IJMMS 2003:49, 3143–3148
PII. S0161171203201277
http://ijmms.hindawi.com

VARIATIONAL APPROACH TO DYNAMICS OF BRIGHT SOLITONS IN LOSSY OPTICAL FIBERS

M. F. MAHMOOD and S. BROOKS

Received 2 January 2002

A variational analysis of dynamics of soliton solution of coupled nonlinear Schrödinger equations with oscillating terms is made, considering a birefringent fiber with a third-order nonlinearity in the anomalous dispersion frequency region. This theoretical model predicts optical soliton oscillations in lossy fibers.

2000 Mathematics Subject Classification: 34A25, 35A15, 35Q55, 37K40, 78A60.

1. Introduction. The propagation of bright solitons in birefringent optical fibers has been the subject of intensive theoretical and experimental investigations during the past two decades. The solitons are nonlinear pulses in fibers when the nonlinearity induced by the optical intensity balances the dispersion of the fiber. Studies of bright soliton propagation in fibers are demanding with reference to the development of soliton-based optical communication, generation of short pulses, and soliton lasers. The idea of exploiting these solitons as natural bits to transmit optical data motivates important research efforts towards the development of models [2, 3, 5, 6, 7, 8, 9] describing solitary wave propagation in optical fibers under different conditions. In lossless, one would not expect solitons to distort in either the time or frequency domains regardless of the distance over which they propagate. This supposition is, however, not true in the case of lossy fibers. In this paper, we follow an adiabatic approach using a variational technique [1] to study dynamics of bright solitons generated from semiconductor lasers in a lossy birefringent fiber.

2. Variational approach to coupled nonlinear Schrödinger (CNLS) equations. The birefringence in fibers arises from the geometric and material contributions [4]. The geometric contribution comes from the ellipticity of the core of the fiber which breaks the cylindrical symmetry. The material contribution comes from the strain within the material forming the core and cladding of the fiber. The birefringence in fibers gives rise to two orthogonal polarization modes that need to be considered. The dynamics of optical solitons in a lossy birefringent fiber is important from a theoretical point of view as well as for the applications, and it is governed by the following CNLS equations [5, 6, 7]:

$$
\begin{aligned}
& i(u_z+\delta u_t)+\frac{1}{2}u_{tt}+(|u|^2+B|v|^2)u+Av^2u^*\exp(-iR\delta z) \\
& \qquad +Du^2v^*\exp\left(\frac{1}{2}iR\delta z\right)+D(2|u|^2+|v|^2)v\exp\left(-\frac{1}{2}iR\delta z\right) \\
& \quad =-i\gamma u, \\
& i(v_z-\delta v_t)+\frac{1}{2}v_{tt}+(B|u|^2+|v|^2)v+Au^2v^*\exp(iR\delta z) \\
& \qquad +Dv^2u^*\exp\left(-\frac{1}{2}iR\delta z\right)+D(|u|^2+2|v|^2)u\exp\left(\frac{1}{2}iR\delta z\right) \\
& \quad =-i\gamma v,
\end{aligned}
\tag{2.1}
$$

where

$$
B=\frac{2(1+\varepsilon\sin^2\theta)}{2+\varepsilon\cos^2\theta},\quad A=\frac{\varepsilon\cos^2\theta}{2+\varepsilon\cos^2\theta},\quad D=\frac{\varepsilon}{2}\tan\theta\quad \text{with } \theta=35^\circ;
\tag{2.2}
$$

u and v are the normalized amplitudes of the fast and slow modes, respectively; t is the time coordinate in a frame moving with the average group velocity of the two modes measured in units of the modulational wavelength; z is the distance along the propagation direction; δ is the normalized birefringence; ε denotes the relative strength of the cross-phase modulation; R is the wave vector mismatch due to modal birefringence of the fiber; and γ denotes the losses in the fiber. The oscillating terms in (2.1) arise from nonlinear polarization and cannot be taken off in the case of fibers with low birefringence as it causes an instability in which the slow moving partial pulse transfers energy to the fast moving partial pulse [7]. Using the transformations (see [5, 6]) $p=\sqrt{B/2}(ue^{i\alpha z}+ve^{-i\alpha z})$ and $q=\sqrt{B/2}(ue^{i\alpha z}-ve^{-i\alpha z})$, we can write the CNLS equations (2.1) as

$$
\begin{aligned}
& i(p_z+\delta q_t)+\frac{1}{2}p_{tt}+\alpha q+p(f|p|^2+h|q|^2)+gp^*q^2+i\gamma p=0, \\
& i(q_z+\delta p_t)+\frac{1}{2}q_{tt}+\alpha p+q(f'|q|^2+h|p|^2)+gq^*p^2+i\gamma q=0,
\end{aligned}
\tag{2.3}
$$

where $\alpha=(1/4)R\delta$, $f=(1/2B)(1+A+B+4D)$, $f'=(1/2B)(1+A+B-4D)$, $h=(1/B)(1-A)$, and $g=(1/2B)(1+A-B)$.

Since losses in the fiber lead to exponential decrease of soliton amplitude, we use the transformations $p\to p'e^{-\gamma z}$ and $q\to q'e^{-\gamma z}$ to write (2.3) in the form

$$
\begin{aligned}
& i(p'_z+\delta q'_t)+\alpha q'+\frac{1}{2}p'_{tt}+p'(f|p'|^2+h|q'|^2)e^{-2\gamma z}+gp'^*q'^2e^{-2\gamma z}=0, \\
& i(q'_z+\delta p'_t)+\alpha p'+\frac{1}{2}q'_{tt}+q'(f'|q'|^2+h|p'|^2)e^{-2\gamma z}+gq'^*p'^2e^{-2\gamma z}=0.
\end{aligned}
\tag{2.4}
$$

The spatiotemporal evolution of the wave amplitudes in the case of bright solitons is governed by the Lagrangian density

$$
\begin{aligned}
L = \int_{-\infty}^{\infty} \Big[& \frac{i}{2}(P_z' P'^* - p_z'^* p') + \frac{i}{2}(q_z' q'^* - q_z'^* q') + \frac{i\delta}{2}(p_t' q'^* - q_t'^* p') \\
& + \frac{i\delta}{2}(q_t' p'^* - p_t'^* q') + \alpha(p'^* q' + p' q'^*) + h|p'|^2|q'|^2 e^{-2\gamma z} \\
& + \frac{1}{2}\left(f|p'|^4 e^{-2\gamma z} - |p_t'|^2\right) + \frac{1}{2}\left(f'|q'|^4 e^{-2\gamma z} - |q_t'|^2\right) \\
& + \frac{1}{2} g(p'^2 q'^{*2} + q'^2 p'^{*2}) e^{-2\gamma z} \Big] dt,
\end{aligned}
\tag{2.5}
$$

where $p_z' = \partial p'/\partial z$, $p_t' = \partial p'/\partial t$, and so on.

The bright soliton solutions of (2.4) using a variational technique is based upon assuming the trial function [6]

$$
\begin{aligned}
\begin{bmatrix} p' \\ q' \end{bmatrix} = 2\eta_r \exp\left[2iV_r(t-\zeta_r) - iC\tanh[2\eta_r(t-\zeta_r)](t-\zeta_r)^2 + iD_r\right] \\
\times \operatorname{sech}[2\eta_r(t-\zeta_r)],
\end{aligned}
\tag{2.6}
$$

that describes the temporal form of the soliton pulses. The evolution parameters η_r, ζ_r, V_r, D_r ($r = 1,2$ correspond to p' and q' solitons, respectively), and C represent amplitude, central position, velocity of soliton's central position as it propagates along the fibre, phase, and initial frequency chirp of the soliton, respectively. We substitute (2.6) into the Lagrangian density (2.5) and use Euler-Lagrange equations to obtain the following system of coupled ordinary differential equations (ODEs) for the evolution of soliton parameters:

$$
\frac{d}{dz}(\eta_r V_r) = \frac{1}{8}\frac{\partial L_1}{\partial \zeta_r} + \frac{1}{8}\frac{\partial L_2}{\partial \zeta_r} + \frac{1}{8}\frac{\partial L_3}{\partial \zeta_r}, \tag{2.7}
$$

$$
8\eta_r \frac{d\zeta_r}{dz} = 16\eta_r V_r - 2C\alpha_1 - 4C - \frac{\partial L_1}{\partial V_r} - \frac{\partial L_3}{\partial V_r}, \tag{2.8}
$$

$$
4\frac{d\eta_r}{dz} = (-1)^r \frac{\partial L_1}{\partial \Lambda} + (-1)^r \frac{\partial L_3}{\partial \Lambda}, \tag{2.9}
$$

$$
\begin{aligned}
\frac{dD_1}{dz} = {} & 2V_1 \frac{d\zeta_1}{dz} + 4f\eta_1^2 e^{-2\gamma z} - 2V_1^2 - 2\eta_1^2 + \frac{C^2}{16\eta_1^2}\alpha_2 \\
& + \frac{C^2}{4\eta_1^2}\alpha_4 + \frac{C^2}{4\eta_1^2}\alpha_5 + \frac{1}{4}\frac{\partial L_1}{\partial \eta_1} + \frac{1}{4}\frac{\partial L_2}{\partial \eta_1} + \frac{1}{4}\frac{\partial L_3}{\partial \eta_1},
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
\frac{dD_2}{dz} = {} & 2V_2 \frac{d\zeta_2}{dz} + 4f'\eta_2^2 e^{-2\gamma z} - 2V_2^2 - 2\eta_2^2 + \frac{C^2}{16\eta_2^2}\alpha_2 \\
& + \frac{C^2}{4\eta_2^2}\alpha_4 + \frac{C^2}{4\eta_2^2}\alpha_5 + \frac{1}{4}\frac{\partial L_1}{\partial \eta_2} + \frac{1}{4}\frac{\partial L_2}{\partial \eta_2} + \frac{1}{4}\frac{\partial L_3}{\partial \eta_2},
\end{aligned}
\tag{2.11}
$$

where

$$\begin{aligned}
\alpha_1 &= \left(\frac{\pi^2}{9}-\frac{2}{3}\right), \qquad \alpha_2 = \left(\frac{7\pi^4}{225}-\frac{\pi^2}{3}+\frac{2}{5}\right),\\
\alpha_3 &= 1, \qquad \alpha_4 = \left(\frac{\pi^2}{18}+\frac{2}{3}\right), \qquad \alpha_5 = \left(\frac{\pi^2}{12}-\frac{1}{2}\right),\\
L_1 &= 4\alpha\eta_2\int_{-\infty}^{\infty}\operatorname{sech}x_1\operatorname{sech}x_2\cos\Lambda\,dx_1 = \frac{8\alpha\eta\rho\cos\Lambda}{\sinh\rho},\\
L_2 &= 8h\eta_1\eta_2^2e^{-2\gamma z}\int_{-\infty}^{\infty}\operatorname{sech}^2x_1\operatorname{sech}^2(x_1+\rho)\,dx_1\\
&= 32h\eta^3e^{-2\gamma z}\frac{\cosh\rho}{\sinh\rho}(\rho-\tanh\rho),\\
L_3 &= 8g\eta_1\eta_2^2e^{-2\gamma z}\int_{-\infty}^{\infty}\operatorname{sech}^2x_1\operatorname{sech}^2(x_1+\rho)\cos2\Lambda\,dx_1\\
&= 32g\eta^3e^{-2\gamma z}\cos\Lambda\frac{\cosh\rho}{\sinh^3\rho}(\rho-\tanh\rho),\\
\Lambda &= 2t(V_1-V_2)-2(V_1\zeta_1-V_2\zeta_2)-(D_2-D_1)\\
&\quad -C\tanh[2\eta_1(t-\zeta_1)](t-\zeta_1)^2+C\tanh[2\eta_2(t-\zeta_2)](t-\zeta_2)^2.
\end{aligned} \tag{2.12}$$

We evaluate the above integrals for nearly equal pulse amplitudes $\eta_1 \cong \eta_2 \cong \eta$, relative phase $\phi = D_2 - D_1$, relative distance between two polarization maxima $\rho = x_2 - x_1$, and $V_1 \cong V2 \cong V$. The relative parameters η_{12}, V_{12}, ϕ, and ρ defined for p' and q' solitons are obtained as follows. Writing $\eta_{12} = \eta_1 - \eta_2$ and using (2.9), we get

$$\begin{aligned}
\frac{d\eta_{12}}{dz} &= -4\alpha\eta\left(\frac{\rho}{\sinh\rho}\right)\sin\left(\frac{V}{\eta}\rho+\phi\right)\\
&\quad -32g\eta^3e^{-2\gamma z}\left[\frac{\cosh\rho}{\sinh^3\rho}(\rho-\tanh\rho)\right]\sin2\left(\frac{V}{\eta}\rho+\phi\right).
\end{aligned} \tag{2.13}$$

Similarly, writing $V_{12} = V_1 - V_2$ and using (2.7), we get

$$\begin{aligned}
\frac{dV_{12}}{dz} &= -\frac{4}{3}\alpha\eta\rho\cos\left(\frac{V}{\eta}\rho+\phi\right)-\frac{64}{15}h\eta^3\rho e^{-2\gamma z}\\
&\quad -\frac{64}{15}g\eta^3\rho e^{-2\gamma z}\cos2\left(\frac{V}{\eta}\rho+\phi\right).
\end{aligned} \tag{2.14}$$

Also, from $\rho = 2\eta(\zeta_1-\zeta_2)$ and by using (2.8), (2.13), and (2.14), we obtain

$$\begin{aligned}
\frac{d^2\rho}{dz^2} &= -\frac{16}{3}\alpha\eta^2\rho\cos\left(\frac{V}{\eta}\rho+\phi\right)-\frac{256}{15}h\eta^4\rho e^{-2\gamma z}\\
&\quad -\frac{256}{15}g\eta^4\rho e^{-2\gamma z}\cos2\left(\frac{V}{\eta}\rho+\phi\right)\\
&\quad -2\left(\frac{\pi^2}{9}+\frac{4}{3}\right)C\alpha\sin\left(\frac{V}{\eta}\rho+\phi\right)\\
&\quad -\frac{16}{3}\left(\frac{\pi^2}{9}+\frac{4}{3}\right)Cg\eta^2e^{-2\gamma z}\sin2\left(\frac{V}{\eta}\rho+\phi\right).
\end{aligned} \tag{2.15}$$

The two solitons with opposite phases form a bound state provided that

$$\frac{V}{\eta}\rho + \phi = \Phi = \pm\pi. \tag{2.16}$$

Thus, we write (2.15) as

$$\frac{d^2\rho}{dz^2} = \frac{16}{3}\alpha\eta^2\rho - \frac{256}{15}h\eta^4\rho e^{-2\gamma z} - \frac{256}{15}g\eta^4\rho e^{-2\gamma z} \tag{2.17}$$

or

$$\frac{d^2\rho}{dz^2} + \left(ae^{-2\gamma z} - b\right)\rho(z) = 0, \tag{2.18}$$

where $a = (256/15)\eta^4(h+g)$ and $b = (16/3)\alpha\eta^2$.

Equation (2.18) can also be written as

$$\sigma^2\frac{d^2\rho}{d\sigma^2} + \sigma\frac{d\rho}{d\sigma} + \left(\sigma^2 - \frac{b}{\gamma^2}\right)\rho = 0, \tag{2.19}$$

where $\sigma = (\sqrt{a}/\gamma)e^{-\gamma z}$.

Equation (2.19) is a Bessel equation. Its general solution is given by

$$\rho(\sigma) = XJ\frac{\sqrt{b}}{\gamma}(\sigma) + YJ - \frac{\sqrt{b}}{\gamma}(\sigma). \tag{2.20}$$

Considering γ to be small, we use asymptotic expansion of Bessel function to write (2.20) as

$$\rho(\sigma) \cong \left(\frac{2\gamma}{\pi\sqrt{a}e^{-\gamma z}}\right)^{1/2}\cos\left(\frac{\sqrt{a}}{\gamma}\left(1 - \gamma z + o(z^2)\right) - \frac{\pi}{2}\frac{\sqrt{b}}{\gamma} - \frac{\pi}{4}\right), \tag{2.21}$$

and the frequency ω of relative oscillations of soliton positions is given by

$$\omega^2 = a = \frac{256}{15}\eta^4\left(\frac{2}{B} - 1\right). \tag{2.22}$$

3. Conclusion. In this paper, we have considered CNLS equations with oscillating terms to develop a theoretical model of a birefringent optical fiber. This theoretical model demonstrates polarized bright soliton dynamics in a lossy birefringent fiber. We used a variational approach to obtain frequency of relative oscillations of soliton positions by taking into account the interaction between different polarizations in a lossy birefringent optical fiber.

References

[1] D. Anderson, M. Lisak, and T. Reichel, *Approximate analytical approaches to nonlinear pulse propagation in optical fibers: a comparison*, Phys. Rev. A **38** (1988), 1618–1622.

[2] K. J. Blow, N. J. Doran, and D. Wood, *Polarization instabilities for solitons in birefringent fibers*, Opt. Lett. **12** (1987), no. 3, 202–204.

[3] M. N. Islam, C. D. Poole, and J. P. Gordon, *Soliton trapping in birefringent optical fibers*, Opt. Lett. **14** (1989), no. 18, 1011–1013.

[4] I. P. Kaminow, *Polarization in optical fibers*, IEEE J. Quant. Electron. **17** (1981), no. 1, 15–22.

[5] M. F. Mahmood, W. W. Zachary, and T. L. Gill, *Chirped solitary pulses in low birefringent optical fibers: a theoretical model*, Opt. Quant. Electron. **28** (1996), 1007–1012.

[6] ———, *Nonlinear pulse propagation in elliptically birefringent optical fibers*, Phys. D **90** (1996), no. 3, 271–279.

[7] C. R. Menyuk, *Pulse propagation in an elliptically birefringent Kerr medium*, IEEE J. Quant. Electron. **25** (1989), no. 12, 2674–2682.

[8] J. M. Soto-Crespo, N. Akhmediev, and A. Ankiewicz, *Soliton propagation in optical devices with two-component fields: a comparative study*, J. Opt. Soc. Amer. B Opt. Phys. **12** (1995), no. 6, 1100–1109.

[9] S. Trillo, S. Wabnitz, W. C. Banyai, N. Finlayson, C. T. Seaton, G. I. Stegeman, and R. H. Stolen, *Observation of ultrafast nonlinear polarization switching induced by polarization instability in a birefringent fiber rocking filter*, IEEE J. Quant. Electron. **25** (1989), no. 1, 104–112.

M. F. Mahmood: Department of Mathematics, Howard University, Washington, DC 20059, USA

E-mail address: mmahmood@howard.edu

S. Brooks: Department of Mathematics, Howard University, Washington, DC 20059, USA