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RC-CONTINUOUS FUNCTIONS AND FUNCTIONS WITH RC-STRONGLY CLOSED GRAPH

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The family of regular closed subsets of a topological space is used to introduce two concepts concerning a function f from a space X to a space Y. The first of them is the notion of f being rc-continuous. One of the established results states that a space Y is extremally disconnected if and only if each continuous function from a space X to Y is rc-continuous. The second concept studied is the notion of a function f having an rc-strongly closed graph. Also one of the established results characterizes rc-compact spaces ($\equiv S$-closed spaces) in terms of functions that possess rc-strongly closed graph.

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1. Introduction. A subset A of a topological space (X,T) is called regular closed if $A = \mathrm{Cl}(\mathrm{Int}(A))$, that is, if A is equal to the closure of its interior. It is called semiopen if there exists $U \in T$ such that $U \subseteq A \subseteq \mathrm{Cl}(U)$. A space (X,T) is called S-closed (see [10]) if every cover $\mathcal{U}$ of (X,T) by semiopen sets contains a finite subfamily $\{U_1,\dots,U_n\}$ such that $X = \bigcup_{i=1}^{n}\mathrm{Cl}(U_i)$. In [1], it is proved that a space (X,T) is S-closed if and only if every cover of X by regular closed subsets contains a finite subcover for X. This result motivated the study of other covering properties by regular closed subsets; for example, rc-Lindelöf spaces (see [2]), almost rc-Lindelöf spaces (see [3]), rc-paracompact spaces (see [2]), and others. In fact, the name rc-compact space is recently used for S-closed spaces (see [6]) in order to have a unified terminology when dealing with covering properties by regular closed sets.

On the other hand, in [5], the rc-convergence of a filterbase or a net in a space (X,T) is formulated using the regular closed sets that contain the limit point. The concept of rc-convergence is proved to be useful in characterizing S-closed (rc-compact) spaces and also extremally disconnected spaces.

In Section 3, we use the family of regular closed subsets to define rc-continuous functions. In our study of this new class of functions, we state a couple of characterizations. Then we show that rc-continuous functions can be used to characterize extremally disconnected spaces.

In Section 4, we introduce, again by using the family of regular closed subsets, the notion of a function f that has an rc-strongly closed graph $G(f)$. Characterizations of a function to have an rc-strongly closed graph are given and then some results connecting them to rc-compact spaces are stated. In particular, for a Hausdorff space (Y,M), it is shown that (Y,M) is an rc-compact space if and only if certain functions to (Y,M), that have an rc-strongly closed graph, must be rc-continuous.

2. Definitions and preliminaries. In what follows, a space always means a topological space with no separation axiom unless it is explicitly stated. For a subset A of a space (X,T), we let $\mathrm{Cl}_T(A)$ (or simply $\mathrm{Cl}(A)$) denote the closure of A while $\mathrm{Int}_T(A)$ (or simply $\mathrm{Int}(A)$) will denote the interior of A, both in the space (X,T). A subset A of (X,T) is called semiopen if there exists $U \in T$ such that $U \subseteq A \subseteq \mathrm{Cl}(U)$. The family of all semiopen subsets of (X,T) is denoted by $\mathrm{SO}(X,T)$. A subset A of (X,T) is called regular closed if $A = \mathrm{Cl}(\mathrm{Int}(A))$. We let $\mathrm{RC}(X,T)$ denote the family of all regular closed subsets of (X,T). It is easy to see that for any space (X,T), it is true that $\mathrm{RC}(X,T) \subseteq \mathrm{SO}(X,T)$. The complement of a regular closed subset of (X,T) is called regular open. Equivalently, a subset U is a regular open subset of (X,T) if $U = \mathrm{Int}(\mathrm{Cl}(U))$. We let $\mathrm{RO}(X,T)$ denote the family of all regular open subsets of (X,T). As a convention, a cover of a space by regular closed subsets will be called an rc-cover.

All definitions and results included in this section are previously known and we quote the reference when possible, while others are easy to establish and we include them for later use.

Definition 2.1 (see [10]). A space (X,T) is called rc-compact ($\equiv S$-closed) if every cover $\mathcal{U}$ of X by semiopen subsets contains a finite subfamily $\{U_1,\dots,U_n\}$ such that $X = \bigcup_{i=1}^{n} \mathrm{Cl}(U_i)$.

Proposition 2.2 (see [1]). *A space (X,T) is rc-compact if and only if every rc-cover of X contains a finite subcover for X.*

Definition 2.3. A subset A of a space (X,T) is called an rc-compact subset of X ($\equiv S$-closed of X in [10]) or an rc-compact subset relative to X ($\equiv S$-closed subset relative to X in [9, 10]) if every rc-family $\mathcal{A}$ of regular closed subsets of X, with $A \subseteq \bigcup \mathcal{A}$, contains a finite subfamily whose union also contains A.

Definition 2.4. A space (X,T) is called extremally disconnected if $\mathrm{Cl}(U)$ is open for each $U \in T$. In an extremally disconnected space (X,T), $\mathrm{RC}(X,T) = \mathrm{RO}(X,T)$.

Definition 2.5 (see [5, Definition 3.1]). (a) A net $(x_\lambda)_{\lambda\in\Lambda}$ in a space (X,T) is said to rc-converge to a point $x_0 \in X$ if each $F \in \mathrm{RC}(X,T)$, with $x_0 \in F$, contains a tail T_λ of (x_λ), that is, the net (x_λ) is eventually in each $F \in \mathrm{RC}(X,T)$ with $x_0 \in F$. Equivalently, for each $U \in T$ such that $x_0 \in \mathrm{Cl}(U)$, $\mathrm{Cl}(U)$ contains a tail of (x_λ). This fact is expressed by writing $x_\lambda \to (\mathrm{rc})x_0$.

(b) A net $(x_\lambda)_{\lambda\in\Lambda}$ in a space (X,T) is said to rc-accumulate at a point $x_0 \in X$ if for each $F \in \mathrm{RC}(X,T)$ and for each $\lambda \in \Lambda$, $F \bigcap T_\lambda \neq \emptyset$, where T_λ is the tail of the net (x_λ) determined by λ, that is, (x_λ) is frequently in each $F \in \mathrm{RC}(X,T)$ that contains x_0.

Proposition 2.6 (see [5]). *A space (X,T) is rc-compact if and only if every net in (X,T) rc-accumulates to some point of X.*

Proposition 2.7 (see [5]). *A space (X,T) is extremally disconnected if and only if each net (x_λ) in X, that is convergent in the usual sense, is also rc-convergent.*

3. rc-continuous functions. For a space (X,T) and a point $x \in X$, we let $\mathrm{ON}(x) = \{U \in T : x \in U\}$, $\overline{\mathrm{ON}}(x) = \{\mathrm{Cl}(U) : x \in U\}$, and $\mathrm{RC}(X,T,x) = \{F \in \mathrm{RC}(X,T) : x \in F\}$. It is clear that $\overline{\mathrm{ON}}(x) \subseteq \mathrm{RC}(X,T,x)$.

We recall that a function $f : (X,T) \to (Y,M)$ is called weakly continuous (see [3, Definition 6]) (or w θ-continuous in [8]) if f is weakly continuous at each $x \in X$ in the sense that for each $W \in \overline{\mathrm{ON}}(f(x))$, there exists $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq W$.

In this section, we introduce rc-continuous functions that form a proper subclass of the class of weakly continuous functions. In studying this new class, we state several characterizations of rc-continuous functions and then we show that a space is extremally disconnected if and only if each continuous function onto that space is rc-continuous.

Definition 3.1. A function $f : (X,T) \to (Y,M)$ is called rc-continuous at a point $x \in X$ if for each $W \in \mathrm{RC}(Y,M,f(x))$, there exists $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq W$. The function f is called an rc-continuous function if it is rc-continuous at each point of its domain.

The next result is a direct consequence of the definitions.

Proposition 3.2. *Every rc-continuous function is weakly continuous.*

However, the converse of this result is not true. To see that, we recall that every continuous function is weakly continuous. On the other hand, it is easy to provide an example of a continuous function that is not rc-continuous. For instance, consider the function f from the set of real numbers $\mathbb{R}$ with the usual topology onto itself given by $f(x) = x+1$. Though this function is continuous, it is not rc-continuous.

The first characterization of rc-continuous functions is obtained through introducing a new topology on the codomain. For a space (X,T), we introduce the regular closed subsets generated topology, abbreviated TRC, to be the topology on X generated by the subbase $\mathrm{RC}(X,T)$. This topology is interesting in its own and will be studied separately elsewhere.

Proposition 3.3. *A function $f : (X,T) \to (Y,M)$ is rc-continuous if and only if the function $f : (X,T) \to (Y,M\mathrm{RC})$ is continuous.*

Proof

Necessity. It is enough to consider the subbase $\mathrm{RC}(Y,M)$ of the topology $M\mathrm{RC}$ on Y and to show that $f^{-1}(W)$ is T-open for each $W \in \mathrm{RC}(Y,M)$. So let $x \in f^{-1}(W)$. Then $f(x) \in W$ with $W \in \mathrm{RC}(Y,M,f(x))$. By definition of rc-continuity, there exists $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq W$. It follows that $x \in U \subseteq f^{-1}(W)$. Thus, $f^{-1}(W)$ is T-open and $f:(X,T) \to (Y,M\mathrm{RC})$ is continuous.

Sufficiency. Assume that $f:(X,T) \to (Y,M\mathrm{RC})$ is continuous. So if $W \in M\mathrm{RC}$, then $f^{-1}(W)$ is T-open. In particular, for any $x \in X$ and any $W \in \mathrm{RC}(Y,M,f(x))$, we see that $f^{-1}(W)$ is T-open. Thus, $f:(X,T) \to (Y,M)$ is rc-continuous at any point $x \in X$. □

Let (X,T) be a given space and let $A \subseteq X$. We say that a point x is in the rc-closure of A (rc-$\mathrm{Cl}(A)$) if $A \bigcap F \neq \varnothing$ for each $F \in \mathrm{RC}(X,T,x)$. We say that A is rc-closed if $A = $ rc-$\mathrm{Cl}(A)$. We state the following characterizations of rc-continuous functions.

Proposition 3.4. *The following conditions are equivalent for a function* $f:(X,T) \to (Y,M)$*:*

(a) *f is rc-continuous,*
(b) *for each rc-closed subset D of Y, $f^{-1}(D)$ is closed,*
(c) *for each $G \in \mathrm{RO}(Y,M)$, $f^{-1}(G)$ is closed,*
(d) *for each $F \in \mathrm{RC}(Y,M)$, $f^{-1}(F)$ is open.*

Proof. (a)⇒(b). To prove that $f^{-1}(D)$ is closed, let $x \in X - f^{-1}(D)$. Then $f(x) \notin D$ and we can find $H \in \mathrm{RC}(Y,M)$ such that $f(x) \in H$ and $H \bigcap D = \varnothing$. By rc-continuity of f, we find $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq H$. It follows that $f(U) \bigcap D = \varnothing$ and so $x \in U \subseteq X - f^{-1}(D)$. This shows that $f^{-1}(D)$ is closed.

(b)⇒(c) follows easily by the fact that if G is regular open then G is rc-closed.

(c)⇒(d) is clear.

(d)⇒(a) follows by direct application of the definition. □

Next, we show that rc-continuity is characterized by rc-convergence of nets.

Proposition 3.5. *A function $f:(X,T) \to (Y,M)$ is rc-continuous at a point $x_0 \in X$ if and only if for each net $(x_\lambda)_{\lambda\in\Lambda}$, if $x_\lambda \to x_0$ in X, then $f(x_\lambda) \to (\mathrm{rc})f(x_0)$ in Y.*

Proof

Necessity. Assume that $f:(X,T) \to (Y,M)$ is rc-continuous at a point $x_0 \in X$. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X such that $x_\lambda \to x_0$. Let $W \in \mathrm{RC}(Y,M,f(x))$. By rc-continuity of f at x_0, there exists $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq W$. But $x_\lambda \to x_0$, so there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for each $\lambda \geq \lambda_0$. It follows that $f(x_\lambda) \in W$ for each $\lambda \geq \lambda_0$, that is, W contains a tail of $f(x_\lambda)$. Thus, $f(x_\lambda) \to (\mathrm{rc})f(x_0)$.

Sufficiency. Suppose that f is not rc-continuous at x_0. Then there exists $W \in \mathrm{RC}(Y,M,f(x_0))$ such that $f(U)\bigcap(Y-W) \neq \varnothing$ for each $U \in T$ with $x_0 \in U$. Let $D = \{(U,t) : x_0 \in U,\ t \in U,\ f(t) \in Y-W\}$. For $d_1, d_2 \in D$, say $d_1 = (U_1,t_1)$ and $d_2 = (U_2,t_2)$, we let $d_2 \leq d_1$ be equivalent to $U_2 \leq U_1$. Then $(D,\leq)$ is a directed set and we define a net $\varphi : D \to X$ as follows: for $d \in D$, say $d = (U,t)$, we let $\varphi(d) = x_d = t$. By its construction, we have that $x_d \to x_0$ in X. However, the net $(f(x_d))$ does not rc-converge to $f(x_0)$, a contradiction. □

We have noted earlier that a continuous function need not be, in general, rc-continuous. However, it is true if the codomain of the function is assumed to be extremally disconnected. In fact, it characterizes extremally disconnected spaces as it is established in our last result of this section.

Theorem 3.6. *A space (Y,M) is extremally disconnected if and only if each continuous function from a space (X,T) into (Y,M) is rc-continuous.*

Proof

Necessity. Let $f : (X,T) \to (Y,M)$ be a continuous function. Let $x \in X$ and $V \in M$ with $f(x) \in \mathrm{Cl}_M(V)$. Since (Y,M) is extremally disconnected, $\mathrm{Cl}_M(V)$ is open. By the continuity of f and since $f(x) \in \mathrm{Cl}_M(V)$, there exists $U \in T$ such that $x \in U$ and $f(U) \subseteq \mathrm{Cl}_M(V)$. This proves that f is rc-continuous at the arbitrary point $x \in X$.

Sufficiency. Suppose that (Y,M) is not extremally disconnected. Then there exists $V \in M$ such that $\mathrm{Cl}_M(V)$ is not open. Choose $x_0 \in \mathrm{Cl}_M(V) - \mathrm{Int}_M(\mathrm{Cl}_M(V))$. Consider the identity function $i_Y : (Y,M) \to (Y,M)$. Although i_Y is continuous, it is not rc-continuous at x_0, which contradicts the hypothesis. This completes the proof. □

4. Functions with rc-strongly closed graph.

The graph of a function $f : X \to Y$ is the set $G(f) = \{(x,y) \in X\times Y : y = f(x)\}$. A function $f : (X,T) \to (Y,M)$ has a closed graph if $G(f)$ is a closed subset of the product space $X\times Y$. On the other hand, f is said to have a strongly closed graph (see [4]) if whenever $(x,y) \in X\times Y$ and $(x,y) \notin G(f)$, there exist $U \in T$ and $V \in M$ such that $x \in U$, $y \in V$, and $(U\times \mathrm{Cl}_M(V))\bigcap G(f) = \varnothing$.

In [4], functions that possess strongly closed graph are related to H-spaces. In this section, we define the concept of a function f to have an rc-strongly closed graph, study its properties, and relate them to rc-compact spaces.

Definition 4.1. A function $f : (X,T) \to (Y,M)$ has an rc-strongly closed graph $G(f)$ if whenever $(x,y) \in X\times Y$ and $(x,y) \notin G(f)$, there exist $U \in T$ and $V \in M$ such that $x \in U$, $y \in \mathrm{Cl}_M(V)$, and $(U\times \mathrm{Cl}_M(V))\bigcap G(f) = \varnothing$.

By the definitions, it directly follows that if a function f possesses a strongly closed graph, then it possesses an rc-strongly closed graph.

Our first characterization of functions that have rc-strongly closed graph is in terms of rc-convergence.

Proposition 4.2. *A function $f:(X,T)\to(Y,M)$ has an rc-strongly closed graph if and only if for any net $(x_\lambda)_{\lambda\in\Lambda}$ in X such that $x_\lambda\to x_0$ while $f(x_\lambda)\to(\mathrm{rc})y_0\in Y$, $f(x_0)=y_0$.*

Proof

Necessity. Assume that f has an rc-strongly closed graph. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X which converges to a point $x_0\in X$ and assume that the net $(f(x_\lambda))$ rc-converges to a point $y_0\in Y$. We show that $f(x_0)=y_0$. Suppose, to the contrary, that $(x_0,y_0)\notin G(f)$. Since $G(f)$ is rc-strongly closed, then there exist $U\in\mathrm{ON}(x_0)$ and $V\in M$ with $y_0\in\mathrm{Cl}_M(V)$ such that $(U\times\mathrm{Cl}_M(V))\bigcap G(f)=\varnothing$. But $x_\lambda\to x_0$ and $f(x_\lambda)\to(\mathrm{rc})y_0$, so there exist $\lambda_1,\lambda_2\in\Lambda$ such that $x_\lambda\in U$ for each $\lambda\geq\lambda_1$ and $f(x_\lambda)\in\mathrm{Cl}_M(V)$ for each $\lambda\geq\lambda_2$. Choose $\lambda_3\in\Lambda$ such that $\lambda_3\geq\lambda_i$, $i=1,2$. Then $(x_\lambda,f(x_\lambda))\in(U\times\mathrm{Cl}_M(V))\bigcap G(f)$, a contradiction.

Sufficiency. Suppose that $G(f)$ is not rc-strongly closed. Then there exists a point $(x_0,y_0)\in X\times Y$, where $(x_0,y_0)\notin G(f)$, such that whenever $U\in T$ and $V\in M$ with $x_0\in U$ and $y_0\in\mathrm{Cl}_M(V)$, then $(U\times\mathrm{Cl}_M(V))\bigcap G(f)\neq\varnothing$. We put $D=\{(U,V,x):x_0\in U\in T,\ V\in M,\ y_0\in\mathrm{Cl}_M(V),\ (x,f(x))\in(U\times\mathrm{Cl}_M(V))\}$. We make D into a directed set in the obvious way and define a net as follows: $\varphi:D\to X$. If $d\in D$, say $d=(U,V,x)$, then we let $\varphi(d)=x_d=x$. By its construction, it is clear that the net (x_d) converges in X to x_0 while the net $f((x_d))$ rc-converges in Y to y_0. However, $f(x_0)\neq y_0$, which contradicts our hypothesis. □

Proposition 4.3. *The following conditions are equivalent for a given function $f:(X,T)\to(Y,M)$:*

(1) *f has an rc-strongly closed graph,*

(2) *given any net (y_λ) in Y that rc-converges to a point $y_0\in Y$, $\limsup f^{-1}(y_\lambda)\subseteq f^{-1}(y_0)$,*

(3) *given any net (y_λ) in Y that rc-converges to a point $y_0\in Y$, $\liminf f^{-1}(y_\lambda)\subseteq f^{-1}(y_0)$.*

Proof. (1)⇒(2). Let $x\in\limsup f^{-1}(y_\lambda)$. Suppose that $x\notin f^{-1}(y_0)$. Then $f(x)\neq y_0$ and we have $(x,y_0)\notin G(f)$. Since $G(f)$ is rc-strongly closed, there exist $U\in\mathrm{ON}(x)$ and $V\in M$ such that $y_0\in\mathrm{Cl}_M(V)$ and $(U\times\mathrm{Cl}_M(V))\bigcap G(f)=\varnothing$. Moreover, since the net (y_λ) rc-converges to $y_0\in Y$, there exists $\lambda_1\in\Lambda$ such that $y_\lambda\in\mathrm{Cl}_M(V)$ for each $\lambda\geq\lambda_1$. Also, as $x\in\limsup f^{-1}(y_\lambda)$, there exists $\lambda_2\geq\lambda_1$ such that $U\bigcap f^{-1}(y_{\lambda_2})\neq\varnothing$, say $x_1\in U\bigcap f^{-1}(y_{\lambda_2})$. It follows that $f(x_1)=y_{\lambda_2}\in\mathrm{Cl}_M(V)$ and so $(x,f(x_1))\in(U\times\mathrm{Cl}_M(V))\bigcap G(f)$, a contradiction.

(2)⇒(3) follows directly by the fact that $\liminf f^{-1}(y_\lambda)\subseteq\limsup f^{-1}(y_\lambda)$.

(3)⇒(1) suppose, to the contrary, that $G(f)$ is not rc-strongly closed. Then, by Proposition 4.2, there exists a net (x_λ) in X such that $x_\lambda\to x_0\in X$, the net $(y_\lambda)=f(x_\lambda)$ rc-converges to a point $y_0\in Y$, and $f(x_0)\neq y_0$. It is clear that $x_0\in\liminf f^{-1}(y_\lambda)$, which implies, by our assumption, that $x_0\in f^{-1}(y_0)$, that is, $f(x_0)=y_0$, a contradiction. □

The graph of a continuous function need not be rc-strongly closed as it is shown in the next example.

Example 4.4. We take $X = R$ with the usual topology T and $Y = R$ with the left-ray topology M. It is clear that the identity function $i_X : (X,T) \to (X,M)$ is continuous. However, the graph $G(i_X)$ is not rc-strongly closed. For instance, $(0,1) \notin G(i_X)$ but for any $U \in T$ and $V \in M$ such that $0 \in U$ and $1 \in \mathrm{Cl}_M(V) = R$, we have $(U \times \mathrm{Cl}_M(V)) \bigcap G(i_X) \neq \emptyset$.

Shortly, we provide a sufficient condition on the codomain of a continuous function to insure that it has an rc-strongly closed graph. But first we need to introduce the following class of spaces.

Definition 4.5. A space (X,T) is called an rc-T_1-space if for every pair of points $x, y \in X$, with $x \neq y$, there exists a regular closed subset F of X such that $x \in F$ but $y \notin F$.

We remark here that the class of rc-T_1-spaces coincides with the class of weakly Hausdorff spaces. Recall that a space (X,T) is called weakly Hausdorff (see [9]) if each singleton $\{x\}$ is the intersection of regular closed subsets. However, our terminology agrees with the recent trend of naming concepts formulated using regular closed subsets, as it is noted in the introduction.

The following are two easily established facts.

Proposition 4.6. *Every Hausdorff space is an rc-T_1-space. Also, every rc-T_1-space is a T_1-space.*

Proposition 4.7. *Every extremally disconnected rc-T_1-space is Hausdorff.*

We are now ready to give a sufficient condition for a continuous function f to have an rc-strongly closed graph.

Theorem 4.8. *Let f be a continuous function from a space (X,T) to an rc-T_1-space (Y,M). Then f has an rc-strongly closed graph.*

Proof. Let $(x,y) \in X \times Y$ with $(x,y) \notin G(f)$. This means that $f(x) \neq y$, and since Y is an rc-T_1-space, there exists $V \in M$ such that $y \in \mathrm{Cl}_M(V)$ and $f(x) \notin \mathrm{Cl}_M(V)$, that is, $f(x) \in Y - \mathrm{Cl}_M(V)$. By continuity of f at x, there exists $U \in \mathrm{ON}(x)$ such that $f(U) \subseteq Y - \mathrm{Cl}_M(V)$. This implies that $f(U) \bigcap \mathrm{Cl}_M(V) = \emptyset$ and it follows that $(x,y) \in U \times \mathrm{Cl}_M(V)$ with $(U \times \mathrm{Cl}_M(V)) \bigcap G(f) = \emptyset$. Thus, $G(f)$ is rc-strongly closed. □

The remaining results of this section relate functions with rc-strongly closed graph to rc-compact spaces.

Theorem 4.9. *Let (Y,M) be an rc-compact space. If a function $f : (X,T) \to (Y,M)$ has an rc-strongly closed graph, then f is rc-continuous.*

Proof. Let $x \in X$ and let $V \subseteq M$ such that $f(x) \in \mathrm{Cl}_M(V)$. For each $y \in Y - \mathrm{Cl}_M(V)$, we have $(x,y) \notin G(f)$. Since $G(f)$ is rc-strongly closed, there exist $U_y \in T$ and $V_y \in M$ such that $x \in U_y$, $y \in \mathrm{Cl}_M(V_y)$, and $(U_y \times \mathrm{Cl}_M(V_y)) \bigcap G(f) = \varnothing$. Note that $f(U_y) \bigcap \mathrm{Cl}_M(V_y) = \varnothing$. Now, the family $\{\mathrm{Cl}_M(V)\} \bigcup \{\mathrm{Cl}_M(V_y) : y \in Y - \mathrm{Cl}_M(V)\}$ forms an rc-cover for the rc-compact space Y. So it contains a finite subcover $\{\mathrm{Cl}_M(V), \mathrm{Cl}_M(V_{y_1}), \ldots, \mathrm{Cl}_M(V_{y_n})\}$ for Y. Put $U_x = \bigcup_{i=1}^n U_{y_i}$. Then $U_x \in T$, $x \in U_x$, and $f(U_x) \bigcap (\bigcup_{i=1}^n \mathrm{Cl}_M(V_{y_i})) = \varnothing$, that is, $f(U_x) \subseteq Y - (\bigcup_{i=1}^n \mathrm{Cl}_M(V_{y_i})) \subseteq \mathrm{Cl}_M(V)$. Thus, f is rc-continuous at the arbitrary point $x \in X$ and so f is rc-continuous. □

Proposition 4.10. *Let $f : (X,T) \to (Y,M)$ have an rc-strongly closed graph. If B is an rc-compact subset relative to Y, then $f^{-1}(B)$ is a closed subset of X.*

Proof. Let $x \in X - f^{-1}(B)$. Then $f(x) \notin B$ and for each $y \in B$, we have $(x,y) \notin G(f)$. Since $G(f)$ is rc-strongly closed, there exist $U_y \in T$ and $V_y \in M$ such that $x \in U_y$, $y \in \mathrm{Cl}_M(V_y)$, and $(U_y \times \mathrm{Cl}_M(V_y)) \bigcap G(f) = \varnothing$. Note that $f(U_y) \bigcap \mathrm{Cl}_M(V_y) = \varnothing$. Now, $\{\mathrm{Cl}_M(V_y) : y \in B\}$ is a family of regular closed subsets of Y whose union contains B. Since B is rc-compact relative to Y, we obtain a finite subfamily $\{\mathrm{Cl}_M(V_{y_1}), \ldots, \mathrm{Cl}_M(V_{y_n})\}$ such that $B \subseteq \bigcup_{i=1}^n \mathrm{Cl}_M(V_{y_i})$. Put $U_x = \bigcup_{i=1}^n U_{y_i}$. Then $U_x \in T$, $x \in U_x$, and $f(U_x) \bigcap (\bigcup_{i=1}^n \mathrm{Cl}_M(V_{y_i})) = \varnothing$. This means that $f(U_x) \bigcap B = \varnothing$ and so $x \in U_x \subseteq X - f^{-1}(B)$, which proves that $f^{-1}(B)$ is closed. □

To state our last result, we point out a class S of spaces introduced in [7], which contains all completely normal, fully normal, and Hausdorff spaces.

Theorem 4.11. *Let (Y,M) be a Hausdorff space. Then (Y,M) is an rc-compact space if and only if for any $(X,T) \in S$, each function $f : (X,T) \to (Y,M)$, that has an rc-strongly closed graph, is rc-continuous.*

Proof

Necessity. Already proved in Theorem 4.9.

Sufficiency. Suppose, to the contrary, that (Y,M) is not rc-compact. Then by Proposition 2.6, there exists a net $\varphi : \Lambda \to Y$ that does not rc-accumulate to any point $y \in Y$. Let $t \notin \Lambda$ and put $X = \Lambda \bigcup \{t\}$. We provide X with a topology T that makes each x an isolated point while a basic open neighborhood of t is of the form $\{t\} \bigcup T_\lambda$, where T_λ is a tail of Λ determined by $\lambda \in \Lambda$. It is well known [7] that the space (X,T) is Hausdorff, fully normal, and completely normal, and so $(X,T) \in S$. Fix a point $y_0 \in Y$ and define a function $f : (X,T) \to (Y,M)$ by the formula

$$f(x) = \begin{cases} \varphi(x), & \text{if } x \in \Lambda, \\ y_0, & \text{if } x = t. \end{cases} \tag{4.1}$$

We first show that f has an rc-strongly closed graph. So let $(x,y) \in X \times Y$ and $(x,y) \notin G(f)$. Consider the case when $x \neq t$. Since $f(x) \neq y$ and Y is a

Hausdorff space, there exists $V \in M$ such that $y \in V$ and $f(x) \notin \mathrm{Cl}_M(V)$. Then $(\{x\} \times \mathrm{Cl}_M(V)) \bigcap G(f) = \emptyset$.

Next, we consider the case when $x = t$. This means that $f(x) = y_0 \neq y$. Again, since (Y, M) is a Hausdorff space, there exists $V \in M$ such that $y \in V$ and $y_0 \notin \mathrm{Cl}_M(V)$. Moreover, the net $\varphi : \Lambda \to Y$ does not rc-accumulate to y. So, there exists $W \in M$ and $\lambda \in \Lambda$ such that $y \in \mathrm{Cl}_M(W)$ and $f(T_\lambda) \bigcap \mathrm{Cl}_M(W) = \emptyset$ for a tail T_λ of Λ. Take $F = \mathrm{Cl}_M(V \bigcap \mathrm{Cl}_M(W))$. Then $y \in F$, $y_0 \notin F$, and F is regular closed as it is easy to see that $F = \mathrm{Cl}_M(V \bigcap \mathrm{Cl}_M(W)) = \mathrm{Cl}_M(V \bigcap W)$. Moreover, $f(T_\lambda) \bigcap F = \emptyset$. Thus, $(T_\lambda \bigcup \{t\} \times F) \bigcap G(f) = \emptyset$. This completes the proof that $G(f)$ is rc-strongly closed. □

Finally, by appealing to Proposition 3.5, we show that f is not rc-continuous. We point here to the fact that the net $(\lambda)_\lambda$ converges in (X, T) to the point t. However, the net $(\lambda)_{\lambda \in \Lambda} = (f(\lambda))_{\lambda \in \Lambda}$ does not rc-accumulate to y_0, which means that it is not rc-convergent to $y_0 = f(t)$. Thus, f is not rc-continuous. The established situation contradicts our hypothesis. Thus, (Y, M) is rc-compact.

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