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A NONUNIFORM BOUND FOR THE APPROXIMATION OF POISSON BINOMIAL BY POISSON DISTRIBUTION

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It is well known that Poisson binomial distribution can be approximated by Poisson distribution. In this paper, we give a nonuniform bound of this approximation by using Stein-Chen method.

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1. Introduction and main result. Let $X_1, X_2, \dots, X_n$ be independent, possibly not identically distributed, Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = p_i$ and let $S_n = X_1 + X_2 + \cdots + X_n$. The sum of this kind is often called a Poisson binomial random variable. In the case where the "success" probabilities are all identical, $p_i = p$, S is the binomial random variable $\mathcal{B}(n,p)$. Let $\lambda = \sum_{i=1}^{n} p_i$ and let $\mathcal{P}_\lambda$ be the Poisson random variable with parameter λ, that is, $P(\mathcal{P}_\lambda = \omega) = e^{-\lambda}\lambda^{\omega}/\omega!$ for all nonnegative integers ω. It has long been known that if p_i's are small, then the distribution of S_n can be approximated by a distribution of $\mathcal{P}_\lambda$ (see, e.g., Chen [2]).

In this paper, we investigate the bound of this approximation. As an illustration, we look at the case of $p_1 = p_2 = \cdots = p_n = p$. There are at least three known uniform bounds: Kennedy and Quine [6] showed that, for $0 < \lambda \le 2 - \sqrt{2}$,

$$\left| P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega) \right| \le 2\lambda\left[(1-p)^{n-1} - e^{-np}\right], \tag{1.1}$$

Barbour and Hall [1] showed that

$$\left| P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega) \right| \le \min(p, \lambda p), \tag{1.2}$$

and Deheuvels and Pfeifer [5] proved that

$$\begin{aligned} &\left| P(S_n \le \omega) - P(\mathcal{P}_\lambda \le \omega) \right| \\ &\quad \le \lambda p e^{-\lambda}\left\{ \frac{(np)^{(a-1)}(a-np)}{a!} - \frac{(np)^{(b-1)}(b-np)}{b!} \right\} + R \end{aligned} \tag{1.3}$$

with $a = [np + 1/2 + \sqrt{np + 1/4}]$, $b = [np + 1/2 - \sqrt{np + 1/4}]$, and $|R| \le (1/2)(2p)^{3/2}/(1-\sqrt{2p})$, for $0 < p < 1/2$, and $[x]$ is understood to be the integer part of x.

For the general case, Le Cam [7] investigated and showed that

$$\sum_{\omega=0}^{\infty}\left|P(S_n=\omega)-\frac{e^{-\lambda}\lambda^{\omega}}{\omega!}\right| \le \frac{16}{\lambda}\sum_{i=1}^{n}p_i^2. \tag{1.4}$$

It can be observed that the constant $16/\lambda$ will be large when λ is small. Stein [10] used the method of Chen [3] to improve the bound and showed that

$$|P(S_n\le\omega)-P(\mathcal{P}_\lambda\le\omega)| \le (\lambda^{-1}\wedge 1)\sum_{i=1}^{n}p_i^2 \tag{1.5}$$

for $\omega=0,1,2,\dots,n$ and $\lambda^{-1}\wedge 1=\min(\lambda^{-1},1)$. In case when λ tends to 0, one can see that (1.5) becomes

$$|P(S_n\le\omega)-P(\mathcal{P}_\lambda\le\omega)| \le \sum_{i=1}^{n}p_i^2. \tag{1.6}$$

In this paper, we consider a nonuniform bound when λ is small, that is, $\lambda\in(0,1]$ and $\omega\in\{1,2,\dots,n-1\}$. Note that, when $\omega\notin\{1,2,\dots,n-1\}$, we can compute the exact probabilities, that is,

$$\begin{gathered} P(S_n=0)=\prod_{i=1}^{n}(1-p_i),\qquad P(S_n=n)=\prod_{j=1}^{n}p_j,\\ P(S_n=\omega)=0,\quad \omega=n+1,n+2,\dots. \end{gathered} \tag{1.7}$$

In finding the uniform bound, there are several techniques which can be used; for example,

(i) the operator method initiated in Le Cam [7],
(ii) the semigroup approach due to Deheuvels and Pfeifer [4],
(iii) the Chen-Stein technique, see Chen [3] and Stein [10],
(iv) direct computations as in Kennedy and Quine [6],
(v) the coupling method, see Serfling [8] and Stein [10].

In the present paper, our argument closely follows the Chen-Stein technique in Chen [3] and Stein [10]. The following theorem is our main result.

Theorem 1.1. *Let $\lambda\in(0,1]$ and $\omega_0\in\{1,2,\dots,n-1\}$. Then*

$$|P(S_n=\omega_0)-P(\mathcal{P}_\lambda=\omega_0)| \le \frac{1}{\omega_0}\sum_{i=1}^{n}p_i^2. \tag{1.8}$$

2. Proof of the main result. Stein [9] gave a new technique to find a bound in the normal approximation to a distribution of a sum of dependent random variables. His technique was free from Fourier methods and relied instead on the elementary differential equation

$$f'(\omega)-wf(\omega)=h(\omega)-N(h), \tag{2.1}$$

where h is a function that is used to test convergence and $N(h) = E[h(Z)]$ where Z is the standard normal. Chen [3] applied Stein's ideas in the Poisson setting. Corresponding to the differential equation in the normal case above, one has an analogous difference equation

$$\lambda f(\omega+1) - \omega f(\omega) = h(\omega) - \mathcal{P}_\lambda(h), \tag{2.2}$$

where $\mathcal{P}_\lambda(h) = E[h(\mathcal{P}_\lambda)]$ and f and h are real-valued functions defined on $\mathbb{Z}^+ \cup \{0\}$. Let $\omega_0 \in \{1,2,\dots,n-1\}$ and define $h, h_{\omega_0} : \mathbb{Z}^+ \cup \{0\} \to \mathbb{R}$ by

$$h(\omega) = \begin{cases} 1, & \text{if } \omega = \omega_0, \\ 0, & \text{if } \omega \neq \omega_0, \end{cases} \qquad h_{\omega_0}(\omega) = \begin{cases} 1, & \text{if } \omega \le \omega_0, \\ 0, & \text{if } \omega > \omega_0. \end{cases} \tag{2.3}$$

Then we see that the solution f of (2.2) can be expressed in the form

$$f_{\omega_0}(\omega) = \begin{cases} \dfrac{(\omega-1)!}{\omega_0!}\lambda^{\omega_0-\omega}\mathcal{P}_\lambda(1-h_{\omega-1}), & \text{if } \omega_0 < \omega, \\ -\dfrac{(\omega-1)!}{\omega_0!}\lambda^{\omega_0-\omega}\mathcal{P}_\lambda(h_{\omega-1}), & \text{if } \omega_0 \ge \omega > 0, \\ 0, & \text{if } \omega = 0, \end{cases} \tag{2.4}$$

$$\lambda E[f_{\omega_0}(S_n+1)] - E[S_n f_{\omega_0}(S_n)] = P(S_n = \omega_0) - P(\mathcal{P}_\lambda = \omega_0). \tag{2.5}$$

Let $S_n^{(i)} = S_n - X_i$ for $i = 1,2,\dots,n$. By using the facts that each X_j takes on values 0 and 1 and that X_j's are independent, we have

$$\begin{aligned} E[S_n f_{\omega_0}(S_n)] &= \sum_{i=1}^{n} p_i E[f(S_n^{(i)}+1)] \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i E[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n+1)] \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i E\{X_i[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n^{(i)}+2)]\} \\ &= \lambda E[f_{\omega_0}(S_n+1)] + \sum_{i=1}^{n} p_i^2 E[f_{\omega_0}(S_n^{(i)}+1) - f_{\omega_0}(S_n^{(i)}+2)], \end{aligned} \tag{2.6}$$

which implies, by (2.5), that

$$P(S_n = \omega_0) - P(\mathcal{P}_\lambda = \omega_0) = \sum_{i=1}^{n} p_i^2 E[f_{\omega_0}(S_n^{(i)}+2) - f_{\omega_0}(S_n^{(i)}+1)]. \tag{2.7}$$

From (2.4), it follows that

$$f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1) = \begin{cases} -\lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}\left[(\omega+1)\mathcal{P}_\lambda(h_{\omega+1})-\lambda\mathcal{P}_\lambda(h_\omega)\right], & \text{if } \omega\le\omega_0-2, \\ \lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}\left[(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})+\lambda\mathcal{P}_\lambda(h_\omega)\right], & \text{if } \omega=\omega_0-1, \\ \lambda^{\omega_0-\omega-2}\dfrac{\omega!}{\omega_0!}\left[(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})-\lambda\mathcal{P}_\lambda(1-h_\omega)\right], & \text{if } \omega\ge\omega_0. \end{cases} \tag{2.8}$$

Case 1 ($\omega\le\omega_0-2$). Since

$$(\omega+1)\mathcal{P}_\lambda(h_{\omega+1})-\lambda\mathcal{P}_\lambda(h_\omega)=e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}(\omega+1-k), \tag{2.9}$$

we have

$$\begin{aligned} \left|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)\right| &= \lambda^{(\omega_0-2)-\omega}\frac{\omega!}{\omega_0!}\left[e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}(\omega+1-k)\right] \\ &\le \frac{(\omega+1)!}{\omega_0!}\left[e^{-\lambda}\sum_{k=0}^{\omega+1}\frac{\lambda^k}{k!}\right] \\ &\le \frac{(\omega_0-1)!}{\omega_0!} \\ &= \frac{1}{\omega_0}, \end{aligned} \tag{2.10}$$

where we have used the facts that $\lambda\in(0,1]$ and $0\le\omega+1-k\le\omega+1$ in the first inequality and the conditions $\omega\le\omega_0-2$ and $e^{-\lambda}\sum_{k=0}^{\omega+1}(\lambda^k/k!)\le 1$ in the second inequality.

Case 2 ($\omega=\omega_0-1$). We have

$$\begin{aligned} \left|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)\right| &= \frac{\lambda^{-1}}{\omega_0}\left[\omega_0 e^{-\lambda}\sum_{k=\omega_0+1}^{\infty}\frac{\lambda^k}{k!}+\lambda e^{-\lambda}\sum_{k=0}^{\omega_0-1}\frac{\lambda^k}{k!}\right] \\ &\le \frac{\lambda^{-1}}{\omega_0}\left[e^{-\lambda}\sum_{k=\omega_0+1}^{\infty}k\frac{\lambda^k}{k!}+e^{-\lambda}\sum_{k=0}^{\omega_0-1}(k+1)\frac{\lambda^{k+1}}{(k+1)!}\right] \\ &= \frac{\lambda^{-1}}{\omega_0}E[\mathcal{P}_\lambda] \\ &= \frac{1}{\omega_0}. \end{aligned} \tag{2.11}$$

Case 3 ($\omega \geq \omega_0$). Since

$$
\begin{aligned}
&\frac{1\lambda^{\omega+2}}{(\omega+2)!}+\frac{2\lambda^{\omega+3}}{(\omega+3)!}+\frac{3\lambda^{\omega+4}}{(\omega+4)!}+\cdots \\
&\quad \leq \lambda^{\omega-\omega_0+2}\left[\frac{\omega_0\lambda^{\omega_0}}{\omega_0!(\omega_0+1)\cdots(\omega+2)}+\frac{(\omega_0+1)\lambda^{\omega_0+1}}{(\omega_0+1)!(\omega_0+2)\cdots(\omega+3)}+\cdots\right] \\
&\quad \leq \frac{\lambda^{\omega-\omega_0+2}}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)}\left[\sum_{k=\omega_0}^{\infty}\frac{k\lambda^k}{k!}\right] \\
&\quad \leq \frac{e^{\lambda}\lambda^{\omega-\omega_0+2}E[\mathcal{P}_\lambda]}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)} \\
&\quad = \frac{e^{\lambda}\lambda^{\omega-\omega_0+3}}{(\omega_0+1)(\omega_0+2)\cdots(\omega+2)}, \\
&(\omega+1)\mathcal{P}_\lambda(1-h_{\omega+1})-\lambda\mathcal{P}_\lambda(1-h_\omega)=-e^{-\lambda}\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(k-(\omega+1))<0,
\end{aligned}
\tag{2.12}
$$

we have

$$
\begin{aligned}
\left|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)\right| &= \lambda^{\omega_0-\omega-2}\frac{\omega!}{\omega_0!}e^{-\lambda}\sum_{k=\omega+2}^{\infty}\frac{\lambda^k}{k!}(k-(\omega+1)) \\
&\leq \frac{\lambda\omega!}{(\omega+2)!} \leq \frac{1}{(\omega+1)(\omega+2)}.
\end{aligned}
\tag{2.13}
$$

From Cases 1, 2, and 3, we conclude that

$$
\left|f_{\omega_0}(\omega+2)-f_{\omega_0}(\omega+1)\right| \leq \frac{1}{\omega_0}. \tag{2.14}
$$

By (2.7) and (2.14), we have

$$
\begin{aligned}
&\left|P(S_n=\omega_0)-P(\mathcal{P}_\lambda=\omega_0)\right| \\
&\quad \leq \left(\sum_{i=1}^{n}p_i^2\right)E\left[\left|f_{\omega_0}(S_n^{(i)}+2)-f_{\omega_0}(S_n^{(i)}+1)\right|\right] \leq \frac{1}{\omega_0}\sum_{i=1}^{n}p_i^2.
\end{aligned}
\tag{2.15}
$$

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