IJMMS 2004:14, 703–719
PII. S0161171204307052
http://ijmms.hindawi.com

T-NEIGHBORHOOD GROUPS

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Received 8 July 2003

We generalize min-neighborhood groups to arbitrary T-neighborhood groups, where T is a continuous triangular norm. In particular, we point out that our results accommodate the previous theory on min-neighborhood groups due to T. M. G. Ahsanullah. We show that every T-neighborhood group is T-uniformizable, therefore, T-completely regular. We also present several results of T-neighborhood groups in conjunction with TI-groups due to J. N. Mordeson.

2000 Mathematics Subject Classification: 54A40, 54H11.

1. Introduction. Menger in [19] introduces an important class of T-uniformities (T being a continuous t-norm) that is generated by a probabilistic metric [21]. Motivated by the Menger's T-uniformities, Höhle [13] brought into light in his celebrated article the idea of probabilistic metrization of fuzzy uniformities. While developing his theory, he showed that a fuzzy T-uniformity is probabilistic metrizable if and only if it is Hausdorff-separated and has a countable base. He also pointed out that when $T = \min$ is considered, his fuzzy T-uniformity reduces to min-fuzzy uniformity of R. Lowen—a fuzzy uniformity widely used over the years. One of the interesting features of this min-fuzzy uniformity [16] is that it gives rise to a fuzzy neighborhood space [17]; an interesting and very well-behaved class of fuzzy topological spaces [15], used by many authors in a wide variety of ways. Among the prominent classes of so called fuzzy neighborhood spaces are, for instance, Katsaras linear fuzzy neighborhood spaces [14], fuzzy metric neighborhood spaces [18], fuzzy neighborhood groups, rings, modules, algebras, and commutative division rings [1, 2, 3, 4, 5].

Very recently, following the famous articles of Menger [19], Höhle [13], Frank [9], Hashem and Morsi [10, 11, 12] introduced a class of fuzzy topological spaces [15] as they put it: fuzzy T-neighborhood spaces herein called T-neighborhood spaces—a natural generalization of min-fuzzy neighborhood spaces of Lowen [17]. Our main target here in this article is to generalize the notion of min-fuzzy neighborhood groups introduced in [2]. We show that every T-neighborhood group is a T-uniform space, and therefore, a T-complete regular space in the sense of Hashem and Morsi [12]. We also generalize the two important characterization theorems, which give necessary and sufficient condition for a T-neighborhood system and a group structure to be compatible, and a prefilter to be a T-neighborhood prefilter.

As an application, we present some results on T-neighborhood groups in conjunction with Mordeson's TI-groups [7], which we believe will open the opportunities to look into further work on fuzzy algebraic structures in connection with the T-neighborhood groups.

2. Preliminaries. Let T be a continuous two-place function (known as continuous triangular norm or t-norm) mapping from the closed unit square to the closed unit interval satisfying certain conditions. In other words, $T : I \times I \to I$, $(\alpha,\beta) \mapsto \alpha T\beta$, satisfying the following conditions:

(Ta) $0T0 = 0$, $\alpha T1 = \alpha$ for all $\alpha \in I$;
(Tb) $\alpha T\beta = \beta T\alpha$ for all $(\alpha,\beta) \in I \times I$;
(Tc) if $\alpha \le \beta$ and $\gamma \le \delta$, then $\alpha T\gamma \le \beta T\delta$ for all $\alpha,\beta,\gamma,\delta \in I$;
(Td) $(\alpha T\beta)T\gamma = \alpha T(\beta T\gamma)$ for all $\alpha,\beta,\gamma \in I$.

Definition 2.1 [6, 16]. A nonempty subset $\mathcal{B} \subset I^G$ is called a prefilterbase if and only if the following conditions are true:

(PB1) $0 \notin \mathcal{B}$;
(PB2) for all $\nu_1,\nu_2 \in \mathcal{B}$, there exists $\nu \in \mathcal{B}$ such that $\nu \le \nu_1 \wedge \nu_2$.

If $\mathcal{B}$ is a prefilterbase in I^G, then by its saturation we understand the following collection:

$$\mathcal{B}^{\sim} = \{\nu : G \longrightarrow I;\ \forall \epsilon > 0\ \exists \nu_\epsilon \in \mathcal{B} \ni \nu_\epsilon - \epsilon \le \nu\}. \tag{2.1}$$

Definition 2.2 [10, 11, 12]. A T-neighborhood space is an I-topological space [15] $(G,^-)$ whose closure operator "$-$" is induced by some indexed family $\Omega = (\Omega(x))_{x\in G}$ of prefilterbases in I^G defined by

$$\bar{\xi}(x) = \inf_{\nu\in\Omega(x)} \sup_{z\in G} \xi(z)T\nu(z) \quad \forall \xi \in I^G,\ x \in G. \tag{2.2}$$

Theorem 2.3 [10, 11, 12]. *A family $\Omega = (\Omega(x))_{x\in G}$ of prefilterbases in I^G is a T-neighborhood base in G if and only if it satisfies the following two properties for all $x \in G$:*

(TB1) *for all $\nu \in \Omega(x)$, $\nu(x) = 1$;*
(TB2) *for all $\nu \in \Omega(x)$, there exists a family $(\nu_{y\epsilon} \in \Omega(y))_{(y,\epsilon)\in G\times I_0}$ which satisfies for all $(y,\epsilon) \in G \times I_0$,*

$$\sup_{z\in G}\left[\nu_{x,\epsilon}(z)T\nu_{z,\epsilon}(y)\right] \le \nu(y) + \epsilon. \tag{2.3}$$

The family Ω is said to be a T-neighborhood basis for $(G,^-)$, and every $\nu \in \Omega(x)$ is called T-neighborhood at x. This I-topology is denoted by $t^T(\Omega)$. However, from now on we will be calling the triple $(G,^-,t(\Omega))$ the T-neighborhood space with Ω a T-neighborhood base in G.

Theorem 2.4 [10, 11, 12]. *Let $(G_1,^-,t(\Omega_1))$ and $(G_2,^-,t(\Omega_2))$ be T-neighborhood spaces with T-neighborhood bases Ω_1 and Ω_2, respectively. Then a function $f : G_1 \to G_2$ is continuous at $x \in G_1$*

$$\begin{aligned}
&\iff \forall \mu \in \Omega_2(f(x)),\ f^{-1}(\mu) \in \Omega_1(x)^{\sim}, \\
&\iff \forall \mu_2 \in \Omega_2(f(x)) \quad \forall \epsilon > 0\ \exists \mu_1 \in \Omega_1(x) \ni \mu_1 - \epsilon \le f^{-1}(\mu_2), \\
&\iff \overline{[f^{-1}(\sigma)]}(x) \le [f^{-1}(\bar{\sigma})](x) \quad \forall \sigma \in I^{G_2}.
\end{aligned} \tag{2.4}$$

If $\Lambda,\Gamma \in I^{G\times G}$ and $\nu \in I^G$, then T-section of Λ over ν is given by

$$\Lambda\langle\nu\rangle_T(x) = \sup_{y\in G} \nu(y) T \Lambda(y,x) \quad \forall x \in G. \tag{2.5}$$

The T-composition of Λ and Γ is defined as

$$\Lambda \circ_T \Gamma(x,y) = \sup_{z\in G} \left[\Gamma(x,z) T \Lambda(z,y)\right] \quad \forall (x,y) \in G\times G. \tag{2.6}$$

Γ is called symmetric if $\Gamma^s = \Gamma$, that is, $\Gamma(y,x) = \Gamma(x,y)$, for all $(x,y) \in G\times G$.

Definition 2.5 [10, 11, 12]. A subset $\mathcal{B} \subset I^{G\times G}$ is called a T-uniform base on a set G if and only if the following properties are fulfilled:
(TUB1) $\mathcal{B}$ is a prefilterbase;
(TUB2) for all $x \in G$, for all $\nu \in \mathcal{B}, \nu(x,x) = 1$;
(TUB3) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon \circ_T \beta_\epsilon - \epsilon \le \beta$;
(TUB4) for all $\beta \in \mathcal{B}$, for all $\epsilon > 0$, there exists $\beta_\epsilon \in \mathcal{B}$ such that $\beta_\epsilon - \epsilon \le \beta$.

The collection $\mathcal{B}$ of fuzzy subsets of $G\times G$ is called T-quasi-uniform base on a set G if and only if it fulfills the preceding conditions (TUB1), (TUB2), and (TUB3), while $\mathcal{U}$ is called T-quasi-uniformity if and only if $\tilde{\mathcal{B}} = \mathcal{U}$. A T-uniformity $\mathcal{U}$ is a saturated T-uniform base $\mathcal{B}$.

Theorem 2.6 [10, 11, 12]. *If $\mathcal{B}$ is a T-quasi-uniform base on a set G, then for all $x \in G$, the family*

$$\Sigma(x) = \{\beta\langle 1_x\rangle \mid \beta \in \mathcal{B}^\sim\} = \{\beta\langle 1_x\rangle \mid \beta \in \mathcal{B}\}^\sim \tag{2.7}$$

is a T-neighborhood system on G.

Proposition 2.7 [10, 11, 12]. *Let $(G,\mathcal{U})$ be a T-quasi-uniform space. Then the closure of the T-neighborhood space $(G,t(\mathcal{U}))$ is given by*

$$\bar{\mu} = \inf_{\sigma\in\mathcal{U}} \sigma\langle\mu\rangle_T \quad \forall \mu \in I^G. \tag{2.8}$$

Theorem 2.8 [10, 11, 12]. *If (G,τ) is a topological space and $\mathcal{V}_\tau = (\mathcal{V}_\tau(x))_{x\in G}$ is its associated neighborhood system in G, then $(G,^-,t(\Omega_\tau))$, a generated topological space, generated by τ, is a T-neighborhood space with a T-neighborhood basis $\Omega = (\Omega(x))_{x\in G}$, where for all $x \in G$,*

$$\begin{aligned} \Omega_1 &:= \Omega_1(x) = \{1_M : G \longrightarrow I;\ M \in \mathcal{V}_\tau(x)\} \subset I^G; \\ \Omega_2 &:= \Omega_2(x) = \{1_M : G \longrightarrow I;\ x \in M \in \tau\} \subset I^G; \\ \Omega_3 &:= \Omega_3(x) = \{\nu : G \longrightarrow I;\ \nu \text{ is l.s.c. in } x \text{ and } \nu(x) = 1\} \subset I^G. \end{aligned} \tag{2.9}$$

Just for the sake of convenience, we provide the proof of the following proposition.

Proposition 2.9. *A function $f : (G,\mathcal{V}_\tau) \to (G',\mathcal{V}'_{\tau'})$ between two topological spaces is continuous at a point $x \in G$ if and only if $f : (G,t(\Omega_\tau)) \to (G',t(\Omega'_{\tau'}))$ is continuous at $x \in G$ between two generated T-neighborhood spaces.*

Proof. Let $f : (G, \mathcal{V}_\tau) \to (G, \mathcal{V}_{\tau'})$ be continuous at $x \in G$ and $\mu' \in \Omega'_{\tau'}(f(x))$; in view of Theorem 2.4, we show that $f^{-1}(\mu') \in \widetilde{\Omega_\tau(x)}$.

Choose $M' \in \mathcal{V}'_\tau(f(x))$ such that $\mu' = 1_{M'}$. This implies that there exists an $M \in \mathcal{V}_\tau(x)$ such that $f(M) \subset M'$, and hence for all $\epsilon > 0$,

$$1_M(x) - \epsilon = 1 - \epsilon \leq 1_{f^{-1}(M')}(x) = f^{-1}(\mu'). \tag{2.10}$$

With $\mu = 1_M$, one obtains $\mu - \epsilon \leq f^{-1}(\mu')$ implying that $f^{-1}(\mu') \in \widetilde{\Omega_\tau(x)}$.

Conversely, we show that the function $f : (G, \mathcal{V}_\tau) \to (G', \mathcal{V}'_{\tau'})$ is continuous at $x \in G$. If $U \in \mathcal{V}'_{\tau'}(f(x))$, then $1_U \in \Omega'_{\tau'}(f(x))$ implies $f^{-1}(1_U) \in \widetilde{\Omega_\tau(x)}$ by continuity of f between the generated spaces.

Thus, for all $\epsilon > 0$, there is a $\mu = \mu_\epsilon \in \Omega_\tau(x)$ such that

$$\mu - \epsilon \leq f^{-1}(1_U). \tag{2.11}$$

This implies that for all $\epsilon > 0$, there exists a $V_\epsilon \in \mathcal{V}_\tau(x)$ such that $1_{V_\epsilon} = \mu = \mu_\epsilon$ and

$$1_{V_\epsilon} - \epsilon \leq 1_{f^{-1}(U)}. \tag{2.12}$$

Now $V_\epsilon \in \mathcal{V}_\tau(x)$ implies $x \in V_\epsilon$ if and only if $1_{V_\epsilon}(x) = 1$. Therefore,

$$\begin{aligned} 0 < 1 - \epsilon = 1_{V_\epsilon}(x) - \epsilon \leq 1_{f^{-1}(U)}(x) \Longrightarrow 1_{f^{-1}(U)}(x) > 0 \Longrightarrow 1_{f^{-1}(U)}(x) \\ = 1 \Longleftrightarrow x \in f^{-1}(U). \end{aligned} \tag{2.13}$$

This means that $V_\epsilon \subseteq f^{-1}(U)$ implies $f^{-1}(U) \in \mathcal{V}_\tau(x)$. That is, f is continuous at $x \in G$. □

Theorem 2.10 [11]. *Let $(G, ^-)$ be an I-topological space. Then $(G, ^-)$ is a T-neighborhood space if and only if $\overline{\alpha T \mu} = \alpha T \bar{\mu}$ for all $\mu \in I^G$ and for all $\alpha \in I$.*

Definition 2.11 [12]. An I-topological space (X, τ) is called T-completely regular if τ is the initial I-topology for the family of all continuous functions from (X, τ) to $(\mathcal{D}^+, t^T(\mathcal{F}_{\mathcal{H}}))$.

Here, $\mathcal{D}^+$ stands for the collection of all distance distribution functions from $\Re^+$ to the unit interval I, and the pair $(\mathcal{D}^+, t^T(\mathcal{F}_{\mathcal{H}}))$ is the T-neighborhood space induced by the well-known Höhle's probabilistic T-metric $\mathcal{F}_{\mathcal{H}}$. For details, we refer to [12, 13].

Theorem 2.12 [12]. *T-complete regularity is equivalent to T-uniformizability.*

3. Some results on T-neighborhood spaces

Theorem 3.1. *Let $(G_1, ^-, t(\Omega_1))$ and $(G_2, ^-, t(\Omega_2))$ be two T-neighborhood spaces with bases $\Omega_1 = (\Omega_1(x))_{x \in G_1}$ and $\Omega_2 = (\Omega_2(x))_{x \in G_2}$ in G_1 and G_2, respectively. Then their T-product $(G_1 \times G_2, -^{\otimes_T}, t(\Omega_1) \otimes_T t(\Omega_2))$ is the T-neighborhood space with base $\Omega = \Omega_1 \otimes_T \Omega_2$ defined by*

$$\Omega(x, y) = \{\nu_1 \otimes_T \nu_2 \mid \nu_1 \in \Omega_1(x),\ \nu_2 \in \Omega_2(y)\}, \tag{3.1}$$

where $\nu_1 \otimes_T \nu_2$ *is given as*

$$\nu_1 \otimes_T \nu_2(x,y) = \nu_1(x) T \nu_2(y) \quad \forall (x,y) \in G_1 \times G_2. \tag{3.2}$$

Moreover,

$$\overline{\nu_1 \otimes_T \nu_2} = \overline{\nu_1} \otimes_T \overline{\nu_2} \quad \forall \nu_1 \in I^{G_1},\ \nu_2 \in I^{G_2}. \tag{3.3}$$

Conversely, if

$$\overline{\nu_1 \otimes_T \nu_2} = \overline{\nu_1} \otimes_T \overline{\nu_2} \quad \forall \nu_1 \in I^{G_1},\ \nu_2 \in I^{G_2}, \tag{3.4}$$

then both the I*-topological spaces* $(G_1, ^-)$ *and* $(G_2, ^-)$ *are* T*-neighborhood spaces.*

Proof. First we show that for all $(x,y) \in G_1 \times G_2$, $\Omega(x,y)$ is a prefilterbase.

(PB1) Obviously, $\Omega \neq \emptyset$ and $0 \notin \Omega$.

(PB2) Let $\xi_1, \xi_2 \in \Omega(x,y)$, then there are $\nu_1, \nu_2 \in \Omega_1(x)$ and $\mu_1, \mu_2 \in \Omega_2(y)$ such that $\xi_1 = \nu_1 \otimes_T \mu_1$ and $\xi_2 = \nu_2 \otimes_T \mu_2$.

Now, $\xi_1 \wedge \xi_2 = (\nu_1 \otimes_T \mu_1) \wedge (\nu_2 \otimes_T \mu_2)$. For any $(x,y) \in G_1 \times G_2$,

$$\begin{aligned} \xi_1(x,y) \wedge \xi_2(x,y) &= (\nu_1 \otimes_T \mu_1)(x,y) \wedge (\nu_2 \otimes_T \mu_2)(x,y) \\ &= (\nu_1(x) T \mu_1(y)) \wedge (\nu_2(x) T \mu_2(y)) \\ &\geq (\nu_1(x) \wedge \nu_2(x)) T (\nu_1(y) \wedge \mu_2(y)) \\ &= (\nu_1 \wedge \nu_2)(x) T (\mu_1 \wedge \mu_2)(y). \end{aligned} \tag{3.5}$$

Therefore, $\xi_1 \wedge \xi_2(x,y) \geq \nu(x) \wedge \mu(y)$ for some $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(y)$, since both $\Omega_1(x)$ and $\Omega_2(y)$ are prefilterbases in G_1 and G_2, respectively.

This implies that $\xi_1 \wedge \xi_2(x,y) \geq \nu \otimes_T \mu(x,y) = \xi(x,y)$ and $\xi \in \Omega(x,y)$, and hence $\xi \leq \xi_1 \wedge \xi_2$, proving that $\Omega(x,y)$ is a prefilterbase in $G_1 \times G_2$.

Now we prove the conditions of Theorem 2.3.

(TB1) If $x \in G$ and $\xi \in \Omega(x,x)$, then for some $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(x)$, we have $\xi(x,x) = \nu \otimes_T \mu(x,x)$, $\nu(x) T \mu(x) = 1T1 = 1$.

(TB2) Let $(x,y) \in G_1 \times G_2$, $\xi \in \Omega(x,y)$, and $\epsilon \in I_0$. Then there exists $\nu \in \Omega_1(x)$ and $\mu \in \Omega_2(y)$ such that $\xi = \nu \otimes_T \mu$.

Consequently, there is a family $(\nu_{y_1\epsilon} \in \Omega_1(y_1))_{(y_1,\epsilon) \in G_1 \times I_0}$, a T-kernel for ν which satisfies for all $(y_1, \epsilon) \in G_1 \times I_0$,

$$\sup_{z_1 \in G_1} [\nu_{x,\epsilon}(z_1) T \nu_{z_1,\epsilon}(y_1)] \leq \nu(y_1) + \epsilon. \tag{3.6}$$

Also, there is a family $(\mu_{y_2\epsilon} \in \Omega_2(y_2))_{(y_2,\epsilon) \in G_2 \times I_0}$, a T-kernel for μ which satisfies for all $(y_2, \epsilon) \in G_2 \times I_0$,

$$\sup_{z_2 \in G_2} [\mu_{y,\epsilon}(z_2) T \mu_{z_2,\epsilon}(y_2)] \leq \mu(y_2) + \epsilon. \tag{3.7}$$

Now for all $(y_1, y_2) \in G_1 \times G_2$,

$$(\nu \otimes_T \mu)(y_1, y_2) + \delta = [\nu(y_1) T \mu(y_2)] + \delta \geq (\nu(y_1) + \epsilon) T (\mu(y_2) + \epsilon) \tag{3.8}$$

with $\epsilon = \epsilon_{T,\delta} > 0$.

This yields that

$$\begin{aligned}
&(\nu \otimes_T \mu)(y_1, y_2) + \delta \\
&\quad \geq \sup_{z_1 \in G_1} [\nu_{x,\epsilon}(z_1) T \nu_{z_1,\epsilon}(y_1)] T \sup_{z_2 \in G_2} [\mu_{y,\epsilon}(z_2) T \mu_{z_2,\epsilon}(y_2)] \\
&\quad = \sup_{(z_1,z_2) \in G_1 \times G_2} [(\nu_{x,\epsilon}(z_1) T \mu_{y,\epsilon}(z_2)) T (\nu_{z_1,\epsilon}(y_1) T \mu_{z_2,\epsilon}(y_2))] \\
&\quad = \sup_{(z_1,z_2) \in G_1 \times G_2} [((\nu_{x,\epsilon} \otimes_T \mu_{y,\epsilon})(z_1, z_2)) T ((\nu_{z_1,\epsilon} \otimes_T \mu_{z_2,\epsilon})(y_1, y_2))] \\
&\quad \Longrightarrow \sup_{(z_1,z_2) \in G_1 \times G_2} [\nu_{x,\epsilon} \otimes_T \mu_{y,\epsilon}(z_1, z_2) T \nu_{z_1,\epsilon} \otimes_T \mu_{z_2,\epsilon}(y_1, y_2)] \\
&\quad \leq \nu \otimes_T \mu(y_1, y_2) + \delta.
\end{aligned} \tag{3.9}$$

In order to prove the final part, we proceed as follows. Let $\nu_1 \in I^{G_1}$, $\nu_2 \in I^{G_2}$, and $(x, y) \in G_1 \times G_2$.

Then in view of Definition 2.2, we have

$$\begin{aligned}
\overline{\nu_1 \otimes_T \nu_2}(x, y) &= \inf_{\xi_1 \in \Omega_1(x)} \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \sup_{z_2 \in G_2} (\nu_1 \otimes_T \nu_2)(z_1, z_2) T (\xi_1 \otimes_T \xi_2)(z_1, z_2) \\
&= \inf_{\xi_1 \in \Omega_1(x)} \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \sup_{z_2 \in G_2} \{\nu_1(z_1) T \nu_2(z_2)\} T \{\xi_1(z_1) T \xi_2(z_2)\} \\
&= \inf_{\xi_1 \in \Omega_1(x)} \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_1 \in G_1} \sup_{z_2 \in G_2} \nu_1(z_1) T \xi_1(z_1) T \nu_2(z_2) T \xi_2(z_2) \\
&= \inf_{\xi_1 \in \Omega_1(x)} \sup_{z_1 \in G_1} \nu_1(z_1) T \xi_1(z_2) T \inf_{\xi_2 \in \Omega_2(y)} \sup_{z_2 \in G_2} \nu_2(z_2) T \xi_2(z_2) \\
&= \overline{\nu_1}(x) T \overline{\nu_2}(y) = \overline{\nu_1} \otimes_T \overline{\nu_2}(x, y).
\end{aligned} \tag{3.10}$$

To prove the converse part, we proceed as follows. Since

$$\overline{\nu_1 \otimes_T \nu_2} = \overline{\nu_1} \otimes_T \overline{\nu_2} \quad \forall \nu_1 \in I^{G_1},\ \nu_2 \in I^{G_2}, \tag{3.11}$$

in view of Theorem 2.10, we have

$$\begin{aligned}
\overline{(\alpha T \nu_1 \otimes_T \nu_2)}(x) &= (\alpha T \overline{\nu_1 \otimes_T \nu_2})(x) \\
&= \alpha(x) T (\overline{\nu_1} \otimes_T \overline{\nu_2})(x) \\
&= \alpha(x) T (\overline{\nu_1}(x) T \overline{\nu_2}(x)).
\end{aligned} \tag{3.12}$$

Since this holds for all x and for all ν_1 and ν_2, with $\nu_2 = 1$, we have

$$\begin{aligned}
\overline{(\alpha T \nu_1 \otimes_T \nu_2)}(x) &= (\alpha T \overline{\nu_1 \otimes_T \nu_2})(x) = \alpha(x) T (\overline{\nu_1}(x) T 1) \\
&= \alpha(x) T \overline{\nu_1}(x) = (\alpha T \overline{\nu_1})(x) = \overline{(\alpha T \nu_1)}(x),
\end{aligned} \tag{3.13}$$

so $(G_1, {}^-)$ is a T-neighborhood space. Similarly, with $\nu_1 = 1$, we see that $(G_2, {}^-)$ is a T-neighborhood space. This completes the proof. □

Proposition 3.2. *Let* $(G_1, {}^-, t(\Omega_1))$ *and* $(G_2, {}^-, t(\Omega_2))$ *be* T*-neighborhood spaces. Then the projections*

$$\begin{aligned} \mathrm{pr}_1 &: (G_1 \times G_2, -^{\otimes_T}, t(\Omega_1) \otimes_T t(\Omega_2)) \longrightarrow (G_1, {}^-, t(\Omega_1)), \qquad (x_1, x_2) \longmapsto x_1, \\ \mathrm{pr}_2 &: (G_1 \times G_2, -^{\otimes_T}, t(\Omega_1) \otimes_T t(\Omega_2)) \longrightarrow (G_2, {}^-, t(\Omega_2)), \qquad (x_1, x_2) \longmapsto x_2, \end{aligned} \tag{3.14}$$

are continuous.

Proof. Let $\nu \in \Omega_1(x_1)$ and $\epsilon > 0$. Then

$$\begin{aligned} \mathrm{pr}_1^{-1}(\nu_1)(x_1, x_2) = \nu_1(\mathrm{pr}_1(x_1, x_2)) = \nu_1(x_1) T 1 \geq \nu_1(x_1) T \nu_2(x_2) \geq \nu_1 \otimes_T \nu_2(x_1, x_2) - \epsilon \\ \Longrightarrow \nu_1 \otimes_T \nu_2 - \epsilon \leq \mathrm{pr}_1^{-1}(\nu_1) \Longrightarrow \mathrm{pr}_1^{-1}(\nu_1) \in \Omega(x_1, x_2)^{\sim}. \end{aligned} \tag{3.15}$$

This implies that $\mathrm{pr}_1 : (G_1 \times G_2, -^{\otimes_T}, t(\Omega_1) \otimes_T t(\Omega_2)) \to (G_1, {}^-, t(\Omega_1))$, $(x_1, x_2) \mapsto x_1$, is continuous, and similarly, one can prove that $\mathrm{pr}_2 : (G_1 \times G_2, -^{\otimes_T}, t(\Omega_1) \otimes_T t(\Omega_2)) \to (G_2, {}^-, t(\Omega_2))$, $(x_1, x_2) \mapsto x_2$, is continuous. □

Definition 3.3. A T-neighborhood space $(G, {}^-, t(\Omega))$ is said to be a TN-regular space if and only if for all $z \in G$, for all $\mu \in \Omega(z)$, and for all $\epsilon > 0$, there exists a $\nu \in \Omega(z)$ closed such that

$$\epsilon + \mu(z) \geq \inf_{\rho \in \Omega(z)} \sup_{t \in G} \nu(t) T \rho(t) \left(= \bar{\nu}(z) \right). \tag{3.16}$$

Theorem 3.4. *Every* T*-quasi-uniform space* (G, Ψ) *is* TN*-regular.*

Proof. Suppose that $z \in G$, $\psi \in \Psi$, and $\epsilon > 0$, and choose $\psi_\epsilon \in \Psi$ such that

$$\psi_\epsilon \circ_T \psi_\epsilon \leq \psi + \epsilon. \tag{3.17}$$

If $t \in G$, then by using Proposition 2.7,

$$\begin{aligned} \overline{\psi_\epsilon \langle z \rangle}^T(t) &= \inf_{\psi'_\epsilon \in \Psi} \sup_{y \in G} \psi_\epsilon \langle z \rangle (y) T \psi'_\epsilon(y, t) \leq \sup_{y \in G} \psi_\epsilon(z, y) T \psi_\epsilon(y, t) \\ &= \psi_\epsilon \circ_T \psi_\epsilon(z, t) \leq \psi(z, t) + \epsilon \\ &= \psi \langle z \rangle (t) + \epsilon. \end{aligned} \tag{3.18}$$

Hence the result follows. □

4. T-neighborhood groups.

In what follows, we consider $(G, \cdot)$ as a multiplicative group with e as the identity element. If $\mu : G \to I$, then $\mu^{-1}(x)$ is defined as $\mu^{-1}(x) = \mu(-x)$, and μ is said to be symmetric if and only if $\mu = \mu^{-1}$.

Definition 4.1. Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with T-neighborhood base Ω on G. Then the quadruple $(G,\cdot,^{-},t(\Omega))$ is called a T- neighborhood group if and only if the following properties are satisfied:

(TG1) the mapping $m:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,^{-},t(\Omega))$, $(x,y)\mapsto xy$, is continuous;

(TG2) the inversion mapping $r:(G,^{-},t(\Omega))\to(G,^{-},t(\Omega))$, $x\mapsto x^{-1}$, is continuous.

A group structure and a T-neighborhood system is said to be compatible if and only if (TG1) and (TG2) are fulfilled.

Remarks 4.2. A T-neighborhood group may not be a fuzzy topological group in the sense of Foster [8] since we have used T-neighborhood topology, which differ from the product fuzzy topology.

Proposition 4.3. *Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with a T-neighborhood base Ω. Then the quadruple $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group if and only if the mapping*

$$h:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\longrightarrow(G,^{-},t(\Omega)),\quad (x,y)\longmapsto xy^{-1}, \tag{4.1}$$

is continuous.

Proof. Observe that the conditions (TG1) and (TG2) are equivalent to the following single condition:

(TG3) the mapping $h:(G\times G,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,^{-},t(\Omega))$, $(x,y)\mapsto xy^{-1}$, is continuous.

In fact, if we let $f(x,y)=(x,y^{-1})$, then by (TG2), f is continuous and hence in conjunction with (TG1), one obtains the continuity of h. On the other hand, (TG3)⇒(TG2) for $x\to ex^{-1}=x^{-1}$ is then continuous; while (TG1) follows from (TG3) and (TG2), because $(x,y)\to x(y^{-1})^{-1}=xy$ is then continuous. □

Proposition 4.4. *Let $(G,\cdot)$ be a group and $(G,^{-},t(\Omega))$ a T-neighborhood space with Ω a T-neighborhood base in G. Then*

(a) *the mapping $m:(G\times G,\cdot,-^{\otimes_T},t(\Omega)\otimes_T t(\Omega))\to(G,\cdot,^{-},t(\Omega))$, $(x,y)\mapsto xy$, is continuous at $(e,e)\in G\times G$ if and only if for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu\le\mu+\epsilon; \tag{4.2}$$

(b) *the inversion mapping $r:(G,\cdot,^{-},t(\Omega))\to(G,\cdot,^{-},t(\Omega))$, $x\mapsto x^{-1}$, is continuous at $e\in G$ if and only if for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists $\nu\in\Omega(e)$ such that*

$$\nu\le\mu^{-1}+\epsilon. \tag{4.3}$$

Proof. (a) In view of Theorem 2.4, continuity at $(e,e)\in G\times G$ is equivalent to

$$\begin{aligned}&\forall\mu\in\Omega(e)\Longrightarrow m^{-1}(\mu)\in(\Omega(e)\otimes_T\Omega(e))^{\sim}\\&\qquad\Longleftrightarrow\forall\mu\in\Omega(e),\quad\forall\epsilon>0,\ \exists\nu=\nu_\epsilon\in\Omega(e)\ni\nu\otimes_T\nu\le m^{-1}(\mu)+\epsilon.\end{aligned}\tag{4.4}$$

But $m(\nu\otimes_T\nu)(z)=\sup_{(x,y)\in m^{-1}(z)}\nu(x)T\nu(y)=\sup_{xy=z}\nu(x)T\nu(y)=\nu\odot_T\nu(z)$. Thus, in this case, continuity at $(e,e)\in G\times G$ is in fact equivalent to

$$\nu\odot_T\nu\le\mu+\epsilon. \tag{4.5}$$

(b) This follows almost in the same way as in (a). □

Corollary 4.5. *If $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group, then the mapping (4.1) is continuous at $(e,e)\in G\times G$ if and only if for all $\mu\in\Omega(e)$ and for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu^{-1}\le\mu+\epsilon. \tag{4.6}$$

Proof. This follows at once from the composition of (a) and (b) in Proposition 4.4. □

Proposition 4.6. *Let $(G,^-,t(\Omega))$ be a T-neighborhood space and $A\subset G$. Then $(A,^-,t(\Omega_{|A}))$ is a T-neighborhood space, a subspace of the T-neighborhood space $(G,^-,t(\Omega))$.*

Proof. The proof follows by easy verification. □

Theorem 4.7. *The triple $(G,\cdot,\tau)$ is a topological group if and only if the quadruple $(G,\cdot,^-,t(\Omega_\tau))$, where Ω is the generated T-neighborhood basis, is a T-neighborhood group.*

Proof. With the help of Proposition 2.9, it follows that the mapping

$$h:(G\times G,\mathcal{V}_\tau\times\mathcal{V}_\tau)\longrightarrow(G,\mathcal{V}_\tau),\quad(x,y)\longmapsto xy^{-1}, \tag{4.7}$$

is continuous if and only if

$$h:(G\times G,t(\Omega_\tau)\otimes_T t(\Omega_\tau))\longrightarrow(G,t(\Omega_\tau)),\quad(x,y)\longmapsto xy^{-1}, \tag{4.8}$$

is continuous, where Ω is the basis for the generated T-neighborhood spaces. □

Lemma 4.8. *Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group and $a\in G$. Then*

(1) *the left translation $\mathcal{L}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $x\mapsto ax$, and the right translation $\mathcal{R}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $x\mapsto ax$, are homeomorphisms;*

(2) *the inner automorphism $\mathcal{I}_a:(G,\cdot,^-,t(\Omega))\to(G,\cdot,^-,t(\Omega))$, $z\mapsto aza^{-1}$, is an isomorphism;*

(3) *$\nu\in\Omega(e)^\sim$ if and only if $\mathcal{L}_a(\nu)\in\Omega(a)^\sim$ if and only if $\mathcal{R}_a(\nu)\in\Omega(a)^\sim$. In other words, if Ω is saturated, then $\nu\in\Omega(e)$ if and only if $1_{\{a\}}\odot_T\nu=a\odot_T\nu\in\Omega(a)$ if and only if $\nu\odot_T a\in\Omega(a)$;*

(4) *$\nu\in\Omega(a)^\sim$ if and only if $\mathcal{L}_{-a}(\nu)\in\Omega(e)^\sim$ if and only if $\mathcal{R}_{-a}(\nu)\in\Omega(e)^\sim$. In other words, if Ω is saturated, then $\nu\in\Omega(a)$ if and only if $1_{\{a^{-1}\}}\odot_T\nu=a^{-1}\odot\nu\in\Omega(e)$ if and only if $\nu\odot_T a^{-1}\in\Omega(e)$;*

(5) *if $\nu\in\Omega(e)$, then $\nu^{-1}\in\Omega(e)$;*

(6) *$\nu\odot_T\nu^{-1}$ is symmetric.*

Proof. (1) follows at once from the definitions while (2) follows from the fact that $\mathcal{I}_a = \mathcal{L}_a \circ \mathcal{R}_{-a} = \mathcal{R}_{-a} \circ \mathcal{L}_a$ for all $a \in G$.

(3) Let $\nu \in \Omega(e)^{\sim} \subset \Omega(e)$, that is, $\nu \in \Omega(a^{-1}a) = \Omega(\mathcal{L}_a^{-1}(a))$. Since $\mathcal{L}_a^{-1}$ is continuous, then in view of Theorem 2.4, $\mathcal{L}_a(\nu) = (\mathcal{L}_a^{-1})^{-1}(\nu) \in \Omega(a)^{\sim}$ implies $\mathcal{L}_a(\nu) \in \Omega(a)^{\sim}$. Conversely, let $\mathcal{L}_a(\nu) \in \Omega(a)^{\sim} \subset \Omega(a)$ implies $\mathcal{L}_a(\nu) \in \Omega(a)$ implies $\mathcal{L}_a(\nu) \in \Omega(a) = \Omega(ae) = \Omega(\mathcal{L}_a(e))$, and since $\mathcal{L}_a : G \to G$ is continuous injection again by Theorem 2.4, $\nu = \mathcal{L}_a^{-1}(\mathcal{L}_a(\nu)) \in \Omega(e)^{\sim}$. For the calculations of the other part, see [7, Theorem 5.1.1].

(4) follows from (3) while (5) follows from the fact that the inversion mapping $r : G \to G$, $x \mapsto x^{-1}$ is a homeomorphism.

(6) We have $\nu \odot_T \nu^{-1} = (\nu \odot_T \nu^{-1})^{-1}$. If $x \in G$, then

$$\begin{aligned}
(\nu \odot_T \nu^{-1})^{-1}(x) = (\nu \odot_T \nu^{-1})(x^{-1}) &= \sup_{ab=x^{-1}} \nu(a) T \nu(b^{-1}) \\
&= \sup_{st^{-1}=x^{-1}} \nu(s) T \nu(t) = \sup_{ts^{-1}=x} \nu(t) T \nu(s) \\
&= \sup_{ts^{-1}=x} \nu(t) T \nu\Big(\big(s^{-1}\big)^{-1}\Big) \\
&= \sup_{ts^{-1}=x} \nu(t) T \nu^{-1}(s^{-1}) \\
&= \nu \odot_T \nu^{-1}(x).
\end{aligned} \tag{4.9}$$

This completes the proof. $\square$

Definition 4.9. A T-neighborhood space $(G,^{-},t(\Omega))$ is called homogeneous space if and only if for all $(a,b) \in G \times G$, there exists a homeomorphism $f : (G,^{-},t(\Omega)) \to (G,^{-},t(\Omega))$ such that $f(a) = b$.

Theorem 4.10. *Every T-neighborhood group is a homogeneous space.*

Proof. This follows from the fact that for all $a,b \in G \times G$, the function

$$\mathcal{R}_{a^{-1}b} : G \longrightarrow G, \quad x \longmapsto xa^{-1}b, \tag{4.10}$$

is a homeomorphism. $\square$

Lemma 4.11. *Let $(G,\cdot)$ be a group, and let, for all $\mu \in I^G$, $\mu_L : G \times G \to I$, $(x,y) \mapsto \mu_L(x,y) = \mu(x^{-1}y)$ (resp., $\mu_R : G \times G \to I$, $(x,y) \mapsto \mu_R(x,y) = \mu(yx^{-1})$) be the vicinities L-associated (resp., R-associated) with μ.*

Then for all $\mu,\theta,\nu \in I^G$, $(x,y) \in G \times G$, and triangular norm $T : I \times I \to I$, the following hold:

(1) $\mu_L\langle\theta\rangle_T = \theta \odot_T \mu$ *(resp., $\mu_R\langle\theta\rangle_T = \mu \odot_T \theta$);*
(2) $\mu_L T \nu_L = (\mu T \nu)_L$ *(resp., $\mu_R T \nu_R = (\mu T \nu)_R$);*
(3) $(\mu_L^s) = (\mu_L)^s$;
(4) $\mu_L \odot_T \nu_L = (\nu \odot_T \mu)_L$ *(resp., $\mu_R \odot_T \nu_R = (\nu \odot_T \mu)_R$).*

Proof. (1) For all $(x,\theta,\mu) \in G \times I^G \times I^G$,

$$\begin{aligned}
\mu_L\langle\theta\rangle_T(x) &= \sup_{y\in G}\big[\theta(y) T \mu_L(y,x)\big] = \sup_{y\in G}\big[\theta(y) T \mu(y^{-1}x)\big] \\
&= \theta \odot_T \mu(x) \quad (\text{by [7, Theorem 5.1.1]}).
\end{aligned} \tag{4.11}$$

(2) and (3) are obvious.

(4) For all $(x,y) \in G\times G$,

$$\begin{aligned} \mu_L \odot_T \nu_L(x,y) &= \sup_{z\in G}\left[\nu_L(x,z)T\mu_L(z,y)\right] = \sup_{z\in G}\left[\nu(x^{-1}z)T\mu(z^{-1}y)\right] \\ &= \sup_{st=x^{-1}zz^{-1}y=x^{-1}y}\left[\nu(s)T\mu(t)\right] = (\nu\odot_T\mu)(x^{-1}y) \\ &= (\nu\odot_T\mu)_L(x,y). \end{aligned} \tag{4.12}$$

□

Theorem 4.12. *Every T-neighborhood group is a T-uniform space.*

Proof. If $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group, then $(G,^-,t(\Omega))$ is a T-neighborhood space with the T-neighborhood basis Ω.

We consider the following collection:

$$\Omega = \{\mu_L \mid \mu\in\Omega(e)\} \subset I^{G\times G}. \tag{4.13}$$

We claim that Ω is a T-uniform basis.

(TUB1) Clearly Ω is a prefilterbasis.

(TUB2) If $\psi\in\Omega$, then there exists a $\mu\in\Omega(e)$ such that $\psi=\mu_L$, and for all $x\in G$,

$$\psi(x,x)=\mu_L(x,x)=\mu(e)=1. \tag{4.14}$$

(TUB3) If $\psi\in\Omega$, then there exists a $\mu\in\Omega(e)$ such that $\psi=\mu_L$.

Thus, by virtue of Proposition 4.4(a), for all $\epsilon>0$, we can find $\nu^\epsilon\in\Omega(e)$ such that

$$\nu^\epsilon\odot_T\nu^\epsilon-\epsilon\le\mu. \tag{4.15}$$

If we let $\nu_L^\epsilon=\psi_\epsilon$, then one obtains

$$\begin{gathered} \psi_\epsilon\odot_T\psi_\epsilon-\epsilon=\nu_L^\epsilon\odot_T\nu_L^\epsilon-\epsilon=\left(\nu^\epsilon\odot_T\nu^\epsilon\right)_L-\epsilon\le\mu_L \\ \Longrightarrow \psi_\epsilon\odot_T\psi_\epsilon-\epsilon\le\psi. \end{gathered} \tag{4.16}$$

(TUB4) If $\psi\in\Omega$, then there is a $\mu\in\Omega(e)$ such that $\psi=\mu_L$. Consequently, by Proposition 4.4(b), for all $\epsilon>0$, there exists a $\nu^\epsilon\in\Omega(e)$ such that

$$\nu^\epsilon-\epsilon\le\mu^{-1}. \tag{4.17}$$

Therefore, $\nu_L^\epsilon-\epsilon\le(\mu^{-1})_L=(\mu_L)^{-1}$ implies $\psi_\epsilon-\epsilon\le\psi_s$.

This shows in accordance with Definition 2.5 that Ω is a T-uniform basis, which in turn gives rise to a left T-uniformity $\mathcal{U}_L=\Omega^\sim$.

In fact, we have for all $x\in G$,

$$\mathcal{U}_L(x)=\{\mu_L\langle 1_x\rangle \mid \mu\in\Omega(e)\}^\sim=\{\mathcal{L}_x(\mu)\mid\mu\in\Omega(e)^\sim\}=\Omega(x)^\sim, \tag{4.18}$$

which is a T-neighborhood system on G and that $(G,t(\Omega)=t(\mathcal{U}_L))$ is a T-uniform space. Similarly, one can obtain right T-uniformity. □

Theorem 4.13. *Every T-neighborhood group is T-completely regular.*

Proof. This follows from the preceding theorem in conjunction with Theorem 2.12 because every T-neighborhood group is T-uniformizable and every T-uniformizable space is T-completely regular. □

Theorem 4.14. *Let $(G,\cdot)$ be a group, $(G,^-,t(\Omega))$ a T-neighborhood space with T-neighborhood base Ω in G. Then the quadruple $(G,\cdot,^-,t(\Omega))$ is a T-neighborhood group if and only if the following are true:*

(1) *for all $a\in G, \Omega(a)^{\sim}=\{\mathcal{L}_a(\mu)\mid \mu\in\Omega(e)\}^{\sim}$ (resp., for all $a\in G, \Omega(a)^{\sim}=\{\mathcal{R}_a(\mu)\mid \mu\in\Omega(e)\}^{\sim}$);*

(2) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\odot_T\nu\le\mu+\epsilon, \tag{4.19}$$

that is, the mapping $m:(x,y)\mapsto xy$ is continuous at $(e,e)\in G\times G$;

(3) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, there exists a $\nu\in\Omega(e)$ such that*

$$\nu\le\mu^{-1}+\epsilon, \tag{4.20}$$

that is, the mapping $r:x\mapsto x^{-1}$ is continuous at $e\in G$;

(4) *for all $\mu\in\Omega(e)$, for all $\epsilon>0$, and for all $a\in G$ such that*

$$a\odot_T\nu\odot_T a^{-1}\le\mu+\epsilon, \tag{4.21}$$

that is, the mapping $\mathcal{I}_a:x\mapsto axa^{-1}$ is continuous at $e\in G$.

Proof. Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group. Then the conditions (1), (2), (3), and (4) are clearly true.

To prove the converse part, we remark that from Corollary 4.5, it follows that the mapping $h:G\times G\to G$; $(x,y)\mapsto xy^{-1}$ is continuous at (e,e), and since the translations $\mathcal{L}_a$ and $\mathcal{R}_a$ are continuous at a and e, respectively, the continuity of m follows from the following chain:

$$G\times G\xrightarrow{\mathcal{L}_{a^{-1}}\times\mathcal{L}_{b^{-1}}}G\times G\xrightarrow{m}G\xrightarrow{\mathcal{I}_b}G\xrightarrow{\mathcal{L}_{ab^{-1}}}G, \tag{4.22}$$

where $(a,b)\to(e,e)\to e\to e\to ab^{-1}$. □

Theorem 4.15. *Let $(G,\cdot)$ be a group and $\mathcal{F}$ a collection of nonempty subsets of I^G, that is, $\varnothing\ne\mathcal{F}\subset I^G$ such that*

(1) *$\mathcal{F}$ is a prefilterbasis and $\mu(e)=1$ for all $\mu\in\mathcal{F}$;*

(2) *for all $\mu\in\mathcal{F}$, for all $\epsilon>0$, there exists a $\nu\in\mathcal{F}$ such that $\nu-\epsilon\le\mu^{-1}$;*

(3) *for all $\mu\in\mathcal{F}$, for all $\epsilon>0$, there exists a $\nu\in\mathcal{F}$ such that $\nu\odot_T\nu-\epsilon\le\mu$;*

(4) *for all $\mu\in\mathcal{F}$, for all $a\in G$, for all $\epsilon>0$, there exists $\nu\in\mathcal{F}$ such that $a\odot_T\nu\odot_T a^{-1}-\epsilon\le\mu$.*

Then there exists a unique T-neighborhood system compatible with the group structure of G such that $\mathcal{F}$ is a T-neighborhood basis at $e\in G$.

Proof. For all $\mu\in\mathcal{F}$, let $\mu_L:G\times G\to I$ be the vicinities L-associated with μ. Evidently, $\mu_L(a,a)=\mu(a^{-1}a)=\mu(e)=1$.

We let

$$\mathcal{B} = \{\mu_L \mid \mu \in \mathcal{F}\} \subset I^{G\times G}. \tag{4.23}$$

We show that $\mathcal{B}$ is a T-quasi-uniform basis for a T-quasi-uniformity. We verify Definition 2.5 upto (TUB3).

(TUB1) $\mathcal{B}$ is a prefilterbasis; for $0 \notin \mathcal{B}$ which is clearly true, since $\mathcal{F}$ is a prefilterbasis. Next, let $\lambda, \xi \in \mathcal{B}$, then $\lambda = \mu_L$ for some $\mu \in \mathcal{F}$ and $\xi = \eta_L$ for some $\eta \in \mathcal{F}$. Since $\mathcal{F}$ is a prefilterbasis, there exists a $\theta \in \mathcal{F}$ such that $\theta \leq \mu \wedge \eta$ and $\theta_L \leq \mu_L \wedge \eta_L = \lambda \wedge \xi$, proving that $\mathcal{B}$ is indeed a prefilterbasis.

(TUB2) For all $x \in G$, and $\psi \in \mathcal{B}$, we have $\psi = \mu_L$ for some $\mu \in \mathcal{F}$ and $\psi(x,x) = \mu_L(x,x) = \mu(e) = 1$ by (1).

(TUB3) Let $\psi \in \mathcal{B}$. Then there exists a $\mu \in \mathcal{F}$ such that $\psi = \mu_L$.

Now by (3), for all $\epsilon > 0$, we can find a $\nu \in \mathcal{F}$ such that

$$\nu \odot_T \nu - \epsilon \leq \mu. \tag{4.24}$$

But then by virtue of Lemma 4.11(4), we get $\nu_L \odot_T \nu_L - \epsilon \leq \mu_L$. So, if we put $\psi_\epsilon = \nu_L$, then

$$\psi_\epsilon \odot_T \psi_\epsilon - \epsilon \leq \psi. \tag{4.25}$$

This completes the proof that $\mathcal{B}$ is a T-quasi-uniform basis which in turn gives rise to a T-quasi-uniformity and hence a T-quasi-uniform space. Then in view of the Theorem 2.6, since every T-quasi-uniform space is a T-neighborhood space, in this case, we have the T-neighborhood system as given by the family

$$\begin{aligned} \{\mu_L\langle 1_x\rangle_T \mid \mu_L \in \mathcal{B}^\sim\} &= \{\mu_L\langle 1_x\rangle_T \mid \mu_L \in \mathcal{B}\}^\sim \\ &= \{1_x \odot_T \mu \mid \mu \in \mathcal{F}\}^\sim \\ &= \{1_x \odot \mu \mid \mu \in \mathcal{F}\}^\sim. \end{aligned} \tag{4.26}$$

Thus one obtains the T-neighborhood system with the following family: $\Omega(x) = \{1_x \odot_T \mu \mid \mu \in \mathcal{F}\}$, a basis for the system in question. □

Theorem 4.16. *Let $(G,\cdot,^-,t(\Omega))$ be a T-neighborhood group. Then for all $\mu : G \to I$,*

$$\bar{\mu} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)^\sim\} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)\}^\sim. \tag{4.27}$$

Proof. Observe that every T-neighborhood group is a T-quasi-uniform space. Therefore, by virtue of Theorem 2.6, we can write, in particular, that

$$\bar{\mu} = \inf\{\nu_L\langle\mu\rangle_T \mid \nu \in \Omega(e)^\sim\}. \tag{4.28}$$

Then by using Lemma 4.11(1), we have the following:

$$\bar{\mu} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)^\sim\} = \inf\{\mu \odot_T \nu \mid \nu \in \Omega(e)\}^\sim. \tag{4.29}$$

□

Corollary 4.17. *In a T-neighborhood group $(G,\cdot,^{-},t(\Omega))$, the following property holds:*

$$\bar{\mu} = \inf\{\nu \odot_T \mu \mid \nu \in \Omega(e)^{\sim}\} = \inf\{\nu \odot_T \mu \mid \mu \in \Omega(e)\}^{\sim}. \tag{4.30}$$

Proof. This follows at once from the preceding results. □

Theorem 4.18. *If $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group, then $(G,^{-},t(\Omega))$ is T-regular.*

Proof. Let $\mu \in \Omega(e)$ and $\epsilon > 0$. Since the map $(x,y) \mapsto xy^{-1}$ is continuous at $(e,e) \in G\times G$, in view of Corollary 4.5, we can find a $\nu \in \Omega(e)$ such that

$$\nu \odot_T \nu^{-1} \le \mu + \epsilon. \tag{4.31}$$

Then using Theorem 4.16, we obtain

$$\bar{\mu}(x) = \inf_{\omega\in\Omega(e)} \nu \odot_T \omega^{-1} \le \nu \odot_T \nu^{-1} \le \mu(x) + \epsilon, \tag{4.32}$$

which ends the proof. □

Proposition 4.19. *If $(G,\cdot,^{-},t(\Omega))$ is a T-neighborhood group, then for all $\mu,\nu \in I^G$, we have the following:*

(i) $\bar{\mu} \odot_T \bar{\nu} \le \overline{\mu \odot_T \nu}$;

(ii) $\overline{\mu^{-1}} = \bar{\mu}^{-1}$;

(iii) $\overline{x \odot_T \mu \odot_T y} = x \odot_T \bar{\mu} \odot_T y$ *for all* $x,y \in G$.

Proof. (i) If $z \in G$, then we have

$$\begin{aligned}
\bar{\mu} \odot_T \bar{\nu}(z) &= \sup_{xy=z} \bar{\mu}(x) T \bar{\nu} y = \sup_{(x,y)\in m^{-1}(z)} [\bar{\mu} \otimes_T \bar{\nu}](x,y) \\
&= m[\bar{\mu} \otimes_T \bar{\nu}](z) = m[\overline{\mu \otimes_T \nu}](z) \\
&\le \overline{m[\mu \otimes_T \nu]}(z) = \overline{\mu \odot_T \nu}(z).
\end{aligned} \tag{4.33}$$

(ii) and (iii) follow immediately. □

Lemma 4.20. *If $(G,\cdot)$ and $(G',\cdot)$ are groups and $f : G \to G'$ is a group homomorphism, then*

$$f(x \odot_T a^{-1} \odot_T \mu) = f(x) \odot_T f(a)^{-1} \odot_T f(\mu). \tag{4.34}$$

Proof. This follows the same way as in [2, Lemma 2.15]; see also [7]. □

Theorem 4.21. *Let $(G,\cdot,^{-},t(\Omega))$ and $(H,\cdot,^{-},t(\Xi))$ be T-neighborhood groups with bases Ω and Ξ in G and H, respectively. If $f : G \to H$ is a group homomorphism, then f is continuous if and only if it is continuous at one point.*

Proof. Let $f : G \to H$ be continuous at the point $a \in G$. We need to show that f is continuous at each $x \in G$. Let $\xi \in \Xi(f(x))$ and $\epsilon > 0$. Then we have $f(x)^{-1} \odot_T \xi \in \Xi(e)$ and hence $f(a) \odot_T f(x)^{-1} \odot_T \xi \in \Xi(f(a))$. Then by Theorem 2.4, the continuity at one point $a \in G$ yields that $f^{-1}(f(a) \odot_T f(x)^{-1} \odot_T \xi) \in \Omega(a)^{\sim}$, which in turn implies that there exists a $\sigma = \sigma_\epsilon \in \Omega(a)$ such that

$$\sigma - \epsilon \leq f^{-1}(f(a) \odot_T f(x)^{-1} \odot_T \xi). \tag{4.35}$$

Now we have $\mu := x \odot_T a^{-1} \odot_T \sigma \in \Omega(x)$.

Thus, one obtains

$$\begin{aligned} \mu(z) - \epsilon &= x \odot_T a^{-1} \odot_T \sigma(z) - \epsilon = \sigma(ax^{-1}z) - \epsilon \\ &\leq f(a) \odot_T f(x)^{-1} \odot_T \xi(f(ax^{-1}z)) \\ &= f(ax^{-1}) \odot_T \xi(f(ax^{-1})f(z)) \\ &= \xi(f(z)) = f^{-1}(\xi)(z), \end{aligned} \tag{4.36}$$

that is, $\mu - \epsilon \leq f^{-1}(\xi)$, which implies that $f^{-1}(\xi) \in \Omega(x)^{\sim}$. □

Now we present some results on T-neighborhood groups in conjunction with Mordeson's TI-group.

5. Application of T-neighborhood groups in TI-groups

Definition 5.1 [7, 20]. An I-subset μ of G is called a TI-subgroup of G if it fulfills the following conditions:

(G1) $\mu(e) = 1$;

(G2) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$;

(G3) $\mu(xy) \geq \mu(x) T \mu(y)$, for all $x, y \in G$.

We denote the set of all TI-subgroups of G by $TI(G)$ and that of the set of all normal TI-subgroups by NTI-subgroups, while by NI-subgroup we mean normal I-subgroups, the one introduced by Rosenfeld [20] in which case $T = \min$ is used.

Proposition 5.2. *Let $(G, \cdot, ^{-}, t(\Omega))$ be a T-neighborhood group and $\mu \in TI(G)$. Then $\bar{\mu}^{t(\Omega)} \in TI(G)$.*

Proof. In view of [7, Theorem 5.1.4], it suffices to prove that

$$\bar{\mu}^{t(\Omega)} \odot_T (\bar{\nu}^{t(\Omega)})^{-1} \leq \bar{\mu}^{t(\Omega)}. \tag{5.1}$$

Since $\mu \in TI(G)$, we have $\mu \odot_T \mu^{-1} \leq \mu$. Then by an easy calculation, one obtains

$$\mu \odot_T \mu^{-1} = h(\mu \otimes_T \mu^{-1}), \tag{5.2}$$

which in conjunction with Theorem 3.1, yields the following:

$$(\bar{\mu}^{t(\Omega)}) \odot_T (\bar{\mu}^{t(\Omega)})^{-1} = h\left[(\bar{\mu}^{t(\Omega)}) \otimes_T (\bar{\mu}^{t(\Omega)})^{-1}\right] \leq \bar{\mu}^{t(\Omega)}, \tag{5.3}$$

which proves that $\bar{\mu}^{t(\Omega)} \in TI(G)$. □

Proposition 5.3. *If* $(G,\cdot,^{-},t(\Omega))$ *is a* T*-neighborhood group and* $\mu \in NTI(G)$*, then* $\bar{\mu}^{t(\Omega)} \in NTI(G)$*.*

Proof. Since $\mu \in NI(G)$, we have $I_x(\mu) = x \odot_T \mu \odot_T x^{-1} = \mu$, where $I_x : G \to G$, $z \mapsto xzx^{-1}$ is an inner automorphism. But then using Proposition 4.19(iii), we obtain

$$x \odot_T \bar{\mu} \odot_T x^{-1} = \overline{x \odot_T \mu \odot x^{-1}} = \bar{\mu}. \tag{5.4}$$

Hence the result follows from [7, Theorem 5.2.1(N5)]. □

Acknowledgments. We express our sincere gratitude to the anonymous referees for scrupulous checking and providing us with their kind suggestions. This work is supported by the College of Science, King Saud University (KSU) Research Center, Project no. MATH/1423/07(2003/07).

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