h}P}\} = \{\chi_P + \lambda_P; \chi_{\mathbf{h}P} + \lambda_{\mathbf{h}P}\}, \text{ and so on.} \tag{4.6}$$

The pullback algebra isomorphism of $\mathscr{C}^\infty(\mathbf{h}P)$ onto $\mathscr{C}^\infty(P)$ induced by $\mathbf{h}$ is usually denoted by $\mathbf{h}_P^*$

$$\mathbf{h}_P^* : \mathscr{C}^\infty(\mathbf{h}P) \longrightarrow \mathscr{C}^\infty(P), \qquad \mathbf{h}_P^*(f) := f \circ \mathbf{h}, \tag{4.7}$$

(where $f \circ \mathbf{h}$ is the germ at $P$ of $f \circ \mathbf{h}_{\mathbf{h}D}$, $D$ being the domain of $f$). Obviously, the inverse of $\mathbf{h}_P^*$ is $\mathbf{h}_{\mathbf{h}P}^* : \mathscr{C}^\infty(P) \to \mathscr{C}^\infty(\mathbf{h}P)$. It is also obvious that $\mathbf{h}_P^*$ and $\mathbf{h}_{\mathbf{h}P}^*$ are different maps.

The involution $\mathbf{h}:\{P,\mathbf{h}P\}\to\{P,\mathbf{h}P\}$ also induces a pullback map defined as follows and denoted, to avoid confusions, by $\mathbf{h}^b$

$$\mathbf{h}^b:\mathscr{C}^\infty(P;\mathbf{h}P)\longrightarrow\mathscr{C}^\infty(P;\mathbf{h}P),\qquad \mathbf{h}^b(\{\chi_P;\chi_{\mathbf{h}P}\}):=\{\mathbf{h}_P^*(\chi_{\mathbf{h}P});\mathbf{h}_{\mathbf{h}P}^*(\chi_P)\}. \tag{4.8}$$

The algebraic properties of $\mathbf{h}_P^*$ imply that $\mathbf{h}^b$ is *an involutive algebra isomorphism.*

We now consider $F\in\mathscr{C}^\infty(\widetilde{P})$. It induces the bi-germ

$$\mathbf{q}^b(F):=\{\mathbf{q}_P^*(F);\mathbf{q}_{\mathbf{h}P}^*(F)\}\in\mathscr{C}^\infty(P;\mathbf{h}P). \tag{4.9}$$

In this way, we obtain the map

$$\mathbf{q}^b:\mathscr{C}^\infty(\widetilde{P})\longrightarrow\mathscr{C}^\infty(P;\mathbf{h}P). \tag{4.10}$$

The main properties of $\mathbf{q}^b$ are given in the following proposition.

**Proposition 4.3.** (i) *The map $\mathbf{q}^b$ is an injective algebra morphism;*
(ii) $\mathbf{h}^b\circ\mathbf{q}^b=\mathbf{q}^b$*, that is, $\mathbf{q}^b$ is $\mathbf{h}^b$-left invariant;*
(iii) $\mathbf{q}^b(\mathscr{C}^\infty(\widetilde{P}))=\{\chi\mid\mathbf{h}^b(\chi)=\chi\}$.

**Proof.** The assertions (i) and (ii) are obvious. According to (ii), $\mathbf{q}^b(\mathscr{C}^\infty(\widetilde{P}))$ consists of bi-germs which are $\mathbf{h}^b$-invariant, that is, $\mathbf{q}^b(\mathscr{C}^\infty(\widetilde{P}))\subseteq\{\chi\mid\mathbf{h}^b(\chi)=\chi\}$.

Reciprocally, let $\chi=\{\chi_P;\chi_{\mathbf{h}P}\}$ be an $\mathbf{h}^b$-invariant bi-germ. We have the following equivalences:

$$\begin{gathered}\mathbf{h}^b(\chi)=\chi\Longleftrightarrow\{\mathbf{h}_P^*(\chi_{\mathbf{h}P});\mathbf{h}_{\mathbf{h}P}^*(\chi_P)\}=\{\chi_P;\chi_{\mathbf{h}P}\}\Longleftrightarrow\mathbf{h}_P^*(\chi_{\mathbf{h}P})=\chi_P,\\ \mathbf{h}_{\mathbf{h}P}^*(\chi_P)=\chi_{\mathbf{h}P}\Longleftrightarrow\mathbf{h}_P^*(\chi_{\mathbf{h}P})=\chi_P\end{gathered} \tag{4.11}$$

(because $\mathbf{h}_{\mathbf{h}P}^*=(\mathbf{h}_P^*)^{-1}$).

If $D$ is a disk or (half-disk) centered at $P$ such that $D\cap\mathbf{h}D=\varnothing$, we consider the map $\mathbf{q}_D^{-1}$, the inverse of $\mathbf{q}_D:D\to\widetilde{D}$.

Let $(\mathbf{q}_D^{-1})_{\widetilde{P}}^*$ be the pullback algebra isomorphism induced by $\mathbf{q}_D^{-1}$. With $\mathbf{h}_P^*(\chi_{\mathbf{h}P})=\chi_P\in\mathscr{C}^\infty(P)$, we get successively:

$$\begin{aligned}\mathscr{C}^\infty(\widetilde{P})\ni F:=(\mathbf{q}_D^{-1})_{\widetilde{P}}^*(\chi_P)&=(\mathbf{q}_D^{-1})_{\widetilde{P}}^*(\mathbf{h}_P^*(\chi_{\mathbf{h}P}))=((\mathbf{q}_D^{-1})_{\widetilde{P}}^*\circ\mathbf{h}_P^*)(\chi_{\mathbf{h}P})\\ &=(\mathbf{h}\circ\mathbf{q}_D^{-1})_{\widetilde{P}}^*(\chi_{\mathbf{h}P})=(\mathbf{q}_{\mathbf{h}D}^{-1})_{\widetilde{P}}^*(\chi_{\mathbf{h}P}),\\ \mathbf{q}^b(F)&=\{\mathbf{q}_P^*(F);\mathbf{q}_{\mathbf{h}P}^*(F)\}=\{\chi_P;\chi_{\mathbf{h}P}\};\end{aligned} \tag{4.12}$$

thus, $\chi\in\mathbf{q}^b(\mathscr{C}^\infty(\widetilde{P}))$. Finally, $\{\chi\mid\mathbf{h}^b(\chi)=\chi\}\subseteq\mathbf{q}^b(\mathscr{C}^\infty(\widetilde{P}))$. □

The main assertion of this section is that finding the germs of $\mathscr{C}^\infty$-functions at $\widetilde{P}$ is equivalent to finding the $\mathbf{h}^b$-invariant elements of $\mathscr{C}^\infty(P;\mathbf{h}P)$.

We will study the $\mathbf{h}^b$-invariant bi-germs in the next section.

**5. Symmetrisation and antisymmetrisation.** We consider an arbitrary bi-germ $\chi \in \mathscr{C}^\infty(P;\mathbf{h}P)$. We define the bi-germ $\mathscr{S}\chi$ by

$$\mathscr{S}\chi := \frac{1}{2}(\chi+\mathbf{h}^b\chi) = \left\{\frac{1}{2}(\chi_P+\mathbf{h}_P^*(\chi_{\mathbf{h}P}));\frac{1}{2}(\chi_{\mathbf{h}P}+\mathbf{h}_{\mathbf{h}P}^*(\chi_P))\right\}. \tag{5.1}$$

It is obvious that the equality $\mathbf{h}^b(\chi) = \chi$ is equivalent to $\mathscr{S}\chi = \chi$, that is, $\mathbf{h}^b$ and $\mathscr{S}$ have the same set of fixed points. Thus, the difference $\chi - \mathscr{S}\chi$ gives an "estimation" of the deviation of $\chi$ from $\mathbf{h}^b$-invariance. This prompts us to define a new operator $\mathscr{A} : \mathscr{C}^\infty(P;\mathbf{h}P) \to \mathscr{C}^\infty(P;\mathbf{h}P)$ by

$$\mathscr{A}\chi := \chi - \mathscr{S}\chi = \frac{1}{2}(\chi - \mathbf{h}^b\chi). \tag{5.2}$$

Clearly, the equality $\mathscr{A}\chi = \chi$ is equivalent to $\mathbf{h}^b(\chi) = -\chi$. In this way, we are led to the following definition.

**Definition 5.1.** A bi-germ $\chi \in \mathscr{C}^\infty(P;\mathbf{h}P)$ is called symmetric or $\mathbf{h}$-invariant (resp., antisymmetric or $\mathbf{h}$-antiinvariant) if and only if $\mathbf{h}^b(\chi) = \chi$ (resp., $\mathbf{h}^b(\chi) = -\chi$).

Thus, the symmetric bi-germs are the fixed points of $\mathscr{S}$ and the antisymmetric ones are the fixed points of $\mathscr{A}$. We denote by $\mathscr{I}$ the identity of $\mathscr{C}^\infty(P;\mathbf{h}P)$.

The following theorem gives the most important properties of the operators $\mathscr{S}$ and $\mathscr{A}$.

**Theorem 5.2.** (A) $(\mathscr{S},\mathscr{A})$ *is a pair of orthogonal projectors of* $\mathscr{C}^\infty(P;\mathbf{h}P)$, *that is, the following five assertions hold:*

(i) $\mathscr{S},\mathscr{A} : \mathscr{C}^\infty(P;\mathbf{h}P) \to \mathscr{C}^\infty(P;\mathbf{h}P)$ *are linear operators;*

(ii) $\mathscr{S}\circ\mathscr{S} = \mathscr{S}$ *and* $\mathscr{A}\circ\mathscr{A} = \mathscr{A}$*;*

(iii) $\mathscr{S}\circ\mathscr{A} = \mathscr{A}\circ\mathscr{S} = 0 =$ *the null operator of* $\mathscr{C}^\infty(P;\mathbf{h}P)$*;*

(iv) $\mathscr{S}+\mathscr{A} = \mathscr{I} =$ *the identity of* $\mathscr{C}^\infty(P;\mathbf{h}P)$*;*

(v) $\mathscr{S}-\mathscr{A} = \mathbf{h}^b =$ *an involution of* $\mathscr{C}^\infty(P;\mathbf{h}P)$.

(B) *With the notations* $\mu_s := \mathscr{S}\mu$ *and* $\mu_a := \mathscr{A}\mu$ *for* $\mu \in \mathscr{C}^\infty(P;\mathbf{h}P)$, *we have*

(vi) $(\mu\lambda)_s = \mu_s\lambda_s + \mu_a\lambda_a$;

(vii) $(\mu\lambda)_a = \mu_s\lambda_a + \mu_a\lambda_s$, *for every* $\mu,\lambda \in \mathscr{C}^\infty(P;\mathbf{h}P)$.

**Proof.** (i) The linearity of $\mathscr{I}$ and $\mathbf{h}^b$, together with the equalities $\mathscr{S} = (1/2)(\mathscr{I}+\mathbf{h}^b)$ and $\mathscr{A} = (1/2)(\mathscr{I}-\mathbf{h}^b)$, implies that $\mathscr{S}$ and $\mathscr{A}$ are linear.

(ii) $\mathscr{S}\circ\mathscr{S} = (1/2)(\mathscr{I}+\mathbf{h}^b)\circ(1/2)(\mathscr{I}+\mathbf{h}^b) = (1/4)(\mathscr{I}+2\mathbf{h}^b+\mathbf{h}^b\circ\mathbf{h}^b) = (1/4)(2\mathscr{I}+2\mathbf{h}^b) = \mathscr{S}$. Analogously, $\mathscr{A}\circ\mathscr{A} = \mathscr{A}$.

(iii) $\mathscr{S}\circ\mathscr{A} = (1/2)(\mathscr{I}+\mathbf{h}^b)\circ(1/2)(\mathscr{I}-\mathbf{h}^b) = (1/4)(\mathscr{I}-\mathbf{h}^b\circ\mathbf{h}^b) = 0$ since $\mathbf{h}^b$ is an involution. Analogously, $\mathscr{A}\circ\mathscr{S} = 0$.

(iv) and (v) are clear. Now we shall prove (vi).

(vi) The assertions

$$\begin{aligned}\mu_s\lambda_s+\mu_a\lambda_a &= \frac{1}{4}(\mu+\mathbf{h}^b\mu)(\lambda+\mathbf{h}^b\lambda)+\frac{1}{4}(\mu-\mathbf{h}^b\mu)(\lambda-\mathbf{h}^b\lambda)\\ &= \frac{1}{4}[\mu\lambda+\mu\mathbf{h}^b\lambda+(\mathbf{h}^b\mu)\lambda+(\mathbf{h}^b\mu)(\mathbf{h}^b\lambda)]\\ &\quad+\frac{1}{4}[\mu\lambda-\mu\mathbf{h}^b\lambda-(\mathbf{h}^b\mu)\lambda+(\mathbf{h}^b\mu)(\mathbf{h}^b\lambda)]\\ &= \frac{1}{4}[2\mu\lambda+2(\mathbf{h}^b\mu)(\mathbf{h}^b\lambda)]\\ &= \frac{1}{2}[\mu\lambda+\mathbf{h}^b(\mu\lambda)]\\ &= \mathcal{S}(\mu\lambda)=(\mu\lambda)_s.\end{aligned} \tag{5.3}$$

In a similar way, we prove (vii). □

Having in view Proposition 4.3(iii) and the properties of $\mathcal{S}$ and $\mathcal{A}$, the following notations are justified:

$$\begin{aligned}\mathscr{C}_s^\infty(P;\mathbf{h}P) &:= \{\chi\in\mathscr{C}^\infty(P;\mathbf{h}P)\mid \mathbf{h}^b\chi=\chi\},\\ \mathscr{C}_a^\infty(P;\mathbf{h}P) &:= \{\chi\in\mathscr{C}^\infty(P;\mathbf{h}P)\mid \mathbf{h}^b\chi=-\chi\}.\end{aligned} \tag{5.4}$$

Clearly, $\mathscr{C}_s^\infty(P;\mathbf{h}P)=\mathcal{S}\mathscr{C}^\infty(P;\mathbf{h}P)$ and $\mathscr{C}_a^\infty(P;\mathbf{h}P)=\mathcal{A}\mathscr{C}^\infty(P;\mathbf{h}P)$. These equalities are consequences of Theorem 5.2 and of the equivalences $\mathbf{h}^b\chi=\chi\Leftrightarrow\mathcal{S}\chi=\chi$ and $\mathbf{h}^b\chi=-\chi\Leftrightarrow\mathcal{A}\chi=\chi$.

The proofs of the following two corollaries of Theorem 5.2 are straightforward.

**Corollary 5.3.** (i) *If $\mu,\lambda\in\mathscr{C}_s^\infty(P;\mathbf{h}P)$ or $\mu,\lambda\in\mathscr{C}_a^\infty(P;\mathbf{h}P)$, then $\lambda\mu\in\mathscr{C}_s^\infty(P;\mathbf{h}P)$;*

(ii) *If $\lambda\in\mathscr{C}_s^\infty(P;\mathbf{h}P)$ and $\mu\in\mathscr{C}_a^\infty(P;\mathbf{h}P)$, then $\lambda\mu\in\mathscr{C}_a^\infty(P;\mathbf{h}P)$.*

**Corollary 5.4.** *The set $\mathscr{C}_s^\infty(P;\mathbf{h}P)$ is a subalgebra of $\mathscr{C}^\infty(P;\mathbf{h}P)$, but $\mathscr{C}_a^\infty(P;\mathbf{h}P)$ is not; however, $\mathscr{C}_a^\infty(P;\mathbf{h}P)$ is a subspace of $\mathscr{C}^\infty(P;\mathbf{h}P)$.*

The statement (ii) in the next theorem gives a representation of the algebra $\mathscr{C}^\infty(\widetilde{P})$ into $\mathscr{C}^\infty(P;\mathbf{h}P)$.

**Theorem 5.5.** (i) *The vector space $\mathscr{C}^\infty(P;\mathbf{h}P)$ has the following direct sum decomposition:*

$$\mathscr{C}^\infty(P;\mathbf{h}P)=\mathscr{C}_s^\infty(P;\mathbf{h}P)\oplus\mathscr{C}_a^\infty(P;\mathbf{h}P). \tag{5.5}$$

(ii) *The natural map $q^b:\mathscr{C}^\infty(\widetilde{P})\to\mathscr{C}_s^\infty(P;\mathbf{h}P)$ is an algebra isomorphism.*

**Proof.** (i) We have seen that $\mathscr{C}_s^\infty(P;\mathbf{h}P)=\mathcal{S}\mathscr{C}^\infty(P;\mathbf{h}P)$ and $\mathscr{C}_a^\infty(P;\mathbf{h}P)=\mathcal{A}\mathscr{C}^\infty(P;\mathbf{h}P)$. It is also clear that $\mathscr{C}_s^\infty(P;\mathbf{h}P)\cap\mathscr{C}_a^\infty(P;\mathbf{h}P)=\{0\}$, that is, the single bi-germ that is both symmetric and antisymmetric is the bi-germ zero.

For every $\mu \in \mathscr{C}^\infty(P;\mathbf{h}P)$, we have

$$\mu = \mathscr{S}\mu + \mathscr{A}\mu \tag{5.6}$$

with $\mathscr{S}\mu = \mu_s \in \mathscr{C}_s^\infty(P;\mathbf{h}P)$ and $\mathscr{A}\mu = \mu_a \in \mathscr{C}_a^\infty(P;\mathbf{h}P)$. Thus, (i) holds true.

(ii) This part of the proposition is a consequence of Proposition 4.3(i), (iii), and the definition of $\mathscr{C}_s^\infty(P;\mathbf{h}P)$. □

As C. Constantinescu noticed, we can prove that the subspace $\mathscr{C}_a^\infty(P;\mathbf{h}P)$ of the algebra $\mathscr{C}^\infty(P;\mathbf{h}P)$ is isomorphic with the vector space of germs of functions of odd type in the sense of G. de Rham (see [5, page 27]).

## 6. Vector fields on $X$ and $\mathbb{O}_2$

**Definition 6.1.** The vector field $\vec{Z}:\mathbb{O}_2 \to \mathbf{T}\mathbb{O}_2$ is called symmetric or $d\mathbf{h}$-invariant (resp., antisymmetric or $d\mathbf{h}$-antiinvariant) if and only if $(dh)(\vec{Z}) = \vec{Z}$ (resp., $(dh)(\vec{Z}) = -\vec{Z}$).

We denote, as usual, by $\chi(\mathbf{X})$ and $\chi(\mathbb{O}_2)$ the sets of vector fields on $\mathbf{X}$ and, respectively, $\mathbb{O}_2$ endowed with their algebraic structures (vector spaces and modules over the algebra of complex functions). We introduce the following notations:

$$\begin{aligned}\chi_s(\mathbb{O}_2) &:= \{\vec{Y} \in \chi(\mathbb{O}_2) \mid (d\mathbf{h})(\vec{Y}) = \vec{Y}\};\\ \chi_a(\mathbb{O}_2) &:= \{\vec{Y} \in \chi(\mathbb{O}_2) \mid (d\mathbf{h})(\vec{Y}) = -\vec{Y}\}.\end{aligned} \tag{6.1}$$

The main result of this paper is assertion (iii) of the next theorem.

**Theorem 6.2.** (i) $\chi_s(\mathbb{O}_2)$ *and* $\chi_a(\mathbb{O}_2)$ *are subspaces of* $\chi(\mathbb{O}_2)$*;*

(ii) *the following direct sum decomposition holds true:*

$$\chi(\mathbb{O}_2) = \chi_s(\mathbb{O}_2) \oplus \chi_a(\mathbb{O}_2); \tag{6.2}$$

(iii) $\chi(\mathbf{X})$ *is canonically isomorphic with* $\chi_s(\mathbb{O}_2)$*.*

We present the proof of this theorem in several steps.

**Step 1** (elementary fields of tangent vectors (EFTV) on $\mathbb{O}_2$). Let $\vec{X}_{\tilde{P}} \in \mathbf{T}_{\tilde{P}}\mathbf{X}$. The differentials $\mathbf{q}_{*,P} = d_P\mathbf{q}$ and $\mathbf{q}_{*,\mathbf{h}P} = d_{\mathbf{h}P}\mathbf{q}$ are bijective mappings. They lead to the tangent vectors $\vec{Y}_P := (d_P\mathbf{q})^{-1}(\vec{X}_{\tilde{P}}) \in \mathbf{T}_P\mathbb{O}_2$ and $\vec{Y}_{\mathbf{h}P} := (d_{\mathbf{h}P}\mathbf{q})^{-1}(\vec{X}_{\tilde{P}}) \in \mathbf{T}_{\mathbf{h}P}\mathbb{O}_2$. Thus, to every tangent vector $\vec{X}_{\tilde{P}} \in \mathbf{T}_{\tilde{P}}\mathbf{X}$ corresponds a nonordered pair of tangent vectors $\{\vec{Y}_P;\vec{Y}_{\mathbf{h}P}\}$ where $\vec{Y}_P \in \mathbf{T}_P\mathbb{O}_2$ and $\vec{Y}_{\mathbf{h}P} \in \mathbf{T}_{\mathbf{h}P}\mathbb{O}_2$. This circumstance suggests the following definition.

**Definition 6.3.** An elementary field of tangent vectors (EFTV) on $\mathbb{O}_2$ (at the pair of symmetric points $\{P;\mathbf{h}P\}$) is a function

$$\vec{Z}:\{P;\mathbf{h}P\}\longrightarrow \mathbf{T}_P\mathbb{O}_2\cup\mathbf{T}_{\mathbf{h}P}\mathbb{O}_2 \tag{6.3}$$

such that $\vec{Z}(Q):=\vec{Z}_Q\in\mathbf{T}_Q\mathbb{O}_2$ for every $Q\in\{P;\mathbf{h}P\}$.

We denote by $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ the set of all EFTV at $\{P;\mathbf{h}P\}$, and the element $\vec{Z}\in\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ will be denoted by $\{\vec{Z}_P;\vec{Z}_{\mathbf{h}P}\}$.

The set $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ will be endowed with its natural structure of vector space: If $\vec{Z}=\{\vec{Z}_P;\vec{Z}_{\mathbf{h}P}\}$, $\vec{W}=\{\vec{W}_P;\vec{W}_{\mathbf{h}P}\}\in\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ and $a\in\mathbb{C}$, then $\vec{Z}+\vec{W}$ and $a\vec{Z}$ are defined as:

$$\begin{aligned}\vec{Z}+\vec{W}&:=\{\vec{Z}_P+\vec{W}_P;\vec{Z}_{\mathbf{h}P}+\vec{W}_{\mathbf{h}P}\},\\ a\vec{Z}&:=\{a\vec{Z}_P;\ a\vec{Z}_{\mathbf{h}P}\}.\end{aligned} \tag{6.4}$$

It is obvious that the differential

$$d\mathbf{h}=\mathbf{h}_*:\mathbf{T}\mathbb{O}_2\longrightarrow\mathbf{T}\mathbb{O}_2 \tag{6.5}$$

is a fixed-point free *involution*. If $\vec{Z}_Q\in\mathbf{T}_Q\mathbb{O}_2$, then $(d\mathbf{h})(\vec{Z}_Q)\in\mathbf{T}_{\mathbf{h}Q}\mathbb{O}_2$.

We see that the restriction of $d\mathbf{h}$ to $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ (still denoted by $d\mathbf{h}$) continues to be an involution. We have

$$(d_P\mathbf{h})^{-1}=d_{\mathbf{h}P}\mathbf{h}:\mathbf{T}_{\mathbf{h}P}\mathbb{O}_2\longrightarrow\mathbf{T}_P\mathbb{O}_2. \tag{6.6}$$

If $\vec{Z}=\{\vec{Z}_P;\vec{Z}_{\mathbf{h}P}\}\in\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$, then $(d\mathbf{h})(\vec{Z})=\{(d\mathbf{h})(\vec{Z}_{\mathbf{h}P});(d\mathbf{h})(\vec{Z}_P)\}\in\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$.

We now formulate the counterpart of Definition 5.1.

**Definition 6.4.** The EFTV $\vec{Z}$ is called symmetric or $d\mathbf{h}$-invariant (resp., antisymmetric or $d\mathbf{h}$-antiinvariant) if and only if $(d\mathbf{h})(\vec{Z})=\vec{Z}$ (resp., $(d\mathbf{h})(\vec{Z})=-\vec{Z}$).

If $\vec{X}_{\tilde{P}}\in\mathbf{T}_{\tilde{P}}\mathbf{X}$, then $(d\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})=\vec{Y}=\{\vec{Y}_P;\vec{Y}_{\mathbf{h}P}\}$, where $\vec{Y}_P=(d_P\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})$ and $\vec{Y}_{\mathbf{h}P}=(d_{\mathbf{h}P}\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})$.

**Proposition 6.5.** (i) *For every $\vec{X}_{\tilde{P}}\in\mathbf{T}_{\tilde{P}}\mathbf{X}$, the EFTV $\vec{Y}=(d\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})$ is $d\mathbf{h}$-invariant.*

(ii) *Conversely, if $\vec{Z}=\{\vec{Z}_P;\vec{Z}_{\mathbf{h}P}\}\in\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ is $d\mathbf{h}$-invariant, then*

$$(d_P\mathbf{q})(\vec{Z}_P)=(d_{\mathbf{h}P}\mathbf{q})(\vec{Z}_{\mathbf{h}P}):=\vec{X}_{\tilde{P}}\in T_{\tilde{P}}X \tag{6.7}$$

*and $(d\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})=\vec{Z}$.*

The proof is simple and will be omitted.

The following simple proposition gives *the rule of derivation* on nonorientable Klein surfaces.

**Proposition 6.6.** *Let* $\vec{X}_{\tilde{P}} \in \mathbf{T}_{\tilde{P}}\mathbf{X}$ *and* $\tilde{f} \in \mathscr{C}^\infty(\tilde{P})$. *If* $\vec{Y} = \{\vec{Y}_P; \vec{Y}_{\mathbf{h}P}\} = (d\mathbf{q})^{-1}(\vec{X}_{\tilde{P}})$ *and* $\{f_P; f_{\mathbf{h}P}\} := q^b(\tilde{f}) = \{q_P^*(\tilde{f}); q_{\mathbf{h}P}^*(\tilde{f})\} \in \mathscr{C}^\infty(P, \mathbf{h}P)$, *then*

$$\vec{X}_{\tilde{P}}(\tilde{f}) = \vec{Y}_P(f_P) = \vec{Y}_{\mathbf{h}P}(f_{\mathbf{h}P}). \tag{6.8}$$

**Proof.** Since $f_P = \mathbf{h}_P^*(f_{\mathbf{h}P})$ and $\vec{Y}$ is $d\mathbf{h}$-invariant, we get

$$\vec{Y}_P(f_P) = \vec{Y}_P(\mathbf{h}_P^*(f_{\mathbf{h}P})) = (d_P\mathbf{h})(\vec{Y}_P)(f_{\mathbf{h}P}) = \vec{Y}_{\mathbf{h}P}(f_{\mathbf{h}P}). \tag{6.9}$$

On the other hand, $\vec{Y}_P(f_P) = \vec{Y}_P(\mathbf{q}_P^*(\tilde{f})) = (d\mathbf{q})(\vec{Y}_P)(\tilde{f}) = \vec{X}_{\tilde{P}}(\tilde{f})$. □

**Step 2** (symmetrisation and antisymmetrisation of EFTV)**.** We consider $\vec{Y} = \{\vec{Y}_P; \vec{Y}_{\mathbf{h}P}\} \in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$. Generally, there is no connection between $\vec{Y}_P$ and $\vec{Y}_{\mathbf{h}P}$. We remember that $(d_P\mathbf{h})^{-1} = d_{\mathbf{h}P}\mathbf{h}$ and that the restriction of $d\mathbf{h}$ to $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ (also denoted by $d\mathbf{h}$) is an involution of $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$.

As in the case of bi-germs, we define the operators $\mathscr{S}, \mathscr{A} : \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \to \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ by

$$\begin{aligned} \mathscr{S}\vec{Y} &:= \frac{1}{2}[\vec{Y} + (d\mathbf{h})(\vec{Y})], \\ \mathscr{A}\vec{Y} &:= \frac{1}{2}[\vec{Y} - (d\mathbf{h})(\vec{Y})] \end{aligned} \tag{6.10}$$

for every $\vec{Y} \in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$.

Clearly, $(d\mathbf{h})(\vec{Y}) = \vec{Y}$ if and only if $\mathscr{S}\vec{Y} = \vec{Y}$ and $(d\mathbf{h})(\vec{Y}) = -\vec{Y}$ if and only if $\mathscr{A}\vec{Y} = \vec{Y}$. Based on Definition 6.1, we call the operators $\mathscr{S}$ and $\mathscr{A}$ *the operators of symmetrisation* and *antisymmetrisation*, respectively, of EFTV on $\mathbb{O}_2$.

We denote by $\mathscr{I}$ the identity of $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$. Formulas (6.10) become

$$\begin{aligned} \mathscr{S} &= \frac{1}{2}[\mathscr{I} + d\mathbf{h}], \\ \mathscr{A} &:= \frac{1}{2}[\mathscr{I} - d\mathbf{h}]. \end{aligned} \tag{6.11}$$

The next theorem is similar to Theorem 5.2 and the proof will be omitted.

**Theorem 6.7.** *The operators* $\mathscr{S}$ *and* $\mathscr{A}$ *are a pair of orthogonal projectors of the space* $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$, *that is, the following five assertions hold:*

(i) $\mathscr{S}, \mathscr{A} : \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \to \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$ *are linear operators;*

(ii) $\mathscr{S} \circ \mathscr{S} = \mathscr{S}$ *and* $\mathscr{A} \circ \mathscr{A} = \mathscr{A}$;

(iii) $\mathcal{S} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{S} = 0 =$ *the null operator of* $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$*;*

(iv) $\mathcal{S} + \mathcal{A} = \mathcal{I} =$ *the identity of* $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$*;*

(v) $\mathcal{S} - \mathcal{A} = d\mathbf{h} =$ *an involution of* $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$*.*

**Step 3** (a representation theorem for $\mathbf{T}_{\tilde{P}}\mathbf{X}$)**.** We introduce the following notations:

$$\begin{aligned} \mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2 &:= \{\vec{Y} \in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \mid (d\mathbf{h})(\vec{Y}) = \vec{Y}\}; \\ \mathbf{ET}_{a;\{P;\mathbf{h}P\}}\mathbb{O}_2 &:= \{\vec{Y} \in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \mid (d\mathbf{h})(\vec{Y}) = -\vec{Y}\}. \end{aligned} \tag{6.12}$$

Now, we can formulate the following theorem.

**Theorem 6.8.** (i) *The sets* $\mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2$ *and* $\mathbf{ET}_{a;\{P;\mathbf{h}P\}}\mathbb{O}_2$ *are subspaces of* $\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$*;*

(ii) *the following direct sum decomposition holds true:*

$$\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 = \mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2 \oplus \mathbf{ET}_{a;\{P;\mathbf{h}P\}}\mathbb{O}_2; \tag{6.13}$$

(iii) *the vector spaces* $\mathbf{T}_{\tilde{P}}\mathbf{X}$ *and* $\mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2$ *are canonically isomorphic.*

**Proof.** (i) The operators $\mathcal{S}$ and $\mathcal{A}$ are linear operators and $\mathcal{S}(\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2) = \mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2$ and $\mathcal{A}(\mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2) = \mathbf{ET}_{a;\{P;\mathbf{h}P\}}\mathbb{O}_2$. This proves (i).

(ii) Obviously, $\mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2 \cap \mathbf{ET}_{a;\{P;\mathbf{h}P\}}\mathbb{O}_2 = \{0\}$ and $\vec{Y} = \mathcal{S}\vec{Y} + \mathcal{A}\vec{Y}$ for every $\vec{Y} \in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$.

(iii) We have seen that if $\vec{Z} = \{\vec{Z}_P, \vec{Z}_{\mathbf{h}P}\} \in \mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2$ then $(d_P\mathbf{q})(\vec{Z}_P) = (d_{\mathbf{h}P}\mathbf{q})(\vec{Z}_{\mathbf{h}P})$. We define

$$\mathbf{q}_b : \mathbf{ET}_{s;\{P;\mathbf{h}P\}}\mathbb{O}_2 \longrightarrow \mathbf{T}_{\tilde{P}}\mathbf{X} \tag{6.14}$$

by $\mathbf{q}_b(\vec{Z}) := (d_P\mathbf{q})(\vec{Z}_P) = (d_{\mathbf{h}P}\mathbf{q})(\vec{Z}_{\mathbf{h}P})$. The mapping $\mathbf{q}_b$ is the natural isomorphism mentioned in (iii). □

**Step 4** (global symmetrisation and antisymmetrisation)**.** The global symmetrisation operator $\mathcal{S}$ and the global antisymmetrisation operator $\mathcal{A}$ (or the operators of $d\mathbf{h}$-*invariance* and $d\mathbf{h}$-*antiinvariance*, respectively) are defined in a way similar to the case of EFTVs, namely, $\mathcal{S}, \mathcal{A} : \chi(\mathbb{O}_2) \to \chi(\mathbb{O}_2)$, where

$$\begin{aligned} \mathcal{S}(\vec{X}) &:= \frac{1}{2}[\vec{X} + (d\mathbf{h})(\vec{X})], \\ \mathcal{A}(\vec{X}) &:= \frac{1}{2}[\vec{X} - (d\mathbf{h})(\vec{X})], \end{aligned} \tag{6.15}$$

for every $\vec{X} \in \chi(\mathbb{O}_2)$ or, equivalently, in terms of $\mathcal{I}$ and $d\mathbf{h}$,

$$\begin{aligned} \mathcal{S} &:= \frac{1}{2}[\mathcal{I} + d\mathbf{h}], \\ \mathcal{A} &:= \frac{1}{2}[\mathcal{I} - d\mathbf{h}]. \end{aligned} \tag{6.16}$$

If $\{P;\mathbf{h}P\}$ is an arbitrary pair of symmetric points on $\mathbb{O}_2$, we denote by $\mathcal{R}$ the operator of restriction to $\{P;\mathbf{h}P\}$ of the vector fields $\vec{X}\in\chi(\mathbb{O}_2)$

$$\mathcal{R}(\vec{X}) := \vec{X}|_{\{P;\mathbf{h}P\}} = \{\vec{X}_P;\vec{X}_{\mathbf{h}P}\}. \tag{6.17}$$

Indeed, the restriction of $\vec{X}$ to the two-element set $\{P;\mathbf{h}P\}$ may be identified with the image by $\vec{X}$ of this set, namely, with $\{\vec{X}_P;\vec{X}_{\mathbf{h}P}\}$. This means that $\mathcal{R}(\vec{X})\in \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2$, that is,

$$\mathcal{R}:\chi(\mathbb{O}_2)\longrightarrow \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2. \tag{6.18}$$

It is obvious that the following two diagrams are commutative:

$$\begin{tikzcd} \chi(\mathbb{O}_2) \arrow[r, "\mathcal{S}"] \arrow[d, "\mathcal{R}"'] & \chi(\mathbb{O}_2) \arrow[d, "\mathcal{R}"] \\ \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \arrow[r, "\mathcal{S}"] & \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2, \end{tikzcd} \qquad \begin{tikzcd} \chi(\mathbb{O}_2) \arrow[r, "\mathcal{A}"] \arrow[d, "\mathcal{R}"'] & \chi(\mathbb{O}_2) \arrow[d, "\mathcal{R}"] \\ \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2 \arrow[r, "\mathcal{A}"] & \mathbf{ET}_{\{P;\mathbf{h}P\}}\mathbb{O}_2. \end{tikzcd} \tag{6.19}$$

As a consequence of the fact that $d\mathbf{h}:\mathbf{T}\mathbb{O}_2\to\mathbf{T}\mathbb{O}_2$ is an involution, we can formulate the following theorem, which is the globalisation of Theorem 6.7. The proof will be omitted.

**Theorem 6.9.** *The pair of operators $(\mathcal{S},\mathcal{A})$ defined by (6.15) is a pair of orthogonal projectors of the vector space $\chi(\mathbb{O}_2)$, that is,*

(i) *$\mathcal{S},\mathcal{A}:\chi(\mathbb{O}_2)\to\chi(\mathbb{O}_2)$ are linear operators;*
(ii) *$\mathcal{S}\circ\mathcal{S}=\mathcal{S}$ and $\mathcal{A}\circ\mathcal{A}=\mathcal{A}$;*
(iii) *$\mathcal{S}\circ\mathcal{A}=\mathcal{A}\circ\mathcal{S}=0=$ the null operator of $\chi(\mathbb{O}_2)$;*
(iv) *$\mathcal{S}+\mathcal{A}=\mathcal{I}=$ the identity of $\chi(\mathbb{O}_2)$;*
(v) *$\mathcal{S}-\mathcal{A}=d\mathbf{h}=$ an involution of $\chi(\mathbb{O}_2)$.*

**Step 5** (final remarks)**.** We can now complete the proof of Theorem 6.2.

(i) The global operators $\mathcal{S}$ and $\mathcal{A}$ are linear operators on the vector space $\chi(\mathbb{O}_2)$. Moreover, $\mathcal{S}\chi(\mathbb{O}_2)=\chi_s(\mathbb{O}_2)$ and $\mathcal{A}\chi(\mathbb{O}_2)=\chi_a(\mathbb{O}_2)$. The same conclusion follows directly from (6.1). However, these equalities highlight the fact that the elements of $\chi_s(\mathbb{O}_2)$ and $\chi_a(\mathbb{O}_2)$ appear, respectively, as the $d\mathbf{h}$-invariant and $d\mathbf{h}$-antiinvariant components of the elements of $\chi(\mathbb{O}_2)$.

(ii) Clearly, $\vec{X}\in\chi_s(\mathbb{O}_2)\cap\chi_a(\mathbb{O}_2)$ if and only if $\vec{X}=\vec{0}=$ the *zero* vector field on $\mathbb{O}_2$. Since $\vec{X}=\mathcal{S}\vec{X}+\mathcal{A}\vec{X}$ for every $\vec{X}\in\chi(\mathbb{O}_2)$ and since $\mathcal{S}\vec{X}\in\chi_s(\mathbb{O}_2)$ and $\mathcal{A}\vec{X}\in\chi_a(\mathbb{O}_2)$, (ii) holds true.

(iii) Let $\vec{M}\in\chi(\mathbf{X})$ be an arbitrary vector field on $\mathbf{X}$ and let $\tilde{P}\in\mathbf{X}$. According to Proposition 6.5(i), $\vec{Y}:=(d\mathbf{q})^{-1}(\vec{M})$ is a $d\mathbf{h}$-invariant EFTV on $\mathbb{O}_2$. Since the $\langle\mathbf{h}\rangle$-orbits on $\mathbb{O}_2$ are either identical or disjoint and since $\mathbb{O}_2=\bigcup_{\tilde{P}\in\mathbf{X}}\{P;\mathbf{h}P\}$, we can define $\vec{X}\in\chi_s(\mathbb{O}_2)$ by its restrictions

$$\vec{X}|_{\{P;\mathbf{h}P\}}:=(d\mathbf{q})^{-1}(\vec{M}_{\tilde{P}}) \tag{6.20}$$

to the set of $\langle \mathbf{h} \rangle$-orbits $\{P;\mathbf{h}P\}$. It is an easy exercise to check that the correspondence $\overrightarrow{M} \leftrightarrow \overrightarrow{X}$ is an isomorphism between the vector spaces $\chi(\mathbf{X})$ and $\chi_s(\mathbb{O}_2)$.

**7. The normal derivative to the border of the Möbius strip.** As an application of the topics that we have dealt with so far, we present, in detail, the normal derivative to the border of the Möbius strip, which is a vector field defined on that border.

We consider the annulus $\mathbf{A}_R$ defined as

$$A_R := \left\{ z \in \mathbb{C} \mid \frac{1}{R} \leq |z| \leq R \right\}, \tag{7.1}$$

where $R > 1$ is fixed. The map $\mathbf{h} : A_R \to A_R$ defined by

$$\mathbf{h}z := \mathbf{h}(z) := -\frac{1}{\overline{z}} = w = u + iv \tag{7.2}$$

for every $z = x + iy \in A_R$ is a fixed-point free antianalytic involution of $A_R$ and the orbit space

$$M_R := A_R / \langle \mathbf{h} \rangle, \tag{7.3}$$

where $\langle \mathbf{h} \rangle = \{Id;\mathbf{h}\}$ is the two element group generated by $\mathbf{h}$, is a Möbius strip.

If $z \in A_R$, its $\mathbf{h}$-orbit is the two-element set

$$\widetilde{z} = \{z;\mathbf{h}z\} = \widetilde{\mathbf{h}z}. \tag{7.4}$$

If $z = \rho e^{i\theta}$, $1/R \leq \rho \leq R$, $0 \leq \theta < \pi$, the $\mathbf{h}$-symmetric point of $z$ is $w = -1/\overline{z} = (1/\rho)e^{i(\theta+\pi)}$.

When $z$ runs over the line segment with end-points $Re^{i\theta}$ and $(1/R)e^{i\theta}$ from $Re^{i\theta}$ towards $(1/R)e^{i\theta}$ (i.e., $\rho$ decreases from $R$ to $1/R$), its symmetric $w$ runs on the line segment with endpoints $(1/R)e^{i(\theta+\pi)}$ and $Re^{i(\theta+\pi)}$ from the first point toward the second.

The border of $M_R$ consists of the orbits $\widetilde{z}$ such that $|z| = R$ or $1/R$, that is,

$$\partial M_R = \left\{ \left\{ Re^{i\theta}; \frac{1}{R} e^{i(\theta+\pi)} \right\} \mid 0 \leq \theta < \pi \right\}. \tag{7.5}$$

If $\mathbf{q} : A_R \to M_R$ is the canonical projection and $z_0 = Re^{i\theta_0}$, then $\mathbf{q}^{-1}(\widetilde{z_0})$ consists of $z_0$ and $(1/R)e^{i(\theta_0+\pi)} = w_0$.

The *unit* normal vector that we use in the definition of the normal derivative to the border of $A_R$ is taken with respect to the Euclidian metric. This metric should be replaced now with its $\mathbf{h}$-invariant component in order to be able to give a meaning to the normal derivative to $\partial M_R$ [4, 6, 8].

The unit inner normal (to $\partial A_R$) vectors with respect to the Euclidian metric $ds^2 = dx^2 + dy^2$ at the points $z_0$ and $w_0 = \mathbf{h}z_0$ are

$$\vec{Y}_{z_0} = \left(\frac{\partial}{\partial \mathbf{n}}\right)_{z_0} = -(\cos\theta_0)\left(\frac{\partial}{\partial x}\right)_{z_0} - (\sin\theta_0)\left(\frac{\partial}{\partial y}\right)_{z_0},$$
$$\vec{Y}_{w_0} = \left(\frac{\partial}{\partial \mathbf{n}}\right)_{w_0} = -(\cos\theta_0)\left(\frac{\partial}{\partial u}\right)_{w_0} - (\sin\theta_0)\left(\frac{\partial}{\partial v}\right)_{w_0}. \tag{7.6}$$

With our earlier notations,

$$\vec{Y} = \{\vec{Y}_{z_0}; \vec{Y}_{w_0}\} \in \mathbf{ET}_{\{z_0;w_0\}} A_R. \tag{7.7}$$

In all that follows, $A_R$ will be endowed with the metric $g_s$ given by

$$g_s(z) = d\sigma^2 = \frac{1}{4}\left(1 + \frac{1}{|z|^2}\right)^2 [dx \otimes dx + dy \otimes dy] \tag{7.8}$$

and $M_R$ with the projection $\widetilde{g_s}$ of this metric by $\mathbf{q}$ (see [8]).

Thus, $\mathbf{h}$ is an *isometric involution* of $(A_R, g_s)$ and

$$\mathbf{q} : (A_R, g_s) \longrightarrow (M_R, \widetilde{g_s}) \tag{7.9}$$

is a *local isometry*.

If $z \in A_R$, we define the scalar product and the norm induced by $g_s$ on $\mathbf{T}_z A_R$ as follows.

If $\vec{U} = U_1(\partial/\partial x)_z + U_2(\partial/\partial y)_z, \vec{V} = V_1(\partial/\partial x)_z + V_2(\partial/\partial y)_z \in \mathbf{T}_z A_R$, their scalar product $\langle \vec{U}, \vec{V} \rangle = \langle \vec{U}, \vec{V} \rangle_{g_s}$ is given by

$$\langle \vec{U}, \vec{V} \rangle = g_s(z)(\vec{U}, \vec{V}) = \frac{1}{4}\left(1 + \frac{1}{|z|^2}\right)^2 [U_1V_1 + U_2V_2]. \tag{7.10}$$

The norm $\|\vec{U}\| = \|\vec{U}\|_{g_s}$ of $\vec{U}$, given by (7.10), is

$$||\vec{U}|| = \langle \vec{U}, \vec{U} \rangle^{1/2} = \frac{1}{2}\left(1 + \frac{1}{|z|^2}\right)\sqrt{U_1^2 + U_2^2}. \tag{7.11}$$

The $g_s$-length of the vectors $\vec{Y}_{z_0}$ and $\vec{Y}_{w_0}$ is given by

$$||\vec{Y}_{z_0}|| = \frac{1}{2}\left(1 + \frac{1}{R^2}\right)\sqrt{\cos^2\theta_0 + \sin^2\theta_0} = \frac{R^2+1}{2R^2};$$
$$||\vec{Y}_{w_0}|| = \frac{1}{2}(1 + R^2)\sqrt{\cos^2\theta_0 + \sin^2\theta_0} = \frac{R^2+1}{2}. \tag{7.12}$$

Thus, $\|\vec{Y}_{z_0}\| < 1 < \|\vec{Y}_{w_0}\|$. As $\vec{Y} = \{\vec{Y}_{z_0}; \vec{Y}_{w_0}\} \notin \mathbf{ET}_{s;\{z_0;w_0\}} A_R$, *it cannot be used to define the unit normal vector to* $\partial M_R$*!*

The relations (7.12) suggest, instead,

$$\vec{N}_0 := \left\{ \frac{2R^2}{R^2+1} \vec{Y}_{z_0}; \frac{2}{R^2+1} \vec{Y}_{w_0} \right\} := \{\vec{N}_{z_0}; \vec{N}_{w_0}\} \in \mathbf{ET}_{\{z_0;w_0\}} A_R. \tag{7.13}$$

Clearly, $\|\vec{N}_{z_0}\| = \|\vec{N}_{w_0}\| = 1$.

If $z = x + iy \in A_R$ and $w = u + iv = -1/\overline{z} = \mathbf{h}z$, then the action of the differential $d_z\mathbf{h} = \mathbf{h}_{*,z}$ on the vectors $(\partial/\partial x)_z, (\partial/\partial y)_z$ of the basis of $\mathbf{T}_z A_R$ is given by

$$\begin{aligned} (d_z\mathbf{h})\left(\left(\frac{\partial}{\partial x}\right)_z\right) &= u'_x(z)\left(\frac{\partial}{\partial u}\right)_w + v'_x(z)\left(\frac{\partial}{\partial v}\right)_w; \\ (d_z\mathbf{h})\left(\left(\frac{\partial}{\partial y}\right)_z\right) &= u'_y(z)\left(\frac{\partial}{\partial u}\right)_w + v'_y(z)\left(\frac{\partial}{\partial v}\right)_w, \end{aligned} \tag{7.14}$$

where $\{(\partial/\partial u)_w; (\partial/\partial v)_w\}$ is the basis of $\mathbf{T}_w A_R$ and $u'_x(z) = (\partial u/\partial x)(z)$, and so on. The Jacobian matrix $J_{\mathbf{h};z}$ of $\mathbf{h}$ at the point $z$ is

$$J_{\mathbf{h};z} = \begin{bmatrix} u'_x(z) & v'_x(z) \\ u'_y(z) & v'_y(z) \end{bmatrix} = \begin{bmatrix} \dfrac{x^2-y^2}{|z|^4} & \dfrac{2xy}{|z|^4} \\ \dfrac{2xy}{|z|^4} & \dfrac{-(x^2-y^2)}{|z|^4} \end{bmatrix}. \tag{7.15}$$

If we take $z = \mathbf{h}^{-1}w = \mathbf{h}w$, the analogues of formulas of (7.14) and (7.15) are

$$\begin{aligned} (d_w\mathbf{h})\left(\left(\frac{\partial}{\partial u}\right)_w\right) &= x'_u(w)\left(\frac{\partial}{\partial x}\right)_z + y'_u(w)\left(\frac{\partial}{\partial y}\right)_z; \\ (d_w\mathbf{h})\left(\left(\frac{\partial}{\partial u}\right)_w\right) &= x'_v(w)\left(\frac{\partial}{\partial x}\right)_z + y'_v(w)\left(\frac{\partial}{\partial y}\right)_z, \end{aligned} \tag{7.16}$$

respectively,

$$J_{\mathbf{h};w} = \begin{bmatrix} x'_u(w) & y'_u(w) \\ x'_v(w) & y'_v(w) \end{bmatrix} = \begin{bmatrix} \dfrac{u^2-u^2}{|w|^4} & \dfrac{2uv}{|w|^4} \\ \dfrac{2uv}{|w|^4} & \dfrac{-(u^2-v^2)}{|w|^4} \end{bmatrix}. \tag{7.17}$$

Let $\vec{X} = \{\vec{X}_z, \vec{X}_w\} \in \mathbf{ET}_{\{z;w\}} A_R$

$$\begin{aligned} \vec{X}_z &= A_z\left(\frac{\partial}{\partial x}\right)_z + B_z\left(\frac{\partial}{\partial y}\right)_z, \quad A_z, B_z \in \mathbb{C}; \\ \vec{X}_w &= A_w\left(\frac{\partial}{\partial u}\right)_w + B_w\left(\frac{\partial}{\partial v}\right)_w, \quad A_w, B_w \in \mathbb{C}. \end{aligned} \tag{7.18}$$

Clearly $(d\mathbf{h})(\vec{X}) = \{(d_w\mathbf{h})(\vec{X}_w);(d_z\mathbf{h})(\vec{X}_z)\} \in \mathbf{ET}_{\{z;w\}}A_R$, and the two elements of $(d\mathbf{h})(\vec{X})$ are obtained from (7.14) and (7.16) and are given by

$$(d_w\mathbf{h})(\vec{X}_w) = [A_w x'_u(w)+B_w x'_v(w)]\left(\frac{\partial}{\partial x}\right)_z + [A_w y'_u(w)+B_w y'_v(w)]\left(\frac{\partial}{\partial y}\right)_z,$$
$$(d_z\mathbf{h})(\vec{X}_w) = [A_z u'_x(z)+B_z u'_y(z)]\left(\frac{\partial}{\partial u}\right)_w + [A_z v'_x(z)+B_z v'_y(z)]\left(\frac{\partial}{\partial v}\right)_w. \tag{7.19}$$

Taking into account (7.18) and (7.19), we can formulate the following proposition.

**Proposition 7.1.** *The EFTVs $\vec{X}$ is $d\mathbf{h}$-invariant, that is, $\vec{X} \in \mathbf{ET}_{s;\{z;w\}}A_R$ if and only if*

$$\begin{aligned} A_w x'_u(w)+B_w x'_v(w) &= A_z;\\ A_w y'_u(w)+B_w y'_v(w) &= B_z. \end{aligned} \tag{7.20}$$

We now consider the particular case $z = z_0 = Re^{i\theta_0}$, $w = w_0 = (1/R)e^{i(\theta_0+\pi)}$ and $\vec{X} = \vec{N}_0 =$ the EFTV given by (7.13).

The coefficients $A_z$, $B_z$, $A_w$, and $B_w$ become, in this case,

$$\begin{aligned} A_z &= \frac{-2R^2\cos\theta_0}{R^2+1}; \qquad & B_z &= \frac{-2R^2\sin\theta_0}{R^2+1}\\ A_w &= \frac{-2\cos\theta_0}{R^2+1}; \qquad & B_w &= \frac{-2\sin\theta_0}{R^2+1}. \end{aligned} \tag{7.21}$$

The corresponding Jacobian matrix is

$$J_{\mathbf{h};w} = \begin{bmatrix} x'_u(w) & y'_u(w)\\ x'_v(w) & y'_v(w)\end{bmatrix} = \begin{bmatrix} R^2\cos 2\theta_0 & R^2\sin 2\theta_0\\ R^2\sin 2\theta_0 & -R^2\cos 2\theta_0\end{bmatrix}. \tag{7.22}$$

We check the validity of relations (7.20) corresponding to the following case:

$$\begin{aligned} A_w x'_u(w)+B_w x'_v(w) &= \frac{-2\cos\theta_0}{R^2+1}R^2\cos 2\theta_0 + \frac{-2\sin\theta_0}{R^2+1}R^2\sin 2\theta_0\\ &= \frac{-2R^2(\cos\theta_0\cos 2\theta_0+\sin\theta_0\sin 2\theta_0)}{R^2+1}\\ &= \frac{-2R^2\cos\theta_0}{R^2+1} = A_z. \end{aligned} \tag{7.23}$$

In the same way, we can see that the second relation of (7.20) is fulfilled.

Thus, the EFTV $\vec{N}_0$ is in $\mathbf{ET}_{s;\{z;w\}}A_R$.

*The (inner)* $\widetilde{g_s}$*-unit normal vector* to the border of $M_R$ at the point $\widetilde{z_0}$ is therefore the image of $\vec{N}_0$ by $d\mathbf{q}$

$$\widetilde{\vec{N}_0} := d\mathbf{q}(\vec{N}_0). \tag{7.24}$$

We see now how $\widetilde{\vec{N}_0}$ acts on germs $\tilde{f}_{\tilde{z}} \in \mathscr{C}^\infty(\tilde{z})$. Here, the answer is given by Proposition 6.6.

$\{f_z, f_w\} = \mathbf{q}^b(\tilde{f}_{\tilde{z}})$ is a symmetric bi-germ and

$$\widetilde{\vec{N}_0}(\tilde{f}_{\tilde{z}}) = \vec{N}_{z_0}(f_z) = \vec{N}_{w_0}(f_w), \tag{7.25}$$

where $\vec{N}_{z_0}$ and $\vec{N}_{w_0}$ are given by (7.13). Thus,

$$\widetilde{\vec{N}_0}(\tilde{f}_{\tilde{z}}) = -\frac{2R^2}{R^2+1}\left[(\cos\theta_0)\frac{\partial f_z}{\partial x}(z_0) + (\sin\theta_0)\frac{\partial f_z}{\partial y}(z_0)\right]. \tag{7.26}$$

A similar result has been obtained in [6, 7] where the derivative, with respect to the exterior normal, was taken.

As a final remark, we should notice that, since any nonorientable surface **S** is “created” by a symmetric Riemann surface **R**, all the objects on **S** should be dependent on “similar” objects on **R**. Although the EFTVs might appear somehow artificial, their existence is dictated by the previously mentioned philosophy.

## References


[1] L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Mathematical Series, no. 26, Princeton University Press, New Jersey, 1960.

[2] N. L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Lecture Notes in Mathematics, vol. 219, Springer-Verlag, Berlin, 1971.

[3] C. Andreian Cazacu, *On the morphisms of Klein surfaces*, Rev. Roumaine Math. Pures Appl. **31** (1986), no. 6, 461–470.

[4] I. Bârză, *Integration on nonorientable Riemann surfaces*, Almost Complex Structures (Sofia, 1992) (K. Sekigawa and S. Dimiev, eds.), World Scientific Publishing, New Jersey, 1994, pp. 63–97.

[5] ______, *Calculus on nonorientable Riemann surfaces*, Libertas Math. **15** (1995), 1–45.

[6] I. Bârză and D. Ghişa, *Boundary value problems on nonorientable surfaces*, Rev. Roumaine Math. Pures Appl. **43** (1998), no. 1-2, 67–79.

[7] ______, *Explicit formulas for Green's functions on the annulus and on the Möbius strip*, Acta Appl. Math. **54** (1998), no. 3, 289–302.

[8] I. Bârză, D. Ghişa, and S. Ianuş, *Some remarks on the nonorientable surfaces*, Publ. Inst. Math. (Beograd) (N.S.) **63(77)** (1998), 47–54.

[9] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Pure and Applied Mathematics, no. 63, Academic Press, New York, 1975.

[10] E. Bujalance, J. Etayo, J. Gamboa, and G. Gromadzki, *Automorphism Groups of Compact Bordered Klein Surfaces*, Lecture Notes in Mathematics, vol. 1439, Springer-Verlag, Berlin, 1990.

[11] S. Helgason, *Differential Geometry and Symmetric Spaces*, Pure and Applied Mathematics, vol. 12, Academic Press, New York, 1962.
[12] S. T. Hu, *Differentiable Manifolds*, Holt, Rinehart and Winston, New York, 1969.
[13] M. Schiffer and D. C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton University Press, New Jersey, 1954.
[14] S. Stoïlow, *Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques*, Gauthier-Villars, Paris, 1937 (French).

Ilie Barza: Department of Engineering Sciences, Physics and Mathematics, Karlstad University, S-651 88-Karlstad, Sweden
*E-mail address*: Ilie.Barza@kau.se

Dorin Ghisa: Department of Mathematics, Glendon College, York University, 2275-Bayview Avenue, Toronto, Canada M4N 3M6
*E-mail address*: dghisa@yorku.ca