HAUSDORFF MEASURE OF THE SINGULAR SET OF QUASIREGULAR MAPS ON CARNOT GROUPS

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Recently, the theory of quasiregular mappings on Carnot groups has been developed intensively. Let ν stand for the homogeneous dimension of a Carnot group and let m be the index of the last vector space of the corresponding Lie algebra. We prove that the $(\nu - m - 1)$-dimensional Hausdorff measure of the image of the branch set of a quasiregular mapping on the Carnot group is positive. Some estimates of the local index of quasiregular mappings are also obtained.

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1. Introduction and statement of main results. Quasiregular mappings, or, in another terminology, mappings with bounded distortion, were firstly introduced and investigated by Reshetnyak in a series of papers that began to appear in 1966 [29, 31]. His work was handed and furthered by several analysts, among them Martio, Rickman, Väisälä [22, 23, 32], and others [1, 28, 33] later. The analytic definition of a quasiregular mapping is similar to the analytic definition of a quasiconformal one, with the exception of the requirement of homeomorphism.

Recently, the analysis on homogeneous groups (the Carnot groups) has been developed intensively. The fundamental role of such groups in analysis was pointed out by Stein [34] in his address to the International Congress of Mathematicians in 1970 (see also his monograph [35]). Briefly, a homogeneous group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [7, 10]. The analysis on the homogeneous group is also a test ground for the study of general subelliptic problems arising from vector fields $X_1, \ldots, X_k$ satisfying the Hörmander hypoellipticity condition [17].

Quasiconformal mappings on a homogeneous group of a special type were initially considered by Mostow [24] in 1971 in connection with rigidity theorems for the rank one symmetric space. Various definitions of quasiconformal mappings can be found in [12, 15, 18, 37, 38, 39]. In the works [13, 16, 19, 20, 37, 39, 40, 42], analytical foundations of the theory of quasiregular mappings

on Carnot groups were developed. We should note that in [13, 16] quasiregular mappings were studied with an additional smoothness condition. An alternative approach to this theory on the Heisenberg group was given in the works [8, 9].

The next analytic definition of a quasiregular mapping on a Carnot group $\mathbb{G}$ is the basic in this paper.

DEFINITION 1.1. Let $\Omega \subset \mathbb{G}$ be a domain. A mapping $f : \Omega \to \mathbb{G}$ is said to be a *quasiregular mapping* (*mapping with bounded distortion*) if
(1) f is continuous open and discrete,
(2) f belongs to $HW_{\nu,\mathrm{loc}}^1(\Omega)$,
(3) the formal horizontal differential $D_H f$ satisfies the condition

$$\left| D_H f(x) \right|^\nu \le K J(x, f) \tag{1.1}$$

for almost all $x \in \Omega$.

The smallest constant K is called the *outer coefficient of quasiregularity* and is denoted by $K_O(f)$. If $f : \Omega \to \mathbb{G}$ is a quasiregular mapping, then there is a constant K' such that

$$J(x, f) \le K' \left(\inf_{\xi \in V_1, \, \rho(\xi)=1} \rho \big(D_H f(x)(\xi) \big) \right)^\nu \tag{1.2}$$

for almost all $x \in \Omega$. The smallest constant K' is called the *inner coefficient of quasiregularity* and is denoted by $K_I(f)$. The relationships $K_O(f) \le C K_I^{\nu-1}(f)$ and $K_I(f) \le C K_O^{\nu-1}(f)$ are realized, where C is some constant. In the case $\mathbb{G} = \mathbb{R}^n$, the constant C is equal to 1. Also we use the notation $K(f) = \max\{K_O(f), K_I(f)\}$ and call $K(f)$ the *coefficient of quasiregularity*. If $\mathbb{G} = \mathbb{R}^n$, then Definition 1.1 is compatible with the definition of the quasiregular mapping in the Euclidean space. The definition of $D_H f$ and more details can be found in Section 2.

We denote by B_f the branch set of f, that is,

$$B_f = \{x \mid f \text{ is not locally homeomorphic at } x\}. \tag{1.3}$$

A result by Černavskiĭ [2, 3] says that a discrete open map f in $\mathbb{R}^n$ satisfies $\dim B_f = \dim f(B_f) \le n - 2$. Homotopy considerations (see [23]) imply that the $(n-2)$-dimensional Hausdorff measure of $f(B_f)$ is strictly positive. By definition, a quasiregular mapping on the Carnot group is a discrete open map, that yields immediately a similar inequality for the set $f(B_f)$. In the present paper, we give a more precise estimate for the Hausdorff measure of $f(B_f)$, namely, we prove the next theorem.

THEOREM 1.2. *Let $\Omega \subset \mathbb{G}$ be an open connected set, and let $f : \Omega \to \mathbb{G}$ be a nonconstant quasiregular mapping such that the set of branch points B_f is not*

empty. Then

$$\mathcal{H}_\rho^{\nu-(m+1)}\big(f(B_f)\big) > 0. \tag{1.4}$$

Here m is an index of the last vector space in the graduated Lie algebra $\mathcal{G}$ of $\mathbb{G}$.

Some estimates for the local index of quasiregular mappings on Carnot groups are also obtained.

2. Notes and preliminary results. Let $\mathcal{G}$ be a Lie algebra and let $\mathbb{G}$ be a corresponding simply connected Lie group. If U and V are some sets from $\mathcal{G}$, then we denote by $[U,V]$ the subspace of the algebra $\mathcal{G}$ generated by the elements $[X,Y] = XY - YX$, $X \in U$, $Y \in V$. By induction, we define the following series:

$$\mathcal{G}_1 = \mathcal{G}, \qquad \mathcal{G}_j = [\mathcal{G}, \mathcal{G}_{j-1}]; \qquad \mathbb{G}_1 = \mathbb{G}, \qquad \mathbb{G}_j = [\mathbb{G}, \mathbb{G}_{j-1}]. \tag{2.1}$$

A Lie algebra is called *nilpotent of step m* if $\mathcal{G}_{m+1} = \{0\}$, but $\mathcal{G}_m \neq \{0\}$.

We call a Lie algebra to be *graduated* if it splits into the direct sum of vector spaces $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus \cdots$. Here $[V_i, V_j] \subset V_{i+j}$, and only finitely many elements do not vanish. A Lie algebra $\mathcal{G}$ is called *stratified* if $\mathcal{G}$ is graduated and the subspace $V_1 \subset \mathcal{G}$ generates $\mathcal{G}$ as an algebra. For the step m nilpotent Lie algebra $\mathcal{G}$, we have

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_m; \qquad [V_1, V_j] = V_{j+1},$$
$$j = 1, \ldots, m-1; \qquad [V_1, V_m] = \{0\}. \tag{2.2}$$

A Lie group is stratified and nilpotent if the corresponding Lie algebra is so.

A Carnot group $\mathbb{G}$ is a stratified simply connected nilpotent Lie group with the Lie algebra $\mathcal{G}$. Let $X_{11}, \ldots, X_{1n_1}$ be a basis of the vector space $V_1 \subset \mathcal{G}$, $n_1 = \dim V_1$. From now on, we call V_1 the *horizontal space*. Since the vector fields $X_{11}, \ldots, X_{1n_1}$ generate the Lie algebra $\mathcal{G}$, we can choose a basis X_{ij}, $1 \leq j \leq n_i = \dim V_i$, $1 < i \leq m$ of space V_i, such that $X_{ij} \subset V_i$ are commutators of the vector fields $X_{1j} \subset V_1$, $j = 1, \ldots, n_1$. We see that the collection $X_{11}, X_{12}, \ldots, X_{1n_1}$ satisfies the Hörmander hypoellipticity condition [17].

It is known [10] that if $\mathbb{G}$ is a simply connected nilpotent Lie group with the Lie algebra $\mathcal{G}$, then the exponential map $\exp : \mathcal{G} \to \mathbb{G}$ is a diffeomorphism from the Lie algebra $\mathcal{G}$ to the Lie group $\mathbb{G}$. Thus, $dx \circ \exp^{-1}$ is a bi-invariant Haare measure on $\mathbb{G}$, where dx is the Lebesgue measure on $\mathcal{G}$. We can identify the elements $x \in \mathbb{G}$ of the group with the elements $x \in \mathcal{G}$ of the algebra, and thus, with $x \in \mathbb{R}^N$, $N = \sum_{i=1}^m \dim V_i$, by the exponential map $x = \exp(\sum x_{ij} X_{ij})$. The numbers $x = (x_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n_i$, are called the coordinates of the point x. There is a natural group of dilations, which is defined by the rule $\delta_r x = (r^i x_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n_i$. It is easy to see that $d(\delta_r x) = r^\nu dx$. The quantity $\nu = \sum_{i=1}^m i \cdot n_i$ is called the *homogeneous dimension* of the group $\mathbb{G}$.

We use the Carnot-Carathéodory metric based on the length of horizontal curves. A piecewise curve $\gamma : [0,b] \to \mathbb{G}$ is said to be *horizontal* if its tangent vector $\dot{\gamma}(s)$ belongs to the space V_1, that is, there exist functions $a_j(s)$, such that

$$\dot{\gamma}(s) = \sum_{j=1}^{n_1} a_j(s) X_{1j}(\gamma(s)). \tag{2.3}$$

The result of [6] implies that one can connect two arbitrary points $x, y \in \mathbb{G}$ by a horizontal curve. We fix on V_1 a nondegenerate quadratic form $\langle \cdot, \cdot \rangle$ so that the vector fields $X_{11}(x), \ldots, X_{1n_1}(x)$ are orthonormal with respect to this form at every $x \in \mathbb{G}$. The length $l(\gamma)$ of a curve γ is defined by the formula

$$l(\gamma) = \int_0^b \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle^{1/2} ds. \tag{2.4}$$

The *Carnot-Carathéodory distance* $d_c(x, y)$ is the infimum of the length over all horizontal curves connecting x and $y \in \mathbb{G}$. Since the quadratic form is left invariant, the Carnot-Carathéodory metric is left invariant as well. The group $\mathbb{G}$ is connected, therefore the metric $d_c(x, y)$ is finite (see [36]). We also use a metric, which is generated by a homogeneous norm. The homogeneous norm on $\mathbb{G}$ is, by definition, a continuous nonnegative function $\rho(\cdot)$ on $\mathbb{G}$ such that $\rho(x) = \rho(x^{-1})$, $\rho(\delta_r(x)) = r\rho(x)$, and $\rho(x) = 0$ if and only if $x = 0$, where 0 denotes the identity element of $\mathbb{G}$. The homogeneous norm is not uniquely defined, however, any two homogeneous norms are equivalent. We fix the homogeneous norm $\rho(x)$ of the element $x = (x_1, \ldots, x_m)$, $x_i \in V_i$, which is prescribed by the formula

$$\rho(x) = \left(\sum_{i=1}^{m} |x_i|^{2m!} \right)^{1/2m!}, \quad |x_i| = \left(\sum_{j=1}^{n_i} x_{ij}^2 \right)^{1/2}. \tag{2.5}$$

The homogeneous norm defines the homogeneous metric by the rule $\rho(x_1, x_2) = \rho(x_1^{-1} x_2)$. The distances $d_c(x_1, x_2)$ and $\rho(x_1, x_2)$ are equivalent.

We let $B(x, r) = \{ y \in \mathbb{G} : \rho(x^{-1} y) < r \}$ denote the open ball with the center at x and of radius $r > 0$ in the metric ρ. Note that $B(x, r)$ is the left translation by x of the ball $B(0, r)$, which is the image under δ_r of the "unit ball" $B(0, 1)$. The Hausdorff dimension of the metric space $(\mathbb{G}, d_c)$ coincides with its homogeneous dimension ν. By $|E|$, we denote the measure of the set E: $|E| = \int_E dx$. Our normalizing condition is such that the balls of radius one are of measure one; $|B(0, 1)| = \int_{B(0,1)} dx = 1$. Since the Jacobian determinant of the dilation δ_r is r^ν, we have that $|B(\cdot, r)| = r^\nu$.

The Euclidean space $\mathbb{R}^n$ with the standard structure is an example of the Abelian group; the exponential map is the identity and the vector fields $X_i = \partial/\partial x_i$, $i = 1, \ldots, n$, have only trivial commutative relations and form the basis of the corresponding Lie algebra.

The simplest example of a non-Abelian group is the Heisenberg group $\mathbb{H}^n$. The underlying space is $\mathbb{R}^{2n+1}$ with the group law of multiplication

$$(x,t)(x',t') = \left(x+x', t+t'+2\sum_{i=1}^{n}(x_{n+j}x'_j - x_j x'_{n+j})\right). \tag{2.6}$$

Here, $x, x' \in \mathbb{R}^{2n}$, $t, t' \in \mathbb{R}$. The Lie algebra $\mathscr{G}$ of the Heisenberg group $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad 1 \le j \le n, \ T = \frac{\partial}{\partial t}. \tag{2.7}$$

Since $[X_j, X_{n+j}] = -4T$, we have that X_j, X_{n+j}, $[X_j, X_{n+j}]$, $1 \le j \le n$, generate the tangent space at each point of the group. The vector fields X_j, $j = 1,\ldots,2n$, form a basis of the horizontal vector space V_1 and $\mathrm{span}\{T\} = V_2$. Thus, the Hiesenberg group is a two-step Carnot group. An example of the homogeneous norm is $\rho(x,t) = (|x|^4 + t^2)^{1/4}$. The homogeneous dimension ν is equal to $2n+2$.

Now we define an absolutely continuous function on curves of the horizontal fibration that is formed by the horizontal vector fields X_{1j}, $j = 1,\ldots,n_1$ (see terminology, for instance, in [18, 37, 43]).

DEFINITION 2.1. A function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, is said to be *absolutely continuous on lines* ($u \in \mathrm{ACL}(\Omega)$) if for any domain $U, \overline{U} \subset \Omega$, and any fibration $\mathscr{X}$ defined by the left-invariant vector fields X_{1j}, $j = 1,\ldots,n_1$, the function u is absolutely continuous on $\gamma \cap U$ with respect to the $\mathscr{H}^1$-Hausdorff measure for $d\gamma$-almost all curves $\gamma \in \mathscr{X}$.

For the function $u \in \mathrm{ACL}(\Omega)$, the derivatives $X_{1j}u$ along the vector fields X_{1j}, $j = 1,\ldots,n_1$, exist almost everywhere in Ω.

The Sobolev space $W_p^1(\Omega)$ ($L_p^1(\Omega)$), $1 \le p < \infty$, consists of locally summable functions $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{G}$, that have distributive derivatives $X_{1j}u$ along the vector fields X_{1j}:

$$\int_\Omega X_{1j}u\varphi\,dx = -\int_\Omega u X_{1j}\varphi\,dx, \quad j = 1,\ldots,n_1, \tag{2.8}$$

for any test function $\varphi \in C_0^\infty$ and of finite norm

$$\|u \mid W_p^1(\Omega)\| = \left(\int_\Omega |u|^p\,dx\right)^{1/p} + \left(\int_\Omega |\nabla_{\mathscr{L}}u|^p\,dx\right)^{1/p},$$
$$\left(\|u \mid L_p^1(\Omega)\| = \left(\int_\Omega |\nabla_{\mathscr{L}}u|^p\,dx\right)^{1/p}\right). \tag{2.9}$$

The quantity $\nabla_{\mathscr{L}} u = (X_{11}u, \ldots, X_{1n_1}u)$ is called the *subgradient* of u. For a smooth function, the subgradient is equal to the projection of the usual Riemannian gradient on V_1. We will say that u belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ ($L_{p,\mathrm{loc}}(\Omega)$) if for any $U, \overline{U} \subset \Omega$, the function u belongs to $W^1_p(U)$ ($L_p(U)$).

DEFINITION 2.2. A mapping $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, belongs to the *horizontal Sobolev class* $HW^1_{p,\mathrm{loc}}(\Omega)$, $1 \le p < \infty$, if

(1) $\rho(f(x)) \in L_{p,\mathrm{loc}}(\Omega)$,

(2) the coordinate functions f_{ij} belong to $\mathrm{ACL}(\Omega)$ for all i and j,

(3) $f_{1j} \in W^1_{p,\mathrm{loc}}(\Omega)$ for $1 \ge j \ge n_1$,

(4) the vector

$$X_{1k}(f(x)) = \sum_{1 \le l \le m,\, 1 \le \omega \le n_l} X_{1k}(f_{l\omega}(x)) \frac{\partial}{\partial x_{l\omega}} \tag{2.10}$$

belongs to the subspace V_1 for almost all $x \in \Omega$ and $k = 1, \ldots, n_1$.

In [37, 40], the reader can find various definitions of the Sobolev space and their relationships.

The matrix $X_{1k}f = (X_{1k}f_{ij})$ defines an operator $D_H f$ which is called the *formal horizontal differential*. It maps the horizontal space into the horizontal space $D_H f : V_1 \to V_1$ [27]. A mapping $f \in HW^1_{p,\mathrm{loc}}(\Omega)$ which possesses this property is called the *(weakly) contact mapping*. The norm of the operator $D_H f$ is defined by

$$|D_H f(x)| = \sup_{\xi \in V_1,\, \rho(\xi)=1} \rho(D_H f(x)(\xi)). \tag{2.11}$$

The norm $|D_H f|$ is equivalent to $|\nabla_{\mathscr{L}} f| = (\sum_{i=1}^{n_1} |X_{1i}f|^2)^{1/2}$. It has been proved in [43] (see also [26, 40]) that the formal horizontal differential $D_H f$ generates a homomorphism $Df : \mathscr{G} \to \mathscr{G}$ of the algebra $\mathscr{G}$, which is called a *formal differential*. The norm of the operator Df is defined in a usual way as

$$|Df(x)| = \sup_{\xi \in \mathscr{G},\, \rho(\xi)=1} \rho(Df(x)(\xi)) \tag{2.12}$$

and is equal to $|D_H f|$ for almost all $x \in \Omega$ [40, 43]. The determinant of the matrix $Df(x)$ is denoted by $J(x, f)$ and is called the *Jacobian* of f.

A map is called *open* if the image of an open set is an open set, and *discrete* if the preimage of any point consists of isolated points. It is known that for $\mathbb{G} = \mathbb{R}^n$ [30, 31] and $\mathbb{G} = \mathbb{H}^n$ [8, 39], inequality (1.1) implies the openness and discreteness of quasiregular mappings. The question about topological properties of quasiregular mappings on an arbitrary Carnot group satisfying only inequality (1.1) is open (see discussion in [9, 13, 39]).

The definition and main results on differentiability on Carnot groups are due to Pansu [26, 27].

DEFINITION 2.3. A mapping $f : \Omega \to \mathbb{G}$ is called $\mathcal{P}$-differentiable in $x \in \Omega$ if the mapping $\delta_{1/t}(f(x)^{-1}f(x\delta_t y))$ converges locally uniformly to a contact homomorphism $L : \mathbb{G} \to \mathbb{G}$ as $t \to 0$.

Some properties of quasiregular mappings are compiled in the next theorem (see [39, 40, 41, 43]).

THEOREM 2.4. *A quasiregular mapping* $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, *possesses the following properties:*

(1) *$\mathcal{P}$-differentiable almost everywhere in Ω;*
(2) *Lusin's $\mathcal{N}$-property: $|A| = 0$ implies $|f(A)| = 0$;*
(3) *$\mathcal{N}^{-1}$-property: $|A| = 0$ implies $|f^{-1}(A)| = 0$;*
(4) *$J(x,f) > 0$ almost everywhere in Ω;*
(5) *$|B_f| = |f(B_f)| = 0$.*

We use the notation $U(x,s)$ for the x-component of the set $f^{-1}(B(f(x),s))$. If $x \in \Omega$, $0 < r < \rho(x,\partial\Omega)$, and $0 < s < \rho(f(x),\partial f(\Omega))$, then

$$
\begin{aligned}
l(x,r) &= \inf_{\rho(x,z)=r} \rho(f(z),f(x)), &\quad L(x,r) &= \sup_{\rho(x,z)=r} \rho(f(z),f(x)), \\
l^\star(x,s) &= \inf_{y\in\partial U(x,f,s)} \rho(x,y), &\quad L^\star(x,s) &= \sup_{y\in\partial U(x,f,s)} \rho(x,y).
\end{aligned}
\tag{2.13}
$$

Making use of the method of the paper [22], we can show that if $C \subset \Omega$ is a compact, then the function $(x,s) \mapsto l^\star(x,s)$ is continuous and $(x,s) \mapsto L^\star(x,s)$ is lower semicontinuous on the set $C \times (0,t)$.

For the mapping $f : \Omega \to \mathbb{G}$, we denote by

$$
H(x,f) = \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)}, \qquad H^\star(x,f) = \limsup_{s\to 0} \frac{L^\star(x,s)}{l^\star(x,s)}
\tag{2.14}
$$

the *linear distortion* and the *inverse linear distortion* of f at $x \in \Omega$, respectively.

A subdomain $D \subset \Omega$ is called a *normal domain* under a mapping $f : \Omega \to \mathbb{G}$ if $f(\partial D) = \partial f(D)$. A neighborhood $U \subset \Omega$ of a point $x \in \Omega$ is called a *normal neighborhood* under a mapping $f : \Omega \to \mathbb{G}$ if it is a normal domain and $U \cap f^{-1}(f(x)) = \{x\}$.

The next topological facts can be proved for a rather broad class of metric spaces. For the Euclidean space $\mathbb{R}^n$, the proof can be found in [14, 22, 32]. By the symbol $\mathbb{C}D$ we will denote the complement to the domain D.

LEMMA 2.5. *Let $f : \Omega \to \mathbb{G}$ be a continuous open discrete and sense preserving mapping. Then for a point $x \in \Omega$, there exists $\sigma_x > 0$ such that for any $0 < s \leq \sigma_x$, the following conditions hold:*

(1) *$U(x,s)$ is the normal neighborhood of x;*
(2) *$U(x,s) = U(x,\sigma_x) \cap f^{-1}(B(f(x),s))$;*
(3) *if $s < \sigma_x$, then $\partial U(x,s) = U(x,\sigma_x) \cap f^{-1}(S(f(x),s))$;*

(4) *the sets $\mathbb{C}U(x,s)$ and $\mathbb{C}\overline{U}(x,s)$ are connected;*
(5) $U(x,s) \setminus \overline{U}(x,t)$ *is a ring-type domain for* $0 < t < s \le \sigma_x$*;*
(6) $l^\star(x, L(x,r)) = L^\star(x, l(x,r)) = r$ *for* $r \in (0, l^\star(x, \sigma_x))$.

If $U = U(x,s)$ is a normal neighborhood of x, then the degree $\mu(y, f, U)$ of continuous open discrete mapping f with respect to $y = f(x)$ is constant for any point $z \in f(U)$ and is denoted by $\mu(f, U)$. The definition of the mapping degree can be found in [30]. In the same work, it was proved that for a compact subdomain $G \subset U$, the degree $\mu(y, f, G)$, $y = f(x)$, $x \in U$, has the same value as for U, which is called the *local index* $i(x, f)$ of the point $x \in U$. The index $i(x, f)$ is equal to 1 if and only if x is not a branch point.

We define the quantity

$$N(f, A) = \sup_{y \in \mathbb{G}} N(y, f, A) = \sup_{y \in \mathbb{G}} \operatorname{card} \{ f^{-1}(y) \cap A \} \tag{2.15}$$

and call it a *multiplicity function*. If a set A is such that $\overline{A}$ is compact and $\overline{A} \subset \Omega$, then the discreteness implies that $N(f, A)$ is finite.

Some relations between the multiplicity function, the mapping degree, and the local index are expressed in the following lemma. Its proof for the Carnot group can be obtained as in [21, Lemma 2.12].

LEMMA 2.6. *Let $f : \Omega \to \mathbb{G}$ be a continuous open discrete and sense preserving mapping. If $U \subset \Omega$ is a normal domain for f, then*
(1) $N(f, U) = \mu(y, f, U) = \mu(f, U)$ *for all $y \in f(U)$;*
(2) $\mu(y, f, U) = \sum_{j=1}^{k} i(x_j, f)$ *for all $y \in f(U)$, where $k = N(y, f, U)$ and $\{x_1, \ldots, x_k\} = f^{-1}(y) \cap U$;*
(3) $i(x, f) = N(f, U)$ *if and only if U is the normal neighborhood of a point $x \in U$.*

3. Estimates of the measure of the image of the branch set. The value

$$\mathcal{H}^\alpha(A) = \liminf_{\varepsilon \to 0} \left\{ \sum_i \operatorname{diam}^\alpha(B_i) \mid A \subset \cup B_i \ \operatorname{diam} B_i < \varepsilon \right\} \tag{3.1}$$

is called *the Hausdorff α-measure*. We denote by $\operatorname{diam}_\rho(B_i)$ the diameter of the set B_i with respect to the homogeneous metric, and by $\operatorname{diam}_{\|\cdot\|}(B_i)$ the diameter under the Euclidean metric. The Hausdorff α-measure with respect to these different metrics we note by $\mathcal{H}_\rho^\alpha$ and $\mathcal{H}_{\|\cdot\|}^\alpha$, respectively.

For the beginning, we give a simple proposition on the positiveness of the Hausdorff $(N-2)$-measure of the image of the branch set of quasiregular mappings on Carnot groups.

LEMMA 3.1. *Let $f : \Omega \to \mathbb{G}$ be a quasiregular mapping on a Carnot group with $B_f \ne \varnothing$. Then $\mathcal{H}_\rho^{N-2}(f(B_f)) > 0$, $N = \sum_{i=1}^{m} \dim V_i$.*

PROOF. From [23], it follows that for an open discrete map in the Euclidean space $\mathbb{R}^N$, the following estimate $\mathcal{H}_{\|\cdot\|}^{N-2}(f(B_f)) > 0$ holds. We cover $f(B_f)$ by open balls $\{B_i\}$, $i = 1, 2, \ldots$. We have the inequality $\|\cdot\| \le \rho$ by the definition of the homogeneous metric ρ. Then

$$
\begin{aligned}
\mathcal{H}_\rho^{N-2}(f(B_f)) &= \liminf_{\varepsilon \to 0} \sum_i \operatorname{diam}_\rho^{N-2}(B_i) \\
&\ge \liminf_{\varepsilon \to 0} \sum_i \operatorname{diam}_{\|\cdot\|}^{N-2}(B_i) = H_{\|\cdot\|}^{N-2}(f(B_f)) > 0.
\end{aligned}
\tag{3.2}
$$
$\square$

Now, we give more precise estimates for the Hausdorff measure of the image of the branch set under quasiregular mappings on Carnot groups. First of all, we describe a set $P_z(E)$. We need this construction in the following lemma. Let z, ω be disjoint points on the group $\mathbb{G}$. We denote by $\sigma_\omega^z(t) = z\delta_t(\omega)$ the one-parametric curve which starts from the point z and passes through the point ω. We let E be a set on $\mathbb{G}$ and let $z \notin E$. The collection of the one-parametric curves $\sigma_\omega^z(t)$, $\omega \in E$, we denote by $P_z(E)$.

LEMMA 3.2. *Let $E \subset \mathbb{G}$ be a set such that $\mathcal{H}_\rho^\alpha(E) = 0$ for any $\alpha > 0$. Then $\mathcal{H}_\rho^{\alpha+m}(P_z(E)) = 0$ for $z \notin E$. Here m is the index of the last vector space in the graduated Lie algebra $\mathscr{G}$ of $\mathbb{G}$.*

PROOF. Since the Haar measure is invariant with respect to the left translation, we can assume that $z = 0$, and from now on we omit the letter z. Moreover, up to the end of the proof, the notations $B_\rho(x, r)$, $S_\rho(x, r)$ mean the ball and the sphere, respectively, under the homogeneous metric ρ, and $B_{\|\cdot\|}(x, r)$, $S_{\|\cdot\|}(x, r)$ mean the ball and the sphere with respect to the Euclidean metric.

We will use the notations $R_k = \{x \in \mathbb{G} : 1/k \le \rho(x) \le k, \ k \in \mathbb{N}\}$, $E_k = E \cap R_k$, and $P_k = P(E_k) \cap R_k$. Since $P(E) = \bigcup_{k=1}^\infty P_k$, the lemma will be proved if we establish the equality $\mathcal{H}_\rho^{\alpha+m}(P_k) = 0$ for any $k \in \mathbb{N}$. Since $\mathcal{H}_\rho^\alpha(E_k) = 0$ for any k, we cover E_k by a system of balls $B_\rho(x_i, r_i)$, such that

 (1) $x_i \in E_k$,
 (2) $r_i < 1/4k^2$,
 (3) $\sum_i r_i^\alpha < \varepsilon$, for any $\varepsilon \in (0, 1)$.

We can assume that the intersection $E_k \cap B_\rho(x_i, r_i)$ is not empty for any index i, otherwise we omit such balls. Let H_i denote the set $P(B_\rho(x_i, r_i)) \cap R_k$. Then $P_k \subset \bigcup_i H_i$. We fix $i \in \mathbb{N}$. Since the homogeneous norm of the point $\overline{x}_i$ satisfies the equalities

$$
\rho(\overline{x}_i) = \rho(\delta_{k/\rho(x_i)}(x_i)) = \frac{k}{\rho(x_i)} \rho(x_i) = k,
\tag{3.3}
$$

the point x_i is mapped onto $\overline{x}_i \in S_\rho(0, k)$ under the dilation $\delta_{k/\rho(x_i)}(x_i)$. Similarly, $\delta_{1/k\rho(x_i)}(x_i) = \underline{x}_i$, $\underline{x}_i \in S_\rho(0, 1/k)$. Thus, if

$$
t \in J_i = \left[\frac{1}{k\rho(x_i)}, \frac{k}{\rho(x_i)} \right] \subset \left[\frac{1}{k^2}, k^2 \right],
\tag{3.4}
$$

then a point $x \in \sigma_{x_i}(t)$ is located between the points $\underline{x}_i \in S_\rho(0, 1/k)$ and $\overline{x}_i \in S_\rho(0, k)$.

We want to estimate the Euclidean length of the part of the curve $\sigma_{x_i}(t)$ which lies in R_k. The parametric equation of $\sigma_{x_i}(t)$ in the Euclidean coordinates of $\mathbb{R}^N$ is of the form

$$\sigma_{x_i}^{(1)}(t) = \frac{t}{\rho(x_i)} x_i^{(1)}, \ldots, \sigma_{x_i}^{(m)}(t) = \left(\frac{t}{\rho(x_i)}\right)^m x_i^{(m)}, \tag{3.5}$$

where $x_i^{(l)} = (x_i^{(l1)}, \ldots, x_i^{(ln_l)})$ are the coordinates of the point x_i in the vector space V_l. The Euclidean length $l(\sigma_{x_i}^k)$ of $\sigma_{x_i}^k = \sigma_{x_i}(t) \cap R_k$ is given by the integral

$$
\begin{aligned}
l(\sigma_{x_i}^k) &= \int_{1/k}^k \left(\left\| \frac{x_i^{(1)}}{\rho(x_i)} \right\|^2 + \left\| \frac{2t x_i^{(2)}}{\rho^2(x_i)} \right\|^2 + \cdots + \left\| \frac{m t^{m-1} x_i^{(m)}}{\rho^m(x_i)} \right\|^2 \right)^{1/2} dt \\
&\leq \int_0^k \left(1 + (2t)^2 + \cdots + (m t^{m-1})^2 \right)^{1/2} dt \leq N m^{3/2} k^m = L
\end{aligned}
\tag{3.6}
$$

and does not depend on $i \in \mathbb{N}$. Here we have used the estimates $\|x_i^{(l)}\| / \rho^l(x_i) \leq 1$, $l = 1, \ldots, m$.

If $x \in B_\rho(x_i, r_i)$, then

$$\rho\big(\delta_{k/\rho(x_i)}(x_i), \delta_{k/\rho(x_i)}(x)\big) = \frac{k}{\rho(x_i)} \rho(x_i, x) = \frac{k}{\rho(x_i)} r_i \leq k^2 r_i < 1. \tag{3.7}$$

We divide the arc $\sigma_{x_i}^k$ into parts of equal Euclidean length which does not exceed $C \cdot (k^2 r_i)^m < 1$ by points $\underline{x}_i = s_{i0}, s_{i1}, \ldots, s_{iN_i} = \overline{x}_i$. Here C is a constant from [25, Proposition 1.1], which depends on k. Let N_i be the biggest positive integer such that $N_i \leq L/(C \cdot (k^2 r_i)^m) + 1$. To this partition of the curve corresponds the following partition of the interval J_i:

$$\frac{1}{k^2} \leq \frac{1}{k\rho(x_i)} = T_0 \leq T_1 \leq \cdots \leq T_{N_i} = \frac{k}{\rho(x_i)} \leq k^2, \quad \delta_{T_j}(x_i) = s_{ij}. \tag{3.8}$$

The collection of the balls $\{B_{\|\cdot\|}(s_{ij}, C \cdot (k^2 r_i)^m), j = 0, \ldots, N_i\}$ covers the curve $\sigma_{x_i}^k$. We claim that

(1) $\sigma_{x_i}^k \subset \bigcup_{j=1}^{N_i} B_\rho(s_{ij}, k^2 r_i)$, $N_i \leq L/C \cdot (k^2 r_i)^m + 1$,

(2) $H_i \subset \bigcup_{j=1}^{N_i} B_\rho(s_{ij}, 2k^2 r_i)$, $N_i \leq L/C \cdot (k^2 r_i)^m + 1$.

The first assertion follows (see [25, Proposition 1.1]) from the inclusions $B_{\|\cdot\|}(s_{ij}, C(k^2 r_i)^m) \subset B_\rho(s_{ij}, k^2 r_i)$. We show the second one. If $y \in H_i$, then there is a point $x \in B_\rho(x_i, r_i)$ such that

$$
\begin{aligned}
y &= \delta_t(x), \qquad \tilde{s} = \delta_t(x_i) = \sigma_{x_i}(t), \\
\rho(\tilde{s}, y) &= \rho(\delta_t(x_i), \delta_t(x)) = t\rho(x_i, x) = t r_i \leq k^2 r_i,
\end{aligned}
\tag{3.9}
$$

for some $t \in [T_{j-1}, T_j]$. Furthermore, the first statement implies that the point $\tilde{s}$ belongs to $B_\rho(s_{ij}, k^2 r_i)$. From this and from the triangle inequality, we have $\rho(s_{ij}, y) \le \rho(s_{ij}, \tilde{s}) + \rho(\tilde{s}, y) \le 2k^2 r_i$. Thus $y \in B_\rho(s_{ij}, 2k^2 r_i)$ and the second assertion is proved.

From the statement above, we obtain that the countable collection of the balls with respect to the homogeneous metric $\{B_\rho(s_{ij}, 2k^2 r_i)\}$, $j = 1, \ldots, N_i$, $i \in \mathbb{N}$, covers the set P_k. Finally, we have

$$
\begin{aligned}
\mathcal{H}_\rho^{\alpha+m}(P_k) &\le \sum_i \sum_{j=1}^{N_i} \left(\operatorname{diam}_\rho B_\rho(s_{ij}, 2k^2 r_i) \right)^{\alpha+m} = \sum_i \sum_{j=1}^{N_i} \left(4k^2 r_i \right)^{\alpha+m} \\
&\le \sum_i \frac{1+L}{C \cdot (k^2 r_i)^m} 4^{\alpha+m} (k^2 r_i)^{\alpha+m} = \frac{1+L}{C} 4^{\alpha+m} k^{2\alpha} \sum_i r_i^\alpha \\
&\le M(m, k)\varepsilon.
\end{aligned}
\tag{3.10}
$$

Since $\varepsilon > 0$ is arbitrary, we get $\mathcal{H}_\rho^{\alpha+m}(P_k) = 0$. $\qquad\square$

The proof of the next theorem is based on the idea of [11, Theorem 1] and [30, Theorem 10.3].

THEOREM 3.3. *Let $\Omega \subset \mathbb{G}$ be an open connected set, and let $f : \Omega \to \mathbb{G}$ be a nonconstant quasiregular mapping such that the set of branch points B_f is not empty. Then*

$$
\mathcal{H}_\rho^{\nu-(m+1)}(f(B_f)) > 0.
\tag{3.11}
$$

Here m is the index of the last vector space in the graduated Lie algebra $\mathcal{G}$ of $\mathbb{G}$.

PROOF. We suppose, contrary to our claim, that $\mathcal{H}_\rho^{\nu-(m+1)}(f(B_f)) = 0$. We choose $x_0 \in B_f$ and we put $y_0 = f(x_0)$. Let $\sigma_{x_0} > 0$ be the number from Lemma 2.5 such that $\overline{B}(x_0, \sigma_{x_0}) \subset \Omega$, and let $l_0 = (1/2)l(x_0, \sigma_{x_0})$. Then $U = U(x_0, l_0)$ is the normal neighborhood of x_0, and the inclusions $B(y_0, l_0) \subset f(B(x_0, \sigma_{x_0}))$, $\overline{U} \subset B(x_0, \sigma_{x_0})$, and $f(U) \subset B(y_0, l_0)$ hold.

We show that $f(B_f \cap U)$ is a closed set in $B(y_0, l_0)$. In fact, if $z \in B(y_0, l_0)$ is an adherent point for $f(B_f \cap U)$, then there exists a sequence $\{t_k\} \in B_f \cap U$, $k = 1, 2, \ldots$, such that $f(t_k) \to z$. We may assume that $\{t_k\}$ converges to $t_0 \in \overline{U}$. Since B_f is closed in Ω and $\overline{U} \subset \Omega$, then $t_0 \in B_f$ and we have $f(t_0) = z$ by the continuity of f. The ball $B(y_0, l_0)$ does not contain points of the boundary ∂U, therefore $t_0 \in U$. Consequently, $z \in f(B_f \cap U)$.

We set $E = f(B_f \cap U)$. By the assumption $\mathcal{H}_\rho^{\nu-(m+1)}(E) = 0$, there is a point $z \in B(y_0, l_0)$ such that $z \notin E$. We construct the set $P_z(E)$. Lemma 3.2 implies that $\mathcal{H}_\rho^{\nu-1}(P_z(E)) = 0$. The set E is closed in $B(y_0, l_0)$, and the intersection $B(y_0, l_0) \cap P_z(E)$ is also closed. Thus $H = B(y_0, l_0) \setminus P_z(E)$ is open. We show that the set H is simple connected. Really, since $z \notin E$, any closed curve γ can

be shrunk into the point z along the one-parametric curves $\sigma_\omega^z(t)$, which are contained in H and connect z with the points $\omega \in \gamma$.

Let $W \subset U$ be a connected component of $f^{-1}(H)$, which is contained in U. Then $\overline{W} \subset B(x_0, \sigma_{x_0})$, $f(W) = H$, $f(\partial W) \subset \partial H$, and the restriction $f|_W$ is a quasiconformal mapping [42, 44].

If $W_1, W_2, \ldots, W_k$ are connected components of $f^{-1}(H) \cap U$ and f_k^{-1} is the inverse mapping to the restriction $f|_{W_k}$, then f_k^{-1} is a quasiconformal mapping on $H = B(y_0, l_0) \setminus P_z(E)$. Since we have $\mathcal{H}_\rho^{\nu-1}(P_z(E)) = 0$, [37, Theorem 8] implies that f_k^{-1} is extended to a quasiconformal mapping $\tilde{f}_k^{-1} : B(y_0, l_0) \to U$. The set $W_0 = \tilde{f}_k^{-1}(B(y_0, l_0)) \subset U$ is open and closed in U, U is connected, thus $W_0 = U$. Therefore, $f : U \to \mathbb{G}$ is a homeomorphism. This contradicts the fact that the set U contains branch points of f. $\qquad\square$

4. Bounds for the local index. We mention some auxiliary definitions and results. Let U be an open set on the Carnot group $\mathbb{G}$ and let V be a compact in U. The ordered pair (U, V) is called a *condenser*.

DEFINITION 4.1. For $p \in [1, \infty)$, the quantity

$$\mathrm{cap}_p(U, V) = \inf \left(\int_U |\nabla_{\mathscr{L}} v|^p \, dx \right)^{1/p} \tag{4.1}$$

is called the *p-capacity* of the condenser (U, V). The infimum is taken over all nonnegative functions $v \in C_0^\infty(U)$ such that $v|_V \geq 1$.

The properties of the p-capacity in the spaces with the geometry of vector fields satisfying the Hörmander hypoellipticity condition can be found in [4, 5]. If U is an open ball and V is a closed ball with the same center and of smaller radius, then such a condenser (U, V) is called the *ring condenser*.

LEMMA 4.2. *Let Ω be a domain in $\mathbb{G}$. Then for the condenser $(B(x, R), \overline{B}(x, r))$, $r < R$, there are constants θ_1, θ_2, independent of r and R, such that*

$$\theta_1 \left(\ln \frac{R}{r} \right)^{1-\nu} \leq \mathrm{cap}_\nu (B(x, R), \overline{B}(x, r)) \leq \theta_2 \left(\ln \frac{R}{r} \right)^{1-\nu}. \tag{4.2}$$

The proof of Lemma 4.2 follows from [5, Theorems 6.6 and 6.9].

In the next lemma, some properties of the ν-capacity of some condensers are given. They have been established in [42] for more general mappings with bounded s-distortion. We formulate them for the case $s = \nu$.

LEMMA 4.3. *Let $f : \Omega \to \mathbb{G}$ be a quasiregular mapping. If $U \subset \Omega$ is a normal domain such that $N(f, U) < \infty$, then*

$$\mathrm{cap}_\nu(U, V) \leq K_O(f) N(f, U) \, \mathrm{cap}_\nu (f(U), f(V)). \tag{4.3}$$

If $U = U(x, r_0)$ is a normal neighborhood of $x \in \Omega$, then, for $0 < r < r_0$,

$$\operatorname{cap}_\nu \left(f(U(x, r_0)), f(\overline{U}(x, r)) \right) \leq \frac{K_I(f)}{i(x, f)} \operatorname{cap}_\nu \left(U(x, r_0), \overline{U}(x, r) \right). \qquad (4.4)$$

We show that the inverse linear distortion $H^\star(x, f)$ of the quasiregular mapping f is bounded above, and the estimate does not depend on $x \in \Omega$.

LEMMA 4.4. *Let $f : \Omega \to \mathbb{G}$ be a quasiregular mapping and $K(f)$ be a coefficient of quasiregularity. Then*

$$H^\star(x, f) \leq C^\star(\nu, K(f)), \qquad (4.5)$$

where the constant $C^\star$ depends on the homogeneous dimension ν and on $K(f)$.

PROOF. Let $x \in \Omega$ and let $\sigma_x > 0$ be a number defined in Lemma 2.5. Let $r > 0$ be such that $L^\star(x, r) < l^\star(x, \sigma_x)$. We claim that $L^\star(x, r)/l^\star(x, r) \leq C^\star(\nu, K(f))$. If we denote $l = l(x, l^\star(x, r))$, $L = L(x, L^\star(x, r))$, then we get the inequality $l \leq r \leq L$. Since $L^\star(x, r) < l^\star(x, \sigma_x)$, we have $L < \sigma_x$. The boundaries $\partial U(x, l), \partial U(x, r)$ of the normal neighborhood intersect the sphere $S(x, l^\star(x, r))$. Besides, $\partial U(x, r)$ and $\partial U(x, L)$ intersect $S(x, L^\star(x, r))$. Thus, [5, Theorem 6.6] implies that the capacities of the condensers $(U(x, r), \overline{U}(x, l))$, $(U(x, L), \overline{U}(x, r))$ are strictly positive and bounded away from zero by a constant a_ν. Since $U(x, r), U(x, L)$ are normal neighborhoods of x, then $N(f, U(x, r)) = N(f, U(x, L)) = i(x, f)$. Lemmas 4.2 and 4.3 imply the inequalities

$$
\begin{aligned}
a_\nu &\leq \operatorname{cap}_\nu \left(U(x, r), \overline{U}(x, l) \right) \\
&\leq K_O(f) i(x, f) \operatorname{cap}_\nu \left(B(f(x), r), \overline{B}(f(x), l) \right) \\
&\leq K_O(f) i(x, f) \theta_2 \left(\ln \frac{r}{l} \right)^{1-\nu}, \\
a_\nu &\leq \operatorname{cap}_\nu \left(U(x, L), \overline{U}(x, r) \right) \\
&\leq K_O(f) i(x, f) \operatorname{cap}_\nu \left(B(f(x), L), \overline{B}(f(x), r) \right) \\
&\leq K_O(f) i(x, f) \theta_2 \left(\ln \frac{L}{r} \right)^{1-\nu}.
\end{aligned}
\qquad (4.6)
$$

From the inclusions $U(x, L) \supset B(x, L^\star(x, r))$, $U(x, l) \subset B(x, l^\star(x, r))$, and from Lemma 4.3, we deduce that

$$
\begin{aligned}
\theta_2 \left(\ln \frac{L^\star(x, r)}{l^\star(x, r)} \right)^{1-\nu} &\geq \operatorname{cap}_\nu \left(B(x, L^\star(x, r)), \overline{B}(x, l^\star(x, r)) \right) \\
&\geq \operatorname{cap}_\nu \left(U(x, L), \overline{U}(x, l) \right) \\
&\geq \frac{i(x, f)}{K_I(f)} \operatorname{cap}_\nu \left(f(U(x, L)), f(\overline{U}(x, l)) \right) \\
&= \frac{i(x, f) \theta_1}{K_I(f)} \left(\ln \frac{L}{l} \right)^{1-\nu}.
\end{aligned}
\qquad (4.7)
$$

Making use of the estimates (4.6), we derive

$$
\left(\ln\frac{L}{l}\right)^{\nu-1} = \left(\ln\frac{L}{r}+\ln\frac{r}{l}\right)^{\nu-1} \le 2^{\nu-2}\left(\left(\ln\frac{L}{r}\right)^{\nu-1}+\left(\ln\frac{r}{l}\right)^{\nu-1}\right)
$$
$$
\le 2^{\nu-1}K_O(f)i(x,f)\theta_2 a_\nu^{-1}. \tag{4.8}
$$

Using the inequality (4.7), we obtain

$$
\left(\ln\frac{L^\star(x,r)}{l^\star(x,r)}\right)^{\nu-1} \le \frac{2^{\nu-1}K_O(f)K_I(f)\theta_2^2}{a_\nu\theta_1}. \tag{4.9}
$$

Thus,

$$
\frac{L^\star(x,r)}{l^\star(x,r)} \le \exp\left(2\left(\frac{K_O(f)K_I(f)\theta_2^2}{a_\nu\theta_1}\right)^{1/(\nu-1)}\right) = C^\star(\nu,K(f)). \tag{4.10}
$$

Letting $r \to 0$, we get the required result. $\qquad\square$

THEOREM 4.5. *Let $f : \Omega \to \mathbb{G}$, $\Omega \subset \mathbb{G}$, be a quasiregular mapping, and let $F \subset B_f$ be a compact set, such that $\mathcal{H}_\rho^\alpha(f(F)) > 0$ for some α. Then*

$$
\inf_{x\in F} i(x,f) < K_I(f)\frac{\theta_2}{\theta_1}\left(\frac{\nu}{\alpha}\right)^{\nu-1}. \tag{4.11}
$$

PROOF. Lemma 4.4, the continuity of the function $L^\star(x,r)$, and the lower continuity of $l^\star(x,r)$ imply that there is a number $i_0 \in \mathbb{N}$ such that the sets

$$
F_i = \left\{x \in F,\ s \in \left(0,\frac{1}{i}\right): \frac{L^\star(x,s)}{l^\star(x,s)} \le C^\star + 1\right\} \tag{4.12}
$$

are compact for all $i \ge i_0$. Since $F = \bigcup_i F_i$, we can find a number $i \ge i_0$, such that $\mathcal{H}_\rho^\alpha(f(F_i)) > 0$. We represent F_i as a union of the sets $F_{i,j} = \{x \in F_i : i(x,f) = j\}$, $j = 1,2,\dots$. We fix j such that $\mathcal{H}_\rho^\alpha(f(F_{i,j})) > 0$. Now we choose a point $x_0 \in F_{i,j}$ and a number σ satisfying the properties
   (1) $0 < \sigma < 1/i$,
   (2) $U(x_0,\sigma)$ is a normal neighborhood of x_0,
   (3) $\mathcal{H}_\rho^\alpha(f(F_{i,j}\cap U(x_0,s))) > 0$ for $0 < s \le \sigma$.
For some $0 < s \le \sigma$, we may select $u > 0$ such that the inclusions

$$
\overline{U}(x_0,s) \subset B(x_0,u) \subset B(x_0,2u) \subset U(x_0,\sigma) \tag{4.13}
$$

hold. We suppose that inequality (4.11) is false. If we established the equality $\mathcal{H}_\rho^\alpha(f(F_{i,j}\cap U(x_0,s))) = 0$, we would get the contradiction with property (3). Since $|B_f| = 0$, we can cover $F_{i,j}\cap U(x_0,s)$ by the balls $B(x_k,r_k)$, $k = 1,2,\dots$ such that
  (P1) $x_k \in F_{i,j}\cap U(x_0,s)$,
  (P2) $\sum_k r_k^\nu < \varepsilon$ for any $\varepsilon \in (0,1)$.

From the equality $i(x_k, f) = i(x_0, f)$ and Lemma 2.6, we deduce that $U(x_k, L_k)$, $L_k = L(x_k, r_k)$, is the normal neighborhood of x_k. Our supposition and Lemma 4.3 imply

$$
\begin{aligned}
\mathrm{cap}_\nu &\left(f(U(x_0, \sigma)), f(\overline{U}(x_k, L_k)) \right) \\
&\leq \frac{K_I(f)}{i(x_0, f)} \mathrm{cap}_\nu \left(U(x_0, \sigma), \overline{U}(x_k, L_k) \right) \\
&\leq \left(\frac{\alpha}{\nu} \right)^{\nu-1} \frac{\theta_1}{\theta_2} \mathrm{cap}_\nu \left(U(x_0, \sigma), \overline{U}(x_k, L_k) \right).
\end{aligned}
\tag{4.14}
$$

The inclusions $f(U(x_0, \sigma)) \subset B(f(x_k), 2\sigma)$ and $\overline{B}(f(x_k), L_k) \subset f(\overline{U}(x_k, L_k))$, in sequel, give us the inequality

$$
\begin{aligned}
\mathrm{cap}_\nu &\left(B(f(x_k), 2\sigma), \overline{B}(f(x_k), L_k) \right) \\
&\leq \mathrm{cap}_\nu \left(f(U(x_0, \sigma)), f(\overline{U}(x_k, L_k)) \right).
\end{aligned}
\tag{4.15}
$$

Analogously, from the insertions $B(x_k, u) \subset U(x_0, \sigma)$ and $B(x_k, L_k^\star) \supset U(x_k, L_k)$, $L_k^\star = L^\star(x_k, L_k)$, we obtain

$$
\mathrm{cap}_\nu \left(U(x_0, \sigma), \overline{U}(x_k, L_k) \right) \leq \mathrm{cap}_\nu \left(B(x_k, u), \overline{B}(x_k, L_k^\star) \right).
\tag{4.16}
$$

By inequalities (4.15), (4.16), and Lemma 4.2, we have

$$
\begin{aligned}
\theta_1 \left(\ln \frac{2\sigma}{L_k} \right)^{1-\nu} &\leq \mathrm{cap}_\nu \left(f(U(x_0, \sigma)), f(\overline{U}(x_k, L_k)) \right) \\
&\leq \left(\frac{\alpha}{\nu} \right)^{\nu-1} \frac{\theta_1}{\theta_2} \mathrm{cap}_\nu \left(U(x_0, \sigma), \overline{U}(x_k, L_k) \right) \\
&\leq \left(\frac{\alpha}{\nu} \right)^{\nu-1} \theta_1 \left(\ln \frac{u}{L_k^\star} \right)^{1-\nu}.
\end{aligned}
\tag{4.17}
$$

By property (6) of Lemma 2.5, the equality $l_k^\star = l^\star(x_k, L_k) = r_k$ holds. Consequently, we get the estimates

$$
L_k \leq 2\sigma u^{-\nu/\alpha} L_k^{\star \nu/\alpha} \leq 2\sigma u^{-\nu/\alpha} (C^\star + 1)^{\nu/\alpha} l_k^{\star \nu/\alpha} = b r_k^{\nu/\alpha}.
\tag{4.18}
$$

The images of $B(x_k, r_k)$ under the mapping f cover the set $f(F_{i,j} \cap U(x_0, s))$. From (4.18) and property (P2), we conclude that

$$
\begin{aligned}
\mathscr{H}_\rho^\alpha \left(f(F_{i,j} \cap U(x_0, s)) \right) &\leq \sum_k \mathrm{diam}_\rho^\alpha \left(f(B(x_k, r_k)) \right) \\
&\leq 2^\alpha \sum_k L_k^\alpha \leq 2^\alpha b^\alpha \sum_k r_k^\nu < 2^\alpha b^\alpha \varepsilon.
\end{aligned}
\tag{4.19}
$$

Thus $\mathscr{H}_\rho^\alpha (f(F_{i,j} \cap U(x_0, s))) = 0$, which contradicts (1.3). $\qquad \square$

Some similar estimates of the index of a point were obtained in [21] for the case $\mathbb{G} = \mathbb{R}^n$.

COROLLARY 4.6. *Let $f : \Omega \to \mathbb{G}$ be a nonconstant quasiregular mapping, and let $F \subset B_f$ be a continuum. Then*

$$\inf_{x \in F} i(x, f) \le \frac{\theta_1}{\theta_2} \nu^{\nu-1} K_I(f) \tag{4.20}$$

and set $E = \{x \in B_f : i(x, f) \ge \nu^{\nu-1}\}$ is totally disconnected.

PROOF. If F is a continuum, then $\mathcal{H}_\rho^1(F) > 0$ and the first assertion follows from Theorem 4.5 with $\alpha = 1$.

If we assume that E contains a continuum, then we get the contradiction with the first statement. $\qquad\square$

COROLLARY 4.7. *Let $\Omega \subset \mathbb{G}$ be an open connected set, and let $f : \Omega \to \mathbb{G}$ be a nonconstant quasiregular mapping such that B_f is not empty. Then*

$$\inf_{x \in F} i(x, f) < K_I(f) \frac{\theta_2}{\theta_1} \left(\frac{\nu}{\nu - (m+1)} \right)^{\nu-1}. \tag{4.21}$$

Here m is the index of the last vector space of the graduated Lie algebra $\mathcal{G}$ of $\mathbb{G}$.

PROOF. Corollary 4.7 follows from Theorems 4.5 and 3.3. $\qquad\square$

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