IJMMS 2004:18, 913–948
PII. S0161171204307039
http://ijmms.hindawi.com

GLOBAL BOUNDEDNESS, INTERIOR GRADIENT ESTIMATES, AND BOUNDARY REGULARITY FOR THE MEAN CURVATURE EQUATION WITH BOUNDARY CONDITIONS

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Received 3 July 2003

We obtain global estimates for the modulus, interior gradient estimates, and boundary Hölder continuity estimates for solutions u to the capillarity problem and to the Dirichlet problem for the mean curvature equation merely in terms of the mean curvature, together with the boundary contact angle in the capillarity problem and the boundary values in the Dirichlet problem.

2000 Mathematics Subject Classification: 53A10, 35J65.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Consider a solution to the mean curvature equation

$$\operatorname{div} Tu = H(x, u(x)) \quad \text{in } \Omega, \tag{1.1}$$

with

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}. \tag{1.2}$$

A solution of the Dirichlet problem can be regarded as a solution of (1.1) subject to the Dirichlet boundary condition

$$u = \varphi, \tag{1.3}$$

where φ is a given function on $\partial\Omega$; a solution of the capillarity problem can be regarded as a solution of (1.1) subject to the "contact angle" boundary condition

$$Tu \cdot \nu = \cos\theta, \tag{1.4}$$

where ν is the outward pointing unit normal of $\partial\Omega$, and where $\cos\theta$ is a given function on $\partial\Omega$. (Thus, in the capillarity problem, we are considering geometrically a function u in $\bar{\Omega}$ whose graph has the prescribed mean curvature H and which meets the boundary cylinder in the prescribed angle θ.) Here, $H = H(x,t)$ is assumed to be a given locally Lipschitz function in $\Omega \times \mathbb{R}$ satisfying the structural conditions

$$\frac{\partial H}{\partial t}(x,t) \geq 0, \quad \text{for } x \in \Omega,\ t \in \mathbb{R}. \tag{1.5}$$

Equation (1.1) is the Euler equation of the functional

$$I(v) = \int_\Omega \sqrt{1+|Dv|^2}dx + \int_\Omega \int_0^v H(x,t)dt\,dx. \tag{1.6}$$

The Dirichlet problem corresponds to the variational problem

$$I(v) \longrightarrow \min, \quad \forall v \in BV(\Omega) \cap \{v|_{\partial\Omega} = \varphi\}. \tag{1.7}$$

The capillarity problem corresponds to the variational problem

$$I(v) - \oint_{\partial\Omega} \beta v \, d\mathcal{H}_{n-1} \longrightarrow \min, \quad \text{with } \beta = \cos\theta, \ \forall v \in BV(\Omega), \tag{1.8}$$

where $\mathcal{H}_k$ is the k-dimensional Hausdorff measure. The work of de Giorgi, Miranda, and Giusti (see, e.g., [13, Chapter 14]) initiates the study of the following generalized version of the variational problem (1.7), namely, to find a solution $u \in H^{1,1}(\Omega)$ of the variational problem

$$I(v) + \oint_{\partial\Omega} |v-\varphi| d\mathcal{H}_{n-1} \longrightarrow \min, \quad v \in BV(\Omega). \tag{1.9}$$

The main purpose of this paper is to obtain global estimates for the modulus of solutions, interior gradient estimates, and boundary Hölder continuity estimates of solutions to the capillarity problem and to the Dirichlet problem merely in terms of the *mean curvature* H, together with the *boundary contact angle* θ in the capillarity problem and the *boundary values* φ in the Dirichlet problem. Since in the capillarity problem and in Dirichlet problem the only prescribed data are the mean curvature H, together with the boundary contact angles θ and the boundary values φ, respectively, estimates which are the most natural and convenient for use take such a form.

We recall that [1] (or later [23, 24]) established the following interior gradient estimates for any solutions u of (1.1) and for any point $y' \in \Omega$:

$$|Du(y')| \le c_1 \cdot \exp\left\{c_2 \cdot \sup_\Omega \frac{(u-u(y'))}{d}\right\}, \tag{1.10}$$

where $d = \operatorname{dist}(y', \partial\Omega)$ and where $c_1 = c_1(n, d\sup_\Omega |DH|)$, $c_2 = c_2(n, d\sup_\Omega |H|, d^2 \sup_\Omega |DH|)$. Thus, once we obtain the global estimates for the modulus of solutions in terms of the above-mentioned quantities, the interior gradient estimates in terms of the same set of quantities follow as an immediate consequence of (1.10).

1.1. Global estimates of the desired type will be obtained and formulated in Sections 3 and 4.3. In Section 3, estimates for $|u||_\Omega$ in terms of $\int_{\partial\Omega} u \, d\mathcal{H}_{n-1}$, H, and the n-dimensional Hausdorff measure of Ω are established under various conditions on H, the geometry of Ω, and the Hausdorff measure of Ω. Estimates which are valid in the most general case are formulated in Theorem 3.8. In particular, these results provide us with global estimates of $|u|$ for solutions to the Dirichlet problem.

In Sections 4.2.1, 4.2.2, and 4.2.3, we estimate $\int_{\partial\Omega} u\, d\mathcal{H}_{n-1}$ in terms of the L^1-norm of $|u|$ and $|Du|$ in case $\partial\Omega$ is piecewise Lipschitz continuous without outward cusps. Using this, in Sections 4.2.4 and 4.5, we formulate and prove Theorem 4.10 which provides us with estimates of the oscillation of the trace of u on $\partial\Omega$ in terms of H and $|\cos\theta|$ for *variational* solutions to the *capillarity problems* with $|\cos\theta|$ being bounded away from both 0 and 1, and $|H(x,t)|$ being bounded in $\bar{\Omega}\times\mathbb{R}$. Combining this with Theorem 3.8, we obtain Theorem 4.11 in Section 4.4, which yields global estimates of the oscillation of u for solutions to the *capillarity problem* with $|\cos\theta|$ bounded away from 0 and 1, and $|H(x,t)|$ bounded in $\bar{\Omega}\times\mathbb{R}$.

For the capillarity problem with $\cos\theta$ *not* bounded away from 0 and/or 1 on $\partial\Omega$, we will treat only the special case where H satisfies certain growth condition and obtain Theorem 4.13 in Section 4.6.

1.2. Simon and Spruck treat in [21] the boundary regularity for the capillarity problems in the case where Ω is C^4, θ in (1.4) is $C^{1,\alpha}$ on $\partial\Omega$ for some $0<\alpha<1$, and $H(x,t)$ is strictly monotone in t:

$$\inf_{x\in\bar{\Omega};t\in\mathbb{R}} \frac{\partial H(x,t)}{\partial t} > 0. \tag{1.11}$$

In case $0<\theta<\pi$, [21] shows the existence of a $C^2(\bar{\Omega})$ solution of (1.1) and (1.3). In case θ is allowed to take the values 0 or π, setting

$$\begin{aligned} S_1^+ &= \{x\in\partial\Omega : \theta\equiv 0 \text{ in some neighborhood of } x\},\\ S_1^- &= \{x\in\partial\Omega : \theta\equiv \pi \text{ in some neighborhood of } x\},\\ S_2 &= \{x\in\partial\Omega : 0<\theta<\pi\} \end{aligned} \tag{1.12}$$

[21] shows the existence of a function u defined on $\bar{\Omega}$ which is of class $C^2(\Omega\cup S_2)$, satisfies (1.1) in Ω, and satisfies (1.3) on S_2; furthermore, u is *Hölder continuous* at each point of $S_1^+\cup S_1^-$, has a restriction to $\partial\Omega$ which is *Lipschitz continuous* at each point of $S_1^+\cup S_1^-$, and satisfies (1.3) on $S_1^+\cup S_1^-$ in the sense that

$$\lim_{\varepsilon\to 0^+}\int_{U\cap\Omega_\varepsilon} |\nu\cdot Tu\pm 1|\,dx = 0 \quad \text{for each } U\subset\Omega \text{ with } \overline{U}\cap\partial\Omega\subset S_1^{\mp}, \tag{1.13}$$

assuming that Tu is extended to some boundary strip Ω_ε with width ε so that it is constant along the normals to $\partial\Omega$. To prove this, a transformation of coordinates near the boundary is performed analogously to that in [20], which, together with a subsequent differentiation of (1.1), (1.3), and an application of (1.11), establishes an estimate of the tangential derivative of u along $\partial\Omega$, under the condition that $|\cos\theta|\le\gamma<1$ for some positive constant γ; in case θ is constant in a neighborhood of the point under consideration, this estimate of tangential derivative is independent of γ. This proves the Lipschitz continuity of the trace of u on $\partial\Omega$, which together with the result in [19] yield the boundary Hölder continuity of u. The disadvantage of their proof is that H is assumed to satisfy the strict inequality (1.11) rather than the less restrictive condition (1.5).

In contrast, as a consequence of the estimates of local oscillation on $\partial\Omega$, under the assumptions

$$\hat{H}(x) = H\left(x, \inf_{\partial\Omega} u\right) \in L^p(\Omega), \qquad \hat{\hat{H}}(x) = H\left(x, \sup_{\partial\Omega} u\right) \in L^p(\Omega), \tag{1.14}$$

for some $p > n$, the *Lipschitz continuity* of the trace of u on $\partial\Omega$ will be established in Section 4.4 at each point x_0 of $\partial\Omega$ which satisfies the following assumptions:

(A1) a small neighborhood $\Omega \cap B_R(x_0)$ exists such that $\cos\theta$ is continuous in $\partial\Omega \cap B_R(x_0)$, and, for some constant $\hat{\beta}$, $1 \geq \hat{\beta} > 0$, we have

$$0 < |\cos\theta| \leq \hat{\beta} \tag{1.15}$$

in $\partial\Omega \cap B_R(x_0)$,

(A2) $\partial\Omega \cap B_R(x_0)$ is either C^2 or the graph of a Lipschitz continuous function with Lipschitz constant L:

$$\hat{\beta} \cdot \sqrt{1 + L^2} < 1. \tag{1.16}$$

The Lipschitz norm of the trace of u on $\partial\Omega$ in such a small neighborhood of x_0 will be shown to depend only on H, $\hat{\beta}$, and the geometry of Ω.

1.3. Furthermore, we will establish in Section 2.1 useful growth lemmas by constructing suitable barriers, adapting the work in [14, II.1.4] and [16, Lemma 4.1]. Theorem 2.7 in Section 2.2 provides us with an interior Hölder seminorm estimate with exponent $\log_4(5/2)$ merely in terms of $(\sup_\Omega u - \inf_\Omega u)/(d(y'))^{\log_4(5/2)}$, under the assumption that H is *nonnegative*. We notice that, in contrast to the classical interior gradient estimate (1.10) which depends exponentially on the quantity $(\sup_\Omega u - \inf_\Omega u)/d(y')$, the interior Hölder seminorm estimate is *linearly* proportional to the quantity $(\sup_\Omega u - \inf_\Omega u)/(d(y'))^{\log_4(5/2)}$.

These growth lemmas also yield boundary Hölder continuity with exponent $1/2$ in the case that H is *nonnegative and bounded above* for solutions to the mean curvature equation with $C^{1/2}$ Dirichlet data φ, *without an assumption on the regularity of the domain*. This result, being formulated as Theorem 2.6 in Section 2.1, improves in some respects a previous work of Korevaar and Simon [15], in which boundary Hölder continuity with exponent $1/2$ is established for solutions with C^2 Dirichlet data φ, also with no dependence on the regularity of the domain. However, the Hölder norm cannot be estimated in our result and is explicitly estimated in [15] in terms of $\sup_\Omega |u|$, the C^2-norm of $|\varphi|$, and the Lipschitz constant of H.

We notice that Theorems 4.12 and 2.6 yield the Hölder continuity with exponent $1/2$ of u up to the boundary locally in $\overline{\Omega} \cap B_R(x_0)$ under the assumption that H *is nonnegative* and *bounded above*.

The results in Sections 3 and 4 are, however, derived without resort to results in Section 2.

2. Growth lemmas, interior gradient estimates, and boundary Hölder continuity. The formulation and proof of the following growth lemma are adapted from [14, II.1.4] and [16, Lemma I.4.1].

Growth Lemma 2.1. *Suppose that D is a domain in $\mathbb{R}^n$ such that ∂D has nonempty intersection with the ball $B_{4R}(x_0)$. Suppose that $H(x,t)$ is nonnegative for all $x \in D \cap B_{4R}(x_0)$ and for all $t \in \mathbb{R}$. Let u be a solution to (1.1) in D which is continuous in $\bar{D} \cap B_{4R}(x_0)$. Suppose that*

$$u|_{\partial D \cap B_{4R}(x_0)} = 0. \tag{2.1}$$

Then, there exists a positive constant $\xi_1 > 1$ such that

$$\sup_{D \cap B_{4R}(x_0)} u(x) \geq \xi_1 \cdot \sup_{D \cap B_R(x_0)} u(x). \tag{2.2}$$

Indeed,

$$\xi_1 = 4. \tag{2.3}$$

Growth Lemma 2.2. *Suppose, in addition to the assumptions in Growth Lemma 2.1, that $H(x,t)$ is bounded above in $(D \cap B_{4R}(x_0)) \times \mathbb{R}$ such that there exists a constant H_{**} for which*

$$H(x,t) \leq H_{**}, \quad \forall x \in D \cap B_{4R}(x_0),\ t \in \mathbb{R}. \tag{2.4}$$

Suppose furthermore that

$$\lim_{r \to 0} \left[\frac{|\inf_{D \cap B_r(x_0)} u|}{r^\alpha} \right] > a \quad \text{for some } \alpha < \frac{1}{2},\ a > 0. \tag{2.5}$$

Then, there exist positive constants $\xi_2 > 1$ and R_0 such that for $R \leq R_0$, there holds

$$\inf_{D \cap B_{4R}(x_0)} u(x) \leq \xi_2 \cdot \inf_{D \cap B_R(x_0)} u(x). \tag{2.6}$$

Indeed, for $R \leq R_0$,

$$\xi_2 = 2. \tag{2.7}$$

Growth Lemma 2.3. *Under the assumptions in Growth Lemma 2.2, let R_0 be the number introduced in Growth Lemma 2.2. For $R \leq R_0$,*

$$\sup_{D \cap B_{4R}(x_0)} u(x) - \inf_{D \cap B_{4R}(x_0)} u \geq \min(\xi_1, \xi_2) \cdot \left[\sup_{D \cap B_R(x_0)} u(x) - \inf_{D \cap B_R(x_0)} u \right]. \tag{2.8}$$

Proof of Growth Lemmas 2.1, 2.2, and 2.3. Let

$$M = \sup_{D \cap B_{4R}(x_0)} u \geq 0, \qquad m = \inf_{D \cap B_{4R}(x_0)} u \leq 0. \tag{2.9}$$

We set

$$v_1(x) = M \cdot \left(\frac{U_1(|x-x_0|)}{U_1(4R)}\right), \qquad v_2(x) = m \cdot \left(\frac{U_2(|x-x_0|)}{U_2(4R)}\right), \tag{2.10}$$

where

$$U_1(r) = r, \qquad U_2(r) = 8H_{**}r^{1/2}, \tag{2.11}$$

and $|x-x_0| = \operatorname{dist}(x,x_0)$. We observe that $U_1(r)$ and $U_2(r)$ are two strictly increasing functions of class $C^2(0,\infty)$ such that

$$\begin{gathered} \operatorname{div} TU_1(|x-x_0|) = 0, \\ \operatorname{div} TU_2(|x-x_0|) \le -H_{**} \quad \text{for } |x-x_0| \text{ sufficiently small.} \end{gathered} \tag{2.12}$$

Indeed, we have

$$\operatorname{div} U_2 \le -2H_{**} \frac{\sum_i |x_i-(x_0)_i|^{-3/2}}{\left[1+\sum_i |x_i-(x_0)_i|^{-1}\right]^{3/2}}, \tag{2.13}$$

which yields

$$\operatorname{div} TU_2(|x-x_0|) \le -H_{**} \quad \text{for } |x-x_0| \text{ sufficiently small.} \tag{2.14}$$

Furthermore, we have

$$U_2(4R) < |m| \quad \text{for } R \text{ sufficiently small,} \tag{2.15}$$

by virtue of assumption (2.5).

Thus, we have, for R sufficiently small,

$$\operatorname{div} Tv_1(x) \le H(x,u(x)), \quad \operatorname{div} Tv_2(x) \ge H(x,u(x)) \quad \text{in } B_{4R}(x_0)\cap D. \tag{2.16}$$

Furthermore, we have

$$\begin{gathered} v_1|_{\partial D\cap B_{4R}(x_0)} \ge 0 = u|_{\partial D\cap B_{4R}(x_0)}, \\ v_2|_{\partial D\cap B_{4R}(x_0)} \le 0 = u|_{\partial D\cap B_{4R}(x_0)}, \end{gathered} \tag{2.17}$$

and, if $\partial B_{4R}(x_0)\cap D$ is nonempty,

$$\begin{gathered} v_1|_{\partial B_{4R}(x_0)\cap D} = M \ge u|_{\partial B_{4R}(x_0)\cap D}, \\ v_2|_{\partial B_{4R}(x_0)\cap D} = m \le u|_{\partial B_{4R}(x_0)\cap D}. \end{gathered} \tag{2.18}$$

Therefore,

$$u \le v_1, \quad u \ge v_2 \quad \text{in } D \cap B_{4R}(x_0). \tag{2.19}$$

Since $M \ge 0$ and $m \le 0$, these yield

$$\begin{aligned} \sup_{D\cap B_R(x_0)} u &\le \sup_{D\cap B_R(x_0)} v_1 \le M \cdot \left(\frac{U_1(R)}{U_1(4R)}\right), \\ \inf_{D\cap B_R(x_0)} u &\ge \inf_{D\cap B_R(x_0)} v_2 \ge m \cdot \left(\frac{U_2(R)}{U_2(4R)}\right). \end{aligned} \tag{2.20}$$

Hence, we can take

$$\xi_1 = \frac{U_1(R)}{U_1(4R)}, \qquad \xi_2 = \frac{U_2(R)}{U_2(4R)}. \tag{2.21}$$

The choice of U_1 and U_2 yield (2.3) and (2.7). $\square$

We note that in the proof of Growth Lemma 2.1, the comparison function $U_1(x)$ can be taken instead to be $|x-x_0|^{\alpha^*}$, for any α^*, $0 < \alpha^* \le 1/2$. A closer examination of the role (2.1) plays yields the following.

Growth Lemma 2.4. *Let D, $B_{4R}(x_0)$, and $H(x,t)$ satisfy the assumptions of Growth Lemma 2.2. Let u be a solution to (1.1) in D which is continuous in $\bar{D}$ with $x_0 \in \bar{\Omega}$ such that*

$$\lim_{r\to 0}\left[\frac{\sup_{\partial D\cap B_{4R}(x_0)} |u(x)-u(x_0)|}{\sqrt{r}}\right] = 0; \tag{2.22}$$

that is, $u|_{\partial D\cap B_{4R}(x_0)}$ is of class $C^{1/2}$ and

$$\lim_{r\to 0}\left[\frac{\inf_{D\cap B_{4R}(x_0)} |u(x)-u(x_0)|}{r^{\alpha}}\right] \ne 0 \quad \textit{for some } \alpha < \frac{1}{2}. \tag{2.23}$$

Then, there hold

$$\begin{aligned} \sup_{D\cap B_{4R}(x_0)} u - u(x_0) &\ge 2\left(\sup_{D\cap B_R(x_0)} u - u(x_0)\right), \\ \inf_{D\cap B_{4R}(x_0)} u - u(x_0) &\le 2\left(\inf_{D\cap B_R(x_0)} u - u(x_0)\right), \end{aligned} \tag{2.24}$$

and thus

$$\sup_{D\cap B_{4R}(x_0)} u - \inf_{D\cap B_{4R}(x_0)} u \ge 2\left(\sup_{D\cap B_R(x_0)} u - \inf_{D\cap B_R(x_0)} u\right) \tag{2.25}$$

for R sufficiently small.

2.1. Growth lemmas and Hölder continuity for solutions to the Dirichlet problem. We now recall the following result from [11, Lemma 8.23], which is also used in [16, Theorem 7.1, page 39].

Lemma 2.5. *Let ω be a nondecreasing function on the interval $(0,R_0]$ satisfying, for $R \le R_0$, the inequality*

$$\omega(\tau R) \le \gamma \omega(R), \tag{2.26}$$

where $0 < \gamma, \tau < 1$. Then, for any $R \le R_0$,

$$\omega(R) \le \frac{1}{\gamma} \cdot \left(\frac{R}{R_0}\right)^{\alpha} \cdot \omega(R_0), \tag{2.27}$$

where

$$\alpha = \log_{1/\tau} \frac{1}{\gamma}. \tag{2.28}$$

From Growth Lemma 2.4 and Lemma 2.5, we obtain the following.

Theorem 2.6. *Let u be a solution to (1.1) which is continuous in $\bar{\Omega}$ and let $x_0 \in \partial\Omega$. Suppose that H is nonnegative and bounded above by a positive constant. If $u|_{\partial\Omega \cap B_R(x_0)}$ is of class $C^{1/2}$ in the sense of (2.22), then u is Hölder continuous with exponent $1/2$ up to the boundary near x_0.*

Indeed, suppose that u is *not* Hölder continuous with exponent $1/2$ up to the boundary near x_0; that is, (2.23) fails to hold. Then (2.22) and (2.23), Growth Lemma 2.4, and Lemma 2.5 yield the Hölder continuity of u on $\Omega \cap B_{R/4}(x_0)$, which contradicts (2.23). This contradiction proves Theorem 2.6.

We notice that since we do not know how small R has to be in Growth Lemma 2.4, we are not able to obtain estimates of the boundary Hölder norm of u from our argument.

2.2. Growth Lemma 2.1 and interior Hölder seminorm estimates. From Growth Lemma 2.1 and Lemma 2.5, we obtain the following estimates of the interior Hölder seminorm.

Theorem 2.7. *Let u be a solution to (1.1) and suppose that H is nonnegative. Then, for*

$$\alpha = \log_4 \frac{5}{2}, \tag{2.29}$$

the estimate

$$|u||_{C^{0,\alpha}} \le \frac{(5/2)(\sup_\Omega u - \inf_\Omega u)}{(\operatorname{dist}(x_0, \partial\Omega))^{\alpha}} \tag{2.30}$$

is valid for $x_0 \in \Omega$.

Indeed, setting $R_0 = \operatorname{dist}(x_0, \partial\Omega)$ and setting, for $R_1 \le R_0/4$,

$$a = \sup_{B_{R_1}(x_0)} u - \inf_{B_{R_1}(x_0)} u, \qquad v = u - \inf_{B_{R_1}(x_0)} u - \frac{a}{2}, \tag{2.31}$$

we let

$$E^+ = \{x \in B_{4R_1}(x_0) : v(x) \ge 0\}. \tag{2.32}$$

Then, $\partial E^+ \cap B_{4R_1}(x_0)$ is nonempty. We obtain from Growth Lemma 2.1 that

$$\begin{aligned}\sup_{B_{4R_1}(x_0)} u - \inf_{B_{4R_1}(x_0)} u &= \sup_{E^+\cap B_{4R_1}(x_0)} v - \inf_{B_{4R_1}(x_0)} v \\ &\geq 4 \sup_{E^+\cap B_{R_1}(x_0)} v - \inf_{B_{R_1}(x_0)} v \\ &\geq \frac{5}{2}\left(\sup_{B_{R_1}(\hat{x}_0)} u - \inf_{B_{R_1}(\hat{x}_0)} u\right),\end{aligned} \tag{2.33}$$

which together with Lemma 2.5 yield Theorem 2.7.

3. Estimates for $|u||_\Omega$ in terms of $u|_{\partial\Omega}$ and global estimates for solutions to the Dirichlet problem. We will establish local and global estimates for the modulus of solutions to the variational problems (1.7) and (1.9). The reasoning below will be adapted from that used in [7, 8, 9] to demonstrate the boundedness of solutions with respect to the capillarity problem or to the Dirichlet problem.

We assume that Ω is a bounded domain with piecewise Lipschitz boundary. We first consider the following variational problem, which is slightly more general than the preceding ones. Namely, let $H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$ be given functions such that (1.5) holds; let $j : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy a Carathéodory condition, that is, it is measurable in x (with respect to the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$) and continuous in the second variable. Then, we consider the functional

$$J(v) = \int_\Omega \sqrt{1+|Dv|^2}\,dx + \int_\Omega \int_0^v H(x,t)\,dt\,dx + \oint_{\partial\Omega} j(x,v)\,d\mathcal{H}_{n-1}. \tag{3.1}$$

We note that by taking

$$j(x,t) = -\beta(x)\cdot t, \qquad j(x,t) = \left|t - \varphi(t)\right|, \tag{3.2}$$

the functionals I in (1.7) and (1.9) are included in the general setting.

3.1. The simplest case where H satisfies a certain growth condition. Under the above assumptions on Ω, H, and j, we will prove that the following holds in the simplest case where H satisfies a certain growth condition.

Proposition 3.1. *Let u be a solution of the variational problem*

$$J(v) \longrightarrow \min \quad \text{in } BV(\Omega). \tag{3.3}$$

Suppose

$$H_{t_0}(x) = H(\cdot,t_0) \geq 0 \tag{3.4}$$

for some $t_0 \in \mathbb{R}$ and all $x \in \Omega$. Then a constant C_1 exists, which is determined completely by $|\Omega|$, n, and t_0 such that the following estimate is valid:

$$u \leq \max\left(\sup_{\partial\Omega} u, t_0\right) + C_1. \tag{3.5}$$

Suppose, instead, that

$$H_{t_0}(x) \le 0 \tag{3.6}$$

for all $x \in \Omega$, then

$$\min\left(\inf_{\partial\Omega} u, t_0\right) - C_1 \le u. \tag{3.7}$$

Indeed, take

$$C_1 = 2^{n+1}(c_*)^{-1}|\Omega|^{n+1}, \tag{3.8}$$

where c_* is a constant depending only on n such that the Sobolev inequality takes the form

$$\|f\|_{n^*} \le c_*\|Df\|_1 \quad \text{for each } f \in W_0^{1,1}(\Omega),\ n^* = \frac{n}{n-1}. \tag{3.9}$$

Here and in the following, we denote by $|\cdot|$ either an n-dimensional or $(n-1)$-dimensional Hausdorff measure.

We notice that such t_0 exists in the case where H satisfies the relations

$$\lim_{t\to\infty}\inf_{\Omega} H(x,t) = +\infty, \tag{3.10}$$

$$\lim_{t\to-\infty}\sup_{\Omega} H(x,t) = -\infty. \tag{3.11}$$

Concus and Finn [2] and Gerhardt [9, Lemma 4.1] obtain a bound for the solution to the capillarity problem with H satisfying the previous two relations in the case where Ω satisfies an interior sphere condition.

In the proof of Proposition 3.1, we will apply a result due to Stampacchia [22, Lemma 4.1], which can be stated as follows.

Lemma 3.2 (Stampacchia). *Suppose that $\varphi(t)$ is a nonnegative nondecreasing function defined on $\mathbb{R}$ such that for some positive constants C, k_0, and γ, there holds*

$$(h-k)\cdot\varphi(h) \le C\cdot[\varphi(k)]^{\gamma} \quad \text{for each } h > k \ge k_0. \tag{3.12}$$

Then

$$h^{1/(1-\gamma)}\cdot\varphi(h) \le 2^{1/(1-\gamma)^2}\cdot\left\{C^{1/(1-\gamma)} + (2k_0)^{1/(1-\gamma)}\cdot\varphi(k)\right\}, \quad \text{if } \gamma < 1, \tag{3.13}$$

$$\varphi(k_0+\tau) = 0, \quad \text{if } \gamma > 1, \tag{3.14}$$

where

$$\tau = 2^{\gamma/(\gamma-1)}\cdot C\cdot[\varphi(k_0)]^{(\gamma-1)}. \tag{3.15}$$

Proof of Proposition 3.1. Let k be a number greater than $\max(\sup_{\partial\Omega} u, t_0)$ and set $u_k = \min(u,k)$. From the minimizing property of u, we obtain

$$J(u) \le J(u_k). \tag{3.16}$$

Hence, using the notation

$$A(k) = \{x \in \Omega : u(x) \geq k\} \tag{3.17}$$

and assuming for a moment that u is smooth, we obtain

$$\int_{A(k)} \sqrt{1+|Du|^2} + \int_\Omega \int_{u_k}^{u} H(x,t)\,dt\,dx \leq |A(k)|. \tag{3.18}$$

We set

$$w = \max(u-k,0) = u - u_k. \tag{3.19}$$

The monotonicity condition (1.5) on $H(x,\cdot)$ yields

$$\int_{u_k}^{u} H(x,t)\,dt \geq H(x,t_0)\cdot(u-u_k) = H_{t_0} w. \tag{3.20}$$

Inserting this into (3.18), we obtain

$$\int_\Omega |Dw|\,dx + \int_\Omega H_{t_0} w\,dx \leq |A(k)|. \tag{3.21}$$

It is easy to see that this is also valid for $u \in BV(\Omega)$ by using an approximation argument. From (3.20) and (3.4), we obtain

$$\int_\Omega |Dw|\,dx \leq |A(k)|. \tag{3.22}$$

From this and the Sobolev inequality (cf. [13, Theorem 1.28, page 24]), we obtain

$$\|w\|_{n^*} \leq c_* \cdot |A(k)|, \tag{3.23}$$

where the constants n^* and c_* are given in the statement of Proposition 3.1. This and Hölder inequality yield

$$\int_{A(k)} (u-k)\,dx = \|w\|_1 \leq c_* \cdot |A(k)|^{1+(1/n)}, \tag{3.24}$$

and hence

$$(h-k)\cdot|A(h)| \leq c_* \cdot |A(k)|^{1+(1/n)} \quad \text{for each } h > k > \max\Big(\sup_{\partial\Omega} u, t_0\Big). \tag{3.25}$$

From this and Lemma 3.2, we obtain

$$u \leq \max\Big(\sup_{\partial\Omega} u, t_0\Big) + 2^{n+1}\cdot c_* |\Omega|^{1/n}. \tag{3.26}$$

By setting $u_k = \max(u,-k)$, $-k \leq \min(\inf_{\partial\Omega} u, t_0)$, $k \geq 0$, in (3.3), a lower bound of u can be obtained in case (3.6) is valid, which completes our proof of Proposition 3.1. □

3.2. The general cases. We are however interested in the cases where (3.4) or (3.6) fails to hold.

3.2.1. Estimates for small domains. Assume for a moment that for some $t_0 \in \mathbb{R}$,

$$H_{t_0} = H(x,t_0) \in L^p(\Omega) \quad \text{for some } p > n, \tag{3.27}$$

$$|\Omega|^{(p-n)/(np)} \le \lambda \frac{(c_*)^{-1}}{||H_{t_0}||_{L^p(\Omega)}} \tag{3.28}$$

for some constant $0 < \lambda < 1$. We will prove the following.

Proposition 3.3. *Let u be a solution of the variational problem (3.3). Suppose (3.28) holds and (3.27) holds for some $t_0 \in \mathbb{R}$. Then, the following inequalities are valid:*

$$\begin{aligned} \sup_\Omega u &\le \max\Big(\sup_{\partial\Omega} u, t_0\Big) + 2^{n+1}(1-\lambda)^{-1}c_* \cdot |\Omega|^{1/n}, \\ \inf_\Omega u &\ge \min\Big(\inf_{\partial\Omega} u, t_0\Big) - 2^{n+1}(1-\lambda)^{-1}c_* \cdot |\Omega|^{1/n}. \end{aligned} \tag{3.29}$$

Choosing $k \ge \sup_{\partial\Omega} u$ and setting $u_k = \min(u,k)$, (3.7) is still valid for each $k \ge t_0$ no matter (3.4) or (3.6) holds or not and, we can still obtain (3.21) from (3.16), (3.18), and (3.20). To treat the second integral in (3.21), we observe that under assumption (3.27), we obtain from Hölder inequality that

$$\begin{aligned} \left| \int_\Omega H_{t_0} w\, dx \right| &\le \|w\|_{n^*} \left\{ \int_{A(k)} |H_{t_0}|^n dx \right\}^{1/n} \\ &\le \|w\|_{n^*} \cdot ||H_{t_0}||_p \cdot |A(k)|^{(p-n)/(np)}. \end{aligned} \tag{3.30}$$

Inserting this into (3.21) and treating the first integral in (3.21) by means of the Sobolev inequality and the Hölder inequality as above, we obtain

$$\Big[(c_*)^{-1} - ||H_{t_0}||_p \cdot |A(k)|^{(p-n)/(np)}\Big] \cdot \|w\|_{n^*} \le |A(k)|. \tag{3.31}$$

Assume that (3.28) holds. Then, by (3.31), we have

$$(1-\lambda) \cdot (c_*)^{-1} \cdot \|w\|_{n^*} \le |A(k)|. \tag{3.32}$$

Inserting (3.23) into (3.32), we obtain

$$(h-k) \cdot |A(h)| \le (1-\lambda)^{-1} \cdot c_* \cdot |A(k)|^{1+(1/n)} \tag{3.33}$$

for each $h > k > \max(\sup_{\partial\Omega} u, t_0)$. Proposition 3.3 follows from this and Lemma 3.2.

3.2.2. Estimates for general domains. In general, (3.28) does not hold and we assume that (3.27) holds for some $t_0 \in \mathbb{R}$ and set

$$C_\lambda = \left[\frac{1}{\lambda} c_* ||H_{t_0}||_{L^p(\Omega)}\right]^{pn/(p-n)} \int_\Omega |u|\, dx. \tag{3.34}$$

The following will be established.

Proposition 3.4. *Let u be a solution of the variational problem (3.3). Suppose (3.27) holds. Then, for the constant C_λ given in (3.34), there hold*

$$\begin{aligned}\sup_{\Omega} u &\le \max\Big(\sup_{\partial\Omega} u, t_0, C_\lambda\Big) + 2^{n+1}(1-\lambda)^{-1}c_*|\Omega|^{1/n},\\ \inf_{\Omega} u &\ge \min\Big(\inf_{\partial\Omega} u, t_0, -C_\lambda\Big) - 2^{n+1}(1-\lambda)^{-1}c_*|\Omega|^{1/n}.\end{aligned} \tag{3.35}$$

Indeed, we observe that

$$|A(k)| \le \frac{1}{k}\int_\Omega |u|\,dx. \tag{3.36}$$

Hence, if (3.27) holds, for C_λ given in (3.34), we obtain (3.32) from (3.31) and (3.30) for each $h > k > \max(\sup_{\partial\Omega} u, t_0, C_\lambda)$. From this and Lemma 3.2, we prove Proposition 3.4.

To estimate C_λ, we proceed to estimate $\|u\|_1$ in terms of $\oint_{\partial\Omega}|u|\,d\mathcal{H}_{n-1}$. For this, we set $H_0 = H(\cdot,0)$ and first assume that

$$\left|\int_E H_0\,dx\right| \le (1-\varepsilon_0)\mathbb{M}(\partial E) \tag{3.37}$$

for some positive number ε_0 independent of E, where E is any measurable subset of Ω and $\mathbb{M}(\partial E)$ denotes the mass of ∂E in the sense of [4, Chapter 4.1.7]. (We may note that in the case that H does not depend on t, Giaquinta [10] demonstrated the existence of solutions for each $\varphi \in L^1(\Omega)$ in $BV(\Omega)$ to problem (1.9) provided that H satisfies condition (3.30).) We assume, in addition, that

$$H(x,0) \in L^1(\Omega) \tag{3.38}$$

holds. The proof of [8, Theorem 5] yields the following.

Proposition 3.5. *Let u be a solution of the variational problem (2.2). Let H satisfy conditions (1.5), (3.37), and (3.38) and let $u|_{\partial\Omega} \in L^1(\partial\Omega)$. Then*

$$\int_\Omega |Du|\,dx + \int_\Omega |u|\,dx \tag{3.39}$$

is bounded by a constant depending only on ε_0, Ω, $\oint_{\partial\Omega}|u|\,d\mathcal{H}_{n-1}$, and $\int_\Omega H(x,0)\,dx$.

The proof of Proposition 3.5 given in [10] is based on the observation that u is a solution of the problem

$$J_*(v) \longrightarrow \min \quad \text{in } BV(\Omega), \tag{3.40}$$

where

$$J_*(v) = \int_\Omega \sqrt{1+|Dv|^2}\,dx + \int_\Omega\int_0^v H(x,t)\,dt\,dx + \int_{\partial\Omega}|v-u|\,d\mathcal{H}_{n-1}. \tag{3.41}$$

Let B be any ball containing $\bar{\Omega}$. Extend $u|_{\partial\Omega}$ to some function φ_B in $H^{1,1}(B\setminus\bar{\Omega})$ having boundary values zero on ∂B (cf. [6]). Then, extend H to $\tilde{H}$ which vanishes outside $\overline{\Omega}$, let

$\tilde{K} = \{v \in BV(\Omega), v|_{B\setminus\bar{\Omega}} = \varphi_B\}$, and set

$$\tilde{J}(v) = \int_B \sqrt{1+|Dv|^2}\,dx + \int_B \int_0^v \tilde{H}(x,t)\,dt\,dx. \tag{3.42}$$

To solve problem (3.40) is equivalent to finding a solution u of the problem

$$\tilde{J}(v) \longrightarrow \min \quad \text{in } \tilde{K}. \tag{3.43}$$

By (1.5),

$$\int_0^u \tilde{H}(x,t)\,dt \ge \tilde{H}_0 \cdot u \quad \text{for } \tilde{H}_0 = \tilde{H}(\cdot,0), \tag{3.44}$$

and by (3.37), it is shown in [10, page 77] that

$$\int_\Omega \tilde{H}_0 \cdot u\,dx \ge -(1-\varepsilon_0)\cdot \int_\Omega |Du|\,dx - (1-\varepsilon_0)\cdot \int_{\partial\Omega} |u|\,d\mathcal{H}_{n-1}. \tag{3.45}$$

Inserting these into (3.42) and using the minimizing property of u for $\tilde{J}$, we obtain

$$\begin{aligned} J_*(0) &+ \int_{B\setminus\bar{\Omega}} \sqrt{1+|D\varphi_B|^2}\,dx \\ &\ge \tilde{J}(u) \ge \varepsilon_0 \int_\Omega |Du|\,dx + \int_{B\setminus\bar{\Omega}} \sqrt{1+|Du|^2}\,dx - (1-\varepsilon_0)\int_{\partial\Omega} |u|\,d\mathcal{H}_{n-1}. \end{aligned} \tag{3.46}$$

This yields a bound of the L^1-norm of $|Du|$, which together with Sobolev inequality yield a uniform bound of the BV-norm of u. This reasoning motivates the work below in Section 3.2.3 for general domains where (3.37) and hence (3.45) do not necessarily hold.

3.2.3. General cases without (3.27) or (3.37). In general, (3.27) does not hold and it is not straightforward to estimate the number ε_0 in (3.37). To treat general cases, we recall the following isoperimetric inequality whose proof is presented in [13, Corollary 1.29].

Lemma 3.6 (isoperimetric inequality). *Let E and A be bounded Caccioppoli sets in $\mathbb{R}^n$. Assume A to be of positive n-dimensional Hausdorff measure and to be sufficiently smooth such that Poincaré inequality*

$$\left(\int_A |f-f_A|^{n/(n-1)}\,dx\right)^{(n-1)/n} \le c_A \int_A |Df| \tag{3.47}$$

holds for every $v \in W^{1,1}(A)$, where c_A is a constant depending only on n and A, and define $f_A = (1/|A|)\int_A f\,dx$. Then

$$\min\{|E\cap A|, |(\mathbb{R}^n\setminus E)\cap A|\}^{(n-1)/n} \le c_A \int_A |D\chi_E|, \tag{3.48}$$

where χ_E is the characteristic function of the set E.

In [25, Chapter 4], Poincaré inequality of the above type is shown to hold in a wide class of domains, called *extension domain* in the sense that there exists for such a domain A a bounded linear operator $L: W^{k,p}(A) \to W^{k,p}(\mathbb{R}^n)$ such that $L(f)|_A = f$ for all $f \in W^{k,p}(\Omega)$. In particular, every Lipschitz domain is an extension domain.

We notice that the reasoning leading to [11, (7.45)] enables us to take

$$c_A = \left(\frac{\omega_n}{|A|}\right)^{1-1/n} (\operatorname{diam} \hat{A})^n, \tag{3.49}$$

where $\hat{A}$ is the convex hull of A and ω_n is the Lebesgue measure of the n-dimensional unit ball.

Select a Caccioppoli set A such that $\partial\Omega \cap A$ is of positive $(n-1)$-dimensional Hausdorff measure and such that

$$|\Omega \cap A| \leq \left|(\mathbb{R}^n \setminus \Omega) \cap A\right|. \tag{3.50}$$

Given δ_*, $0 < \delta_* < 1$, suitably choose A so that

$$c_A\Big(\sup_{\Omega} |H_0(x)|\Big)|\Omega \cap A|^{1/n} \leq \delta_* \tag{3.51}$$

under the additional assumption

$$\Big(\sup_{\Omega} |H_0(x)|\Big) < \infty. \tag{3.52}$$

Inequalities (3.50) and (3.51) yield

$$\int_{\Omega\cap A} H_0 u\, dx \geq -\delta_* \int_{\Omega\cap A} |Du|\, dx - \delta_* \int_{\partial\Omega\cap A} |u|\, d\mathcal{H}_{n-1}. \tag{3.53}$$

Indeed, we obtain from (3.50) and Lemma 3.6 that

$$|\Omega \cap A| \leq c_A |\partial\Omega \cap A| \tag{3.54}$$

and, for every Caccioppoli set $E \subset \Omega$,

$$|E \cap A| \leq c_A |\partial E \cap A|; \tag{3.55}$$

hence,

$$\begin{aligned} \left|\int_{\Omega\cap A} H_0\, dx\right| &\leq \Big(\sup_{\Omega} |H_0(x)|\Big)|\Omega \cap A| \\ &\leq c_A\Big(\sup_{\Omega} |H_0(x)|\Big)|\Omega \cap A|^{1/n}|\partial\Omega \cap A|; \end{aligned} \tag{3.56}$$

and, for every Caccioppoli set $E \subset \Omega$,

$$\left|\int_{E\cap A} H_0\, dx\right| \leq \Big(\sup_{\Omega} |H_0(x)|\Big)|E \cap A| \leq c_A\Big(\sup_{\Omega} |H_0(x)|\Big)|E|^{1/n}|\partial E \cap A|. \tag{3.57}$$

From the reasoning in [10, page 77], we have

$$\int_{\Omega\cap A} H_0 u\,dx = \int_0^\infty dt \int_{\Sigma_t\cap A} H_0\,dx, \tag{3.58}$$

where

$$\Sigma_t = \{x \in \Omega : u(x) > t\}; \tag{3.59}$$

hence, by (3.56), (3.57), and (3.58),

$$\left|\int_{\Omega\cap A} H_0 u\,dx\right| \le c_A\Big(\sup_{\Omega}|H_0(x)|\Big)\cdot|\Omega\cap A|^{1/n}\cdot\left[\int_{\Omega\cap A}|Du|dx + \int_{\partial\Omega\cap A}|u|dx\right], \tag{3.60}$$

which, together with (3.51), yields (3.53).

We intend to estimate $\int_{\Omega\cap A}|Du|dx$ by using (3.53) and adapting the reasoning leading to Proposition 3.5. Setting $d(x) = \operatorname{dist}(x,\partial\Omega)$ for $x \in \Omega$ and letting

$$\partial^*\Omega_t = \{x : x \in \Omega,\ \operatorname{dist}(x,\partial\Omega) = t\}, \quad \text{for } t > 0, \tag{3.61}$$

we let the boundary of $\Omega\cap A$ be made up of three parts:

$$\partial(\Omega\cap A) = (\partial\Omega\cap A)\cup(\partial^* A)\cup\big((\partial A\cap\Omega)\setminus\partial^* A\big) \tag{3.62}$$

such that $\partial\Omega\cap A$ is either Lipschitz continuous or of zero n-dimensional Hausdorff measure, $\partial^* A$ is an $(n-1)$-dimensional Lipschitz continuous surface included in $\partial^*\Omega_{\delta_0}$ and several connected $(n-1)$-dimensional surfaces on which if $\partial\Omega\cap A$ is C^2, then we have

$$Dd\cdot\nu_{\Omega\cap A}|_{(\partial A\cap\Omega)\setminus\partial^* A} = 0, \tag{3.63}$$

where we let $\nu_{(\Omega\cap A)}$ be the unit outward normal to $\partial(A\cap\Omega)$ and if $\partial\Omega\cap A$ can be represented as the graph of a Lipschitz continuous function f, then we have that

(B) $(\partial A\cap\Omega)\setminus\partial^* A$ is orthogonal to the coordinate plane of f.

We require $|\partial\Omega\cap A|$, $|\partial A\cap\Omega|$, and δ_0 to be so small and in such a suitable proportion to each other that (3.51) is satisfied for given δ_*. For example, we may take

$$\begin{gathered} \operatorname{diam}\partial\Omega\cap A \le (\delta_0)^{n/(n-1)}, \qquad \operatorname{diam}\partial^* A \le (\delta_0)^{n/(n-1)}, \\ |\partial\Omega\cap A| \ge \frac{1}{2}(\delta_0)^n, \qquad \text{or} \qquad |\partial^* A| \ge \frac{1}{2}(\delta_0)^n. \end{gathered} \tag{3.64}$$

This assures us of the validity of (3.53).

We have the estimate

$$\begin{aligned} \int_{\Omega\cap A}|Du|dx \le{}& \frac{1}{1-\delta_*}|\Omega\cap A| + \frac{1}{1-\delta_*}\int_{\Omega\cap A} H(x,0)dx \\ &+ \frac{1+\delta_*}{1-\delta_*}\int_{\partial\Omega\cap A}|u|d\mathcal{H}_{n-1} - \frac{1}{1-\delta_*}\int_{\partial A\cap\Omega}\beta_A u\,d\mathcal{H}_{n-1} \end{aligned} \tag{3.65}$$

with

$$\beta_{\Omega\cap A} = Tu \cdot \nu_{(\Omega\cap A)} = \frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu_{\Omega\cap A} \tag{3.66}$$

and with $\nu_{\Omega\cap A}$ being the unit outward normal of $\partial A \cap \Omega$. Indeed, motivated by the reasoning leading to Proposition 3.5, we observe that $u|_{\Omega\cap A}$ is a solution of the variational problem

$$J_A^*(v) \longrightarrow \min \quad \text{in } BV(\Omega\cap A), \tag{3.67}$$

where we set

$$\begin{aligned} J_A^*(v) = \int_{\Omega\cap A} \sqrt{1+|Dv|^2}dx + \int_{\Omega\cap A}\int_0^v H(x,t)dt\,dx \\ + \int_{\partial\Omega\cap A} |v-u|\,d\mathcal{H}_{n-1} - \int_{\partial A\cap\Omega} \beta_A v\,d\mathcal{H}_{n-1}. \end{aligned} \tag{3.68}$$

We extend $u|_{\partial(\Omega\cap A)}$ to some function $\varphi_{A,B}$ in $H^{1,1}(B\setminus(\bar{\Omega}\cap\bar{A}))$, for some smooth set $B \supset (\Omega\cap A)$ with $\partial B \supset \partial A\cap\Omega$ such that $\varphi_{A,B}$ has boundary value zero on $\partial B\setminus(\partial A\cap\Omega)$. Then we set

$$\begin{aligned} \tilde{H}_A(x,t) &= \begin{cases} H(x,t), & \text{if } x\in\Omega\cap A, \\ 0, & \text{if } x\in\mathbb{R}^n\setminus(\Omega\cap A), \end{cases} \\ \tilde{K}_A &= \{v\in BV(B) : v|_{B\setminus(\bar{\Omega}\cap\bar{A})} = \varphi_{A,B}\}. \end{aligned} \tag{3.69}$$

Then we let

$$\begin{aligned} \tilde{J}_A(v) &= \int_B \sqrt{1+|Dv|^2}dx + \int_B\int_0^v \tilde{H}_A(x,t)dt\,dx - \int_{\partial A\cap\Omega} \beta_A v\,d\mathcal{H}_{n-1}, \\ J_A(v) &= \int_{\Omega\cap A} \sqrt{1+|Dv|^2}dx + \int_{\Omega\cap A}\int_0^v H(x,t)dt\,dx - \int_{\partial A\cap\Omega} \beta_A v\,d\mathcal{H}_{n-1}. \end{aligned} \tag{3.70}$$

Thus, we have

$$\tilde{J}_A(v) = J_A(v) + \int_{B\setminus(\Omega\cap A)} \sqrt{1+\left|D\varphi_{A,B}\right|^2}dx, \tag{3.71}$$

and to solve problem (3.67) is equivalent to finding a solution of the problem

$$\tilde{J}_A(v) \longrightarrow \min \quad \text{in } \tilde{K}. \tag{3.72}$$

We obtain from (3.53) that

$$\begin{aligned} \tilde{J}_A(u) \ge \int_{B\setminus(\bar{\Omega}\cap\bar{A})} \sqrt{1+\left|D\varphi_{A,B}\right|^2}dx + (1-\delta_*)\int_{\Omega\cap A}|Du|dx \\ -\delta_*\int_{\partial\Omega\cap A}|u|\,d\mathcal{H}_{n-1} - \int_{\partial A\cap\Omega}\beta_{\Omega\cap A}u\,d\mathcal{H}_{n-1}. \end{aligned} \tag{3.73}$$

Now that $\tilde{J}_A(u)$ is estimated from above by

$$J_A(0) + \int_{B\setminus(\bar{\Omega}\cap\bar{A})} \sqrt{1+\left|D\varphi_{A,B}\right|^2}dx, \tag{3.74}$$

we obtain that

$$\begin{aligned}(1-\delta_*)\int_{\Omega\cap A}|Du|dx &\le J_A(0)+\delta_*\int_{\partial\Omega\cap A}|u|d\mathcal{H}_{n-1}-\int_{\partial A\cap\Omega}\beta_{\Omega\cap A}u\,d\mathcal{H}_{n-1}\\ &\le |\Omega\cap A|+\int_{\Omega\cap A}H(x,0)dx\\ &\quad+(1+\delta_*)\int_{\partial\Omega\cap A}|u|d\mathcal{H}_{n-1}-\int_{\partial A\cap\Omega}\beta_{\Omega\cap A}u\,d\mathcal{H}_{n-1},\end{aligned}\tag{3.75}$$

which is (3.65).

To treat the last boundary integral in (3.65) and gain estimates of $\int_\Omega |Du|dx$, we consider a tubular neighborhood of the boundary $\partial\Omega$:

$$\Omega^0=\{x : x\in\Omega,\ \operatorname{dist}(x,\partial\Omega)\le f_0(x)\}\tag{3.76}$$

with $f_0(x)$ being nonnegative, piecewise Lipschitz continuous and $f_0(x)$ being so small that Ω^0 can be covered by sets $A^0_\alpha\cap\Omega$, $\alpha\in\mathbb{I}_0$, with $\mathbb{I}_0$ being a set of indices such that sets with distinct indices can intersect at most on their boundaries and each set A^0_α satisfies condition (3.51), and is of the type described in the previous paragraph for $A=A^0_\alpha$.

Then, for each $\alpha\in\mathbb{I}_0$, (3.65) is valid for $A=A^0_\alpha$. By our choice of the covering $\{A^0_\alpha\cap\Omega\}_{\alpha\in\mathbb{I}_0}$, we decompose the set

$$\bigcup_{\alpha\in\mathbb{I}_0}(\partial A^0_\alpha\cap\Omega)\setminus\partial^*A^0_\alpha\tag{3.77}$$

in such a way that each element in this decomposition belongs to the boundary of exactly two elements of this covering. Since the unit outward normal points in opposite directions along $(\partial A^0_\alpha\cap\Omega)\setminus\partial^*A^0_\alpha$ for each pair of two elements of the covering meeting there, the integral along $(\partial A^0_\alpha\cap\Omega)\setminus\partial^*A^0_\alpha$ vanishes by summing over $\alpha\in\mathbb{I}_0$, and we obtain

$$\begin{aligned}\int_{\Omega^0}|Du|dx\le{}&\frac{1}{1-\delta_*}|\Omega_{\delta_0}|+\frac{1}{1-\delta_*}\int_{\Omega^0}H(x,0)dx\\ &+\frac{1+\delta_*}{1-\delta_*}\int_{\partial\Omega}|u|d\mathcal{H}_{n-1}-\frac{1}{1-\delta_*}\int_{\partial^*\Omega^0}\beta_{\Omega_{\delta_0}}\cdot u\,d\mathcal{H}_{n-1},\end{aligned}\tag{3.78}$$

where we set $\partial^*\Omega^0=\partial\Omega^0\cap\Omega$.

Setting

$$\partial^*\Omega^0_t=\{x : x\in\Omega\setminus\bar{\Omega}^0,\ \operatorname{dist}(x,\partial^*\Omega^0)=t\},\quad \text{for } t>0,\tag{3.79}$$

we now consider a tubular neighborhood of $\partial\Omega^0$:

$$\Omega^1=\{x : x\in\Omega\setminus\bar{\Omega}^0,\ \operatorname{dist}(x,\partial\Omega^0)\le f_1(x)\}\tag{3.80}$$

with f_1 being nonnegative, piecewise Lipschitz continuous and $f_1(x)$ being so small that Ω^1 can be decomposed into sets $\Omega^1\cap A^1_\alpha$, $\alpha\in\bar{\mathbb{I}}_1$, with $\mathbb{I}_1$ being a set of indices. For each $\alpha\in\mathbb{I}_2$, the boundary of $\Omega\cap A^1_\alpha$ consists of $\partial^*\Omega^0\cap A^1_\alpha$, an $(n-1)$-dimensional surface $\partial^* A_\alpha^1$ included in $\partial^*\Omega_{\delta_1}^0$, and several connected $(n-1)$-connected $(n-1)$-dimensional surfaces such that, if $\partial^* A$ is C^2, setting $d_1 = \operatorname{dist}(x, \partial\Omega^1)$, then we have

$$Dd_1 \cdot \nu|_{(\partial A_\alpha^1 \cap \Omega^1)\setminus \partial^* A_\alpha^1} = 0, \tag{3.81}$$

and if $\partial\Omega \cap A$ can be represented as the graph of a Lipschitz continuous function f, then $(\partial A \cap \Omega) \setminus \partial^* A_\alpha^1$ is orthogonal to the coordinate plane of f. Furthermore, δ_1 is sufficiently small that there holds, analogously to (3.51),

$$c_{A_\alpha^1} |\Omega^1 \cap A_\alpha^1| \le \delta_*. \tag{3.82}$$

We further require that each two distinct elements in this covering can intersect at most at their boundaries and $\partial^*\Omega^0$ is decomposed in such a way that each element in this decomposition belongs to the boundary of exactly one element of this covering. Then, we obtain analogously

$$\begin{aligned}\int_{\Omega^1} |Du| dx \le \frac{1}{1-\delta_*} |\Omega^1| + \frac{1}{1-\delta_*}\int_{\Omega^1} H(x,0)dx + \frac{\delta_*}{1-\delta_*}\int_{\partial^*\Omega^0} |u| d\mathcal{H}_{n-1} \\ - \frac{1}{1-\delta_*}\int_{\partial^*\Omega^0} \beta_{\Omega_{\delta_0}} \cdot u\, d\mathcal{H}_{n-1} - \frac{1}{1-\delta_*}\int_{\partial^*\Omega^1} \beta_{\Omega^1_{\delta_1}} \cdot u\, d\mathcal{H}_{n-1},\end{aligned} \tag{3.83}$$

where we set $\partial^*\Omega^1 = \partial\Omega^1 \setminus \partial^*\Omega^0$.

Adding (3.78) and (3.83), the integral of $\beta_{\Omega_{\delta_*}} u$ along $\partial^*\Omega^0$ vanishes and we obtain

$$\begin{aligned}\int_{\Omega^0\cup\Omega^1} |Du| dx \le \frac{1}{1-\delta_*} |\Omega^0 \cup \Omega^1| + \frac{1}{1-\delta_*}\int_{\Omega^0\cup\Omega^1} H(x,0)dx \\ + \frac{1+\delta_*}{1-\delta_*}\int_{\partial\Omega} |u| d\mathcal{H}_{n-1} + \frac{\delta_*}{1-\delta_*}\int_{\partial^*\Omega^0} |u| d\mathcal{H}_{n-1} \\ - \frac{1}{1-\delta_*}\int_{\partial^*\Omega^1} \beta_{\Omega^1_{\delta_1}} \cdot u\, d\mathcal{H}_{n-1}.\end{aligned} \tag{3.84}$$

We then set iteratively Ω^{m+1}, $\mathbb{N}$ to be a tubular neighborhood of $\partial^*\Omega$:

$$\Omega^{m+1} = \{x : x \in \Omega \setminus \bar\Omega^m,\ \operatorname{dist}(x, \partial^*\Omega^m) \le f_{m+1}(x)\}, \tag{3.85}$$

where $\partial^*\Omega^m = (\partial\Omega^m)\setminus\partial^*\Omega^m$, f_{m+1} is nonnegative, piecewise Lipschiz continuous, and with $f_{m+1}(x)$ being sufficiently small such that Ω^{m+1} can be decomposed into $A_\alpha^{m+1} \cap \Omega^{m+1}$, $\alpha \in \mathbb{I}_m$, in a manner analogous to that for Ω^0 and Ω^1 described above. After a finite iteration, the set Ω^{m_*+1} is empty for some $m_* \in \mathbb{N}$, and we finally arrive at the inequality

$$\begin{aligned}\int_\Omega |Du| dx \le |\Omega| + \frac{1}{1-\delta_*}\int_\Omega H(x,0)dx + \frac{1+\delta_*}{1-\delta_*}\int_{\partial\Omega} |u| d\mathcal{H}_{n-1} \\ + \frac{\delta_*}{1-\delta_*}\sum_{m=0}^{m_*}\int_{\partial^*\Omega^m} |u| d\mathcal{H}_{n-1}.\end{aligned} \tag{3.86}$$

To treat the last integral along $\partial\Omega^i$, $0 \le i \le M_*$, we appeal to the results in Sections 4.2.1 and 4.2.2 to conclude that *the last integral in (3.86) approaches* 0 *as* $\delta_* \to 0$. Indeed, by

(4.25), (4.27), (4.35), and (4.38), and our choice of Ω^m, we have

$$\begin{aligned}
&\delta_* \int_{\partial^*\Omega^m} |u|\,d\mathcal{H}_{n-1}\\
&\quad\le \delta_*\sqrt{1+L^2}\int_{\Omega^{m+1}\setminus\Omega^m}|Du|\,dx+\delta_*\sum_{\alpha\in\mathbb{I}_m}\hat{C}_{\partial^*A_\alpha^m,\delta_m}\cdot\int_{A_\alpha^m\cap\Omega}|u|\,dx\\
&\quad\le \delta_*\int_{\Omega^{m+1}\setminus\Omega^m}|Du|\,dx+\delta_*\sum_{\alpha\in\mathbb{I}_m}\hat{C}_{\partial^*A_\alpha^m,\delta_m}\cdot\left|A_\alpha^m\cap\Omega^m\right|^{1/n}\cdot\int_{A_\alpha^m\cap\Omega}|u|^{1+1/n}dx,
\end{aligned}\tag{3.87}$$

where $L=\max_{\alpha\in\mathbb{I}_m}L_\alpha$, L_α is the Lipschitz constant of $\Omega\cap\partial A_\alpha^m$, $\hat{C}_{\partial^*A_\alpha^m,\delta_m}=2(\delta_m)^{-1}$ if $\partial^*A_\alpha^m$ can be represented as the graph of a Lipschitz continuous function, and $\hat{C}_{\partial^*A_\alpha^m}=2\mathcal{K}_{\partial^*A_\alpha^m}+2(\delta_m)^{-1}$ if $\partial^*A_\alpha^m$ is C^2. By (3.51), we have

$$\begin{aligned}
&\delta_*\cdot\hat{C}_{\partial^*A_\alpha^m,\delta_m}\cdot\left|A_\alpha^m\cap\Omega^m\right|^{1/n}\\
&\quad\le 2(\delta_*)^2\Big[\mathfrak{K}_{\partial^*A_\alpha^m}+\max\left(\left|\partial^*A_\alpha^m\right|,\left|\partial A_\alpha^m\cap\partial\Omega^{m-1}\right|\right)\Big]\cdot(c_A)^{-1}\cdot\Big(\sup_\Omega\left|H_0(x)\right|\Big)^{-1},
\end{aligned}\tag{3.88}$$

where

$$\mathfrak{K}_{\partial^*A_\alpha^m}=\begin{cases}\mathcal{K}_{\partial^*A_\alpha^m}, & \text{if } \partial^*A_\alpha^m \text{ is } C^2,\\ 0, & \text{if } \partial^*A_\alpha^m \text{ can be represented as the graph}\\ & \text{of a Lipschitz continuous function.}\end{cases}\tag{3.89}$$

By the Sobolev embedding theorem (cf. [11, (7.30)]), we know that $u\in L^{1+1/n}(\Omega)$. Hence, the last integral in (3.87) approaches zero as $\delta_*\to 0$.

We obtain the following by setting $\delta_*\to 0$ in (3.86) and an application of the modified Sobolev inequality, Lemma 4.8, and Hölder inequality.

Theorem 3.7. *Let u be a solution to the variational problem (3.3) and suppose H satisfies (3.27), (3.38), and (3.52) for some $t_0\in\mathbb{R}$. Suppose Ω is piecewise Lipschitz continuous without outward cusps such that the decomposition $\Omega=\bigcup_{m=0}^{m_*}\Omega_{\delta_m}^m$ of Ω indicated above can be constructed. Then the L^1-norm of $|Du|$ and u can be estimated in terms of $|\Omega|$, $\int_\Omega H(x,0)\,dx$, $\int_{\partial\Omega}|u|\,d\mathcal{H}_{n-1}$, namely,*

$$\begin{aligned}
\int_\Omega |Du|\,dx &\le |\Omega|+\int_\Omega H(x,0)\,dx+\int_{\partial\Omega}|u|\,d\mathcal{H}_{n-1},\\
\int_\Omega |u|\,dx &\le \frac{n}{\omega_n}\cdot|\Omega|^{1+1/n}+\frac{n}{\omega_n}\cdot|\Omega|^{1/n}\cdot\int_\Omega H(x,0)\,dx\\
&\quad+2\frac{n}{\omega_n}\cdot|\Omega|^{1/n}\cdot\int_{\partial\Omega}|u|\,d\mathcal{H}_{n-1}.
\end{aligned}\tag{3.90}$$

From Proposition 3.4 and Theorem 3.7, we obtain the following theorem.

Theorem 3.8. *Under the assumption made on u, H, and Ω, estimates (3.35) are valid for each λ, $0<\lambda<1$, with the constant C_λ,*

$$\begin{aligned} C_\lambda \le &\left[\frac{1}{\lambda}\cdot c_*\cdot ||H_{t_0}||_{L^p(\Omega)}\right]^{(p-n)/(np)} \\ &\cdot\left[\frac{n}{\omega_n}|\Omega|^{1+1/n}+\frac{n}{\omega_n}|\Omega|^{1/n}\int_\Omega H(x,0)dx+2\frac{n}{\omega_n}\left|\Omega^n\right|^{1/n}\int_{\partial\Omega}|u|d\mathcal{H}_{n-1}\right]. \end{aligned} \tag{3.91}$$

4. Estimates for capillary surfaces

4.1. Global estimates for the oscillation of $|u|$ in terms of H and the contraction of j. Estimates for capillary surfaces with $|\cos\theta|$ bounded away from 1. In order to gain estimates for $\sup_{\partial\Omega}u$, $\inf_{\partial\Omega}u$, and $\sup_{\partial\Omega}u-\inf_{\partial\Omega}u$, with u being a minimizing function of the functional $J(u)$, we impose some *additional* restrictions on the domain Ω and the functions $H(x,t)$ and $j(x,t)$. Namely, the function $j(x,\cdot)$ is assumed to be a contraction for $\mathcal{H}_{n-1}$, almost every x in Ω, that is, for some constant a,

$$0\le a\le 1, \tag{4.1}$$

which is independent of x, we have

$$\left|j(x,r)-j(x,s)\right|\le(1-a)\cdot|r-s|. \tag{4.2}$$

Moreover, we assume that

$$j(\cdot,0)\in L^1(\partial\Omega). \tag{4.3}$$

We assume the existence of two positive constants μ and C_Ω depending only on Ω such that in the case where $a>0$, there hold

$$(1-a)\cdot\mu\le 1, \tag{4.4}$$

$$\oint_{\partial\Omega}v\,d\mathcal{H}_{n-1}\le\mu\cdot\int_\Omega|Dv|dx+C_\Omega\cdot\int_\Omega|v|dx \tag{4.5}$$

for all $v\in BV(\Omega)$.

We note that [12, Lemma 1.1] establishes (4.5) for $\mu=1$ in the special case where Ω is a bounded domain with C^2 boundary, and we formulate a generalized version of this result in Section 4.2.1 as Lemma 4.2. An inequality of type (4.5) appears first in [3] with $\mu=\sqrt{1+L^2}$ for any Lipschitz domain with Lipschitz constant L. (See also [17, page 203].) In [5, pages 141–143], this result is extended to include domains in which one or more corners with inward opening angle appear. As pointed out in [5, page 197], this extended result permits inward cusps and even boundary segments that may physically coincide but are adjacent to different parts of Ω. However, it is pointed out in [5, page 143] that an outward cusp or a vertex of an outward corner is not permitted. A modified version of this result will be presented in Section 4.2.3; in particular, the results in Lemma 4.7 permit domains with vertices of outward corners.

Under the assumption that (4.5) holds, a modified Sobolev inequality

$$\|f\|_{n_*}\le c_{**}\cdot\|Df\|_1+\hat{c}_{**}\cdot\|f\|_1 \tag{4.6}$$

is valid for all $f \in W^{1,1}(\Omega)$; here c_{**} and $\hat{c}_{**}$ depend only on n and Ω. This can be reduced to Friedrich's inequality. We will show this in Section 4.2.3.

Global estimates for the oscillation of u can be obtained in the special situation indicated below. Results which are valid in the general situations will be indicated in Section 4.3.

Proposition 4.1. *Under the above assumptions on Ω, H, and j and under the assumption that (3.27) holds for some $t_0 \in \mathbb{R}$, let $a > 0$ and u be a solution of the variational problem (3.3). Furthermore, set*

$$\begin{aligned} a_* = {} & [1-(1-a)\mu]\cdot(c_{**})^{-1} - \|H_{t_0}\|_{L^p(\Omega)}\cdot|\Omega|^{(p-n)/(np)} \\ & - \Big[(1-a)\cdot C_\Omega - [1-(1-a)\cdot\mu]\cdot\hat{c}_{**}\cdot(c_{**})^{-1}\Big]\cdot|\Omega|^{1/n} \end{aligned} \tag{4.7}$$

with

$$c_{**} = \frac{n(\mu+1)}{\omega_n}, \qquad \hat{c}_{**} = \frac{nC_\Omega}{\omega_n}. \tag{4.8}$$

Suppose that

$$a_* > 0. \tag{4.9}$$

Then there exists a constant C_3 determined completely by a_*, n, t_0, $|\Omega|$, and the geometry of Ω such that

$$\sup_\Omega u - \inf_\Omega u \le C_3. \tag{4.10}$$

Proof of Proposition 4.1. Let k be a number greater than $\max(\inf_\Omega u, t_0)$. We set $u_k = \min(u,k)$. Then u_k belongs to $BV(\Omega)$ and the minimizing property of the function u yields

$$J(u) \le J(u_k). \tag{4.11}$$

Adopting again the notation $A(k) = \{x \in \Omega : u(x) \ge k\}$ and for a moment assuming that u is smooth, we obtain

$$\begin{aligned} & \int_{A(k)} \sqrt{1+|Du|^2}\,dx + \int_\Omega \int_{u_k}^{u} H(x,t)\,dt\,dx \\ & \quad + \oint_{\partial\Omega} [j(x,u) - j(x,u_k)]\,d\mathcal{H}_{n-1} \le |A(k)|. \end{aligned} \tag{4.12}$$

Condition (4.2) yields

$$\oint_{\partial\Omega} [j(x,u) - j(x,u_k)]\,d\mathcal{H}_{n-1} \le (1-a)\cdot\oint_{\partial\Omega} |u-u_k|\,d\mathcal{H}_{n-1}, \tag{4.13}$$

which together with (4.5) yield

$$\begin{aligned} & \oint_{\partial\Omega} [j(x,u) - j(x,u_k)]\,d\mathcal{H}_{n-1} \\ & \quad \le (1-a)\cdot\mu\cdot\int_{A(k)} |Dw|\,dx + (1-a)\cdot C_\Omega\cdot\int_{A(k)} |w|\,dx, \end{aligned} \tag{4.14}$$

where we have set

$$w = \max(u-k, 0). \tag{4.15}$$

Inserting (3.38) and (4.14) into (4.12), we obtain

$$[1-(1-a)\cdot\mu]\cdot\int_\Omega |Dw|\,dx + \int_\Omega H_{t_0}\cdot w\,dx - (1-a)\cdot C_\Omega\cdot\int_{A(k)} w\,dx \le |A(k)|. \tag{4.16}$$

This is also valid for $u \in BV(\Omega)$ using an approximation argument.

By the modified Sobolev inequality (4.6) and Hölder's inequality, we obtain

$$\begin{aligned}&\Big[(1-(1-a)\mu)(c_{**})^{-1} - \|H_{t_0}\|_{L^p(\Omega)}\cdot|A(k)|^{(p-n)/np}\\ &\quad -\Big[(1-a)\cdot C_\Omega - (1-(1-a)\mu)\cdot\hat{c}_{**}\cdot(c_{**})^{-1}\Big]\cdot|A(k)|^{1/n}\Big]\cdot\|w\|_{n^*} \le |A(k)|.\end{aligned} \tag{4.17}$$

Thus, by (4.7) and (4.9), we have

$$a_*\cdot\|w\|_{n_*} \le |A(k)|, \tag{4.18}$$

which together with Hölder's inequality imply

$$(h-k)\cdot|A(h)| \le (a_*)^{-1}\cdot|A(k)|^{1+1/n} \tag{4.19}$$

for each $h > k > \max(\inf_\Omega u, t_0)$. This and Lemma 3.6 yield

$$\sup_\Omega u \le \max\Big(\inf_\Omega u, t_0\Big) + 2^{n+1}\cdot(a_*)^{-1}\cdot|\Omega|^{1/n}. \tag{4.20}$$

This completes the proof of Proposition 4.1. □

4.2. Local estimates for the oscillation of u near the boundary. Estimates for capillary surfaces with $|\cos\theta|$ bounded away from 1. We are interested in the situations where in some local sense a modified version of (4.5) holds for some proper subset Γ of $\partial\Omega$; however, (4.9) does not necessarily hold. We will follow the approach taken in [8, pages 176–179]. We consider the capillarity problem (1.8), rather than the variational problem (3.3). Under the assumption that

$$0 < |\cos\theta| = |\beta| < 1-a \quad \text{with } 0 < a < 1, \tag{4.21}$$

for some constant a, we will arrive at local estimates for the oscillation of u indicated in Theorem 4.10, which will give us estimates for the Lipschitz constant near the boundary as indicated in Theorem 4.12. In case (4.4) holds locally on $\partial\Omega$ in the sense indicated in Section 4.3 below, we arrive at estimates of the oscillation of u along the boundary, which, together with Theorem 3.8, gives us global estimates of the oscillation of u as indicated in Theorem 4.11.

We first pay some special attention to the case where (4.4) holds locally on $\partial\Omega$. For this, we present some preliminary results in Sections 4.2.1, 4.2.2, and 4.2.3. These results were used to derive (3.87).

4.2.1. Boundary integrals along piecewise C^2 boundary. The proof of the following lemma can be modified from that of [12, Lemma 1.1] in an obvious way.

Lemma 4.2. *Let E be a Caccioppoli set in $\mathbb{R}^n$, let Γ be a subset of ∂E which is a C^2 manifold, and let $d(x) = \operatorname{dist}(x, \partial E)$ for $x \in E$. Let*

$$E_{\Gamma,t} = \{x \in E : \operatorname{dist}(x,\Gamma) \le t\} \quad \text{for } t > 0. \tag{4.22}$$

Let ε_Γ be so small that the function $d(x)$ is of class C^2 in $E_{\Gamma,\varepsilon_\Gamma}$, and consider, for $0 < \varepsilon' < \varepsilon_\Gamma$, a domain $E^*_{\Gamma,\varepsilon_\Gamma}$:

$$E_{\Gamma,\varepsilon'} \subseteq E^*_{\Gamma,\varepsilon_\Gamma} \subseteq E_{\Gamma,\varepsilon_\Gamma} \tag{4.23}$$

such that on a portion of its boundary, $\partial^* E^*_{\Gamma,\varepsilon_\Gamma} \subset E_{\Gamma,\varepsilon_\Gamma} \setminus E_{\Gamma,\varepsilon'}$, and on the remaining portion of its boundary in Ω,

$$Dd \cdot \nu|_{(\partial E^*_{\Gamma,\varepsilon_\Gamma} \cap \Omega) \setminus \partial^* E^*_{\Gamma,\varepsilon_\Gamma}} = 0 \tag{4.24}$$

with ν being the unit outward normal to $\partial E^*_{\Gamma,\varepsilon_\Gamma}$. Then, there exists a constant $C_{\Gamma,\varepsilon'}$ depending only on Γ and ε' such that the inequality

$$\int_\Gamma w \, d\mathcal{H}_{n-1} \le \int_{E^*_{\Gamma,\varepsilon'}} |Dw| dx + C_{\Gamma,\varepsilon'} \int_{E^*_{\Gamma,\varepsilon'}} |w| dx \tag{4.25}$$

holds for all $w \in BV(E_{\Gamma,\varepsilon_\Gamma})$. In fact, let $\eta_{\varepsilon'}$ be a C^∞ function with

$$0 \le \eta_{\varepsilon'} \le 1, \qquad \eta_{\varepsilon'} = 1 \quad \text{on } \Gamma, \qquad \eta_{\varepsilon'} = 0 \quad \text{in } E \setminus E_{\Gamma,\varepsilon'}, \tag{4.26}$$

then inequality (4.25) holds with

$$C_{\Gamma,\varepsilon'} = \sup_{E^*_{\Gamma,\varepsilon_\Gamma}} |\operatorname{div}(\eta_{\varepsilon'} Dd)|. \tag{4.27}$$

If, in addition,

$$w|_{\partial^* E^*_{\Gamma,\varepsilon_\Gamma}} = 0, \tag{4.28}$$

then inequality (4.25) holds with

$$C_{\Gamma,\varepsilon'} = \sup_{E^*_{\Gamma,\varepsilon_\Gamma}} |\operatorname{div}(Dd)| \tag{4.29}$$

for all $w \in BV(E_{\Gamma,\varepsilon_\Gamma})$.

In order to apply Lemma 4.2, we have to estimate the value of $C_{\Gamma,\varepsilon'}$ in (4.27) and (4.29). For this, we formulate the following result which is well known and can be found, for example, in [11, pages 420–422].

Lemma 4.3. *Let $\Gamma \subseteq \partial E$ be of class C^2 whose principal curvatures are bounded in absolute value by $\mathcal{K}_\Gamma$. Then $d(x) = \operatorname{dist}(x,\Gamma)$ is of class C^2 in $E_{\Gamma,\varepsilon_\Gamma}$, for $\varepsilon_\Gamma \le 1/\mathcal{K}_\Gamma$, where $E_{\Gamma,\varepsilon_\Gamma}$ is given in (4.22).*

Furthermore, for points $\bar{x}$ in $E_{\Gamma,\varepsilon_\Gamma}$, $\varepsilon_\Gamma \le 1/\mathcal{K}_{\partial E}$, define $\bar{y} = y(\bar{x})$ to be the (unique) nearest point of Γ to $\bar{x}$. Consider the special coordinate frame in which the x_n-axis is oriented along the inward normal to Γ at $\bar{y}$ and the coordinates $x_1,\dots,x_{n-1}$ lie along the principal directions of Γ at the point $\bar{y}$. In these special coordinates, there hold at $\bar{x}$,

$$Dd = (0,\dots,0,1), \tag{4.30}$$

$$D^2 d = \text{diagonal}\left[\frac{-k_1}{1-k_1 d},\dots,\frac{-k_{n-1}}{1-k_{n-1}d},0\right], \tag{4.31}$$

where $k_1,\dots,k_{n-1}$ are the principal curvatures of Γ at $\bar{y}$.

Inserting (4.30) and (4.31) into (4.27) and (4.29), we obtain the following.

Lemma 4.4. *Let $\Gamma \subseteq \partial E$ be of class C^2 whose principal curvatures are bounded in absolute value by $\mathcal{K}_\Gamma$. Then, for $\varepsilon_\Gamma \le 1/\mathcal{K}_\Gamma$ and for each δ, $0 < \delta \le 1$, there holds in (4.27) that*

$$C_{\Gamma,\varepsilon'} \le |D\eta_{\varepsilon'}| + 2(n-1)\mathcal{K}_\Gamma \le \left(\frac{1+\delta}{\varepsilon'}\right) + 2(n-1)\mathcal{K}_\Gamma, \tag{4.32}$$

and in (4.29) that

$$C_{\Gamma,\varepsilon'} \le 2(n-1)\mathcal{K}_\Gamma. \tag{4.33}$$

4.2.2. Domains with piecewise Lipschitz continuous boundary. In this section, we formulate some results in connection with (4.4) for piecewise Lipschitz continuous domains. We first consider a portion of $\partial\Omega$ which can be represented as the graph of a Lipschitz function.

Lemma 4.5. *Let E be a Caccioppoli set in $\mathbb{R}^n$ and let Γ be a subset of ∂E which can be represented over some $(n-1)$-dimensional domain D by a Lipschitz function $f(x_1,\dots,x_{n-1})$ with Lipschitz constant L_Γ. Suppose the strip*

$$\mathcal{S}_{\Gamma,\varepsilon} = \{(x_1,\dots,x_{n-1}) \in D,\ -\varepsilon < x_n - f(x_1,\dots,x_{n-1}) < 0\} \tag{4.34}$$

lies in E for $0 < \epsilon < \tilde{\varepsilon}_\Omega$. Then, for $\varepsilon' < \tilde{\varepsilon}_\Omega$, there exists a constant $\tilde{C}_{\Gamma,\varepsilon'}$ depending only on Γ and ε' such that the inequality

$$\int_\Gamma w\, d\mathcal{H}_{n-1} \le \sqrt{1+(L_\Gamma)^2}\int_{\mathcal{S}^*_{\Gamma,\varepsilon'}} |Dw|\,dx + \tilde{C}_{\Gamma,\varepsilon'}\int_{\mathcal{S}^*_{\Gamma,\varepsilon'}} |w|\,dx \tag{4.35}$$

holds for all $w \in BV(S^*_{\Gamma,\varepsilon'})$, with domains $\mathcal{S}^*_{\Gamma,\varepsilon'}$ satisfying

$$\mathcal{S}_{\Gamma,\varepsilon'} \subseteq \mathcal{S}^*_{\Gamma,\varepsilon'} \subseteq \mathcal{S}_{\Gamma,\tilde{\varepsilon}_\Omega}. \tag{4.36}$$

In fact, letting $\eta_{\varepsilon'}$ be a C^∞ function with

$$0 \le \eta_{\varepsilon'} \le 1, \qquad \eta_{\varepsilon'} = 1 \quad \text{on } \Gamma, \qquad \eta_{\varepsilon'} = 0 \quad \text{in } \partial\mathcal{S}_{\Gamma,\varepsilon'} \setminus \partial E, \tag{4.37}$$

then inequality (4.35) holds with

$$\tilde{C}_{\Gamma,\varepsilon'} = \sup_{\mathcal{S}_{\Gamma,\varepsilon'}} |D\eta_{\varepsilon'}| \le 2(\varepsilon')^{-1}. \tag{4.38}$$

If $w = 0$ on $\partial\mathcal{S}^*_{\Gamma,\varepsilon} \setminus (\partial\mathcal{S}_{\Gamma,\varepsilon'} \cap \mathcal{S}_{\Gamma,\tilde{\varepsilon}_\Omega})$, then inequality (4.35) holds with

$$\tilde{C}_{\Gamma,\varepsilon'} = 0. \tag{4.39}$$

We now formulate a global result for domains with piecewise Lipschitz continuous boundary.

Lemma 4.6. *Let $E \in \mathbb{R}^n$ be bounded with ∂E being piecewise Lipschitz continuous. Suppose the tubular neighborhood E_ε of ∂E,*

$$E_\varepsilon = \{x : x \in E,\ \operatorname{dist}(x, \partial E) \le \varepsilon\}, \tag{4.40}$$

is covered by a partition of unity with particular properties; namely, suppose E_ε is covered by a finite number N of sets E_i, $1 \le i \le N$, each of which is open in $\bar{E}$ and to each of which is associated a nonnegative function $\phi_i \in C_0^\infty(\mathbb{R}^n)$ such that $\sum_j \phi_j(x) = 1$ for all $x \in E_\varepsilon$ and each $\partial E \cap E_i$ can be represented by a Lipschitz function f_i of $(n-1)$ variables for which strips with width ε,

$$\begin{aligned} \mathcal{S}_{\partial E \cap E_i,\varepsilon} = \{&(x_1,\ldots,x_{i,n-1},x_{i,n}),(x_{i,1},\ldots,x_{i,n-1}) \in D_i, \\ &-\varepsilon < x_{i,n} - f(x_{i,1},\ldots,x_{i,n-1}) < 0\}, \end{aligned} \tag{4.41}$$

are disjoint to each other. Let the Lipschitz constant of f_i be L_i, $1 \le i \le N$. Then, the inequality

$$\oint_{\partial E} v\, d\mathcal{H}_{n-1} \le \mu \int_E |Dv|\,dx + C_E \int_E |v|\,dx, \tag{4.42}$$

with

$$\mu = \sqrt{1 + \left(\max_j L_j\right)^2}, \qquad C_E = \sum_j \sup_{\operatorname{supp}\phi_j} |D\phi_j| + 2\varepsilon^{-1}, \tag{4.43}$$

is valid for all $v \in BV(E)$.

If there is a set E_* including the tubular neighborhood E_ε of ∂E such that $v = 0$ on $\partial E_* \cap E$, then inequality (4.42) is valid with $\mu = \sqrt{1 + (\max_j L_j)^2}$ and

$$C_E = \sum_j \sup_{\operatorname{supp}\phi_j} |D\phi_j|; \tag{4.44}$$

the same is true for those $w \in BV(E)$ with $\{x : x \in E, w(x) = 0\}$ being of positive $(n-1)$-dimensional Hausdorff measure and dividing $\partial\Omega$ into two connected portions intersecting with each other at their endpoints.

Other estimates for constants in (4.42) are available for domains with piecewise Lipschitz continuous boundary.

Lemma 4.7. *Let $E \in \mathbb{R}^n$ be bounded with ∂E being piecewise Lipschitz continuous; that is, ∂E can be decomposed into*

$$\partial E = \bigcup_{i=1}^{N_0} \partial_i E \tag{4.45}$$

such that $\partial_i E$, $1 \le i \le N_0$, can be represented as the graph of a Lipschitz function f_i of $(n-1)$ variables $(x_{i,1},\dots,x_{i,n-1})$ over an $(n-1)$-dimensional domain D_i. Suppose that the tubular neighborhood E_ε of ∂E, where E_ε is given in (4.40), can be covered by strips of width ε,

$$\begin{aligned} \mathcal{S}_{\partial_i E,\varepsilon} = \{&(x_1,\dots,x_{i,n-1},x_{i,n}),(x_{i,1},\dots,x_{i,n-1}) \in D_i, \\ &-\varepsilon < x_{i,n} - f(x_{i,1},\dots,x_{i,n-1}) < 0\}, \end{aligned} \tag{4.46}$$

and that each point in E_ε is included in at most N_1 such strips. Denoting by L_i the Lipschitz constant of f_i, then inequality (4.42) with

$$\mu = N_1 \sqrt{1 + \Big(\max_{1\le j\le N_0} L_j\Big)^2}, \qquad C_E = 2N_1\varepsilon^{-1} \tag{4.47}$$

is valid for all $v \in BV(E)$.

If there is a set E_* including the tubular neighborhood E_ε of ∂E such that $v = 0$ on $\partial E_* \cap E$, then inequality (4.42) is valid with $\mu = N_1\sqrt{1+(\max_{1\le j\le N_0} L_j)^2}$ and

$$C_E = 0; \tag{4.48}$$

the same is true for those $w \in BV(E)$ with $\{x : x \in E, w(x) = 0\}$ being of positive $(n-1)$-dimensional Hausdorff measure and dividing $\partial\Omega$ into two connected portions intersecting with each other at their endpoints.

4.2.3. Modified Sobolev inequality. We will give a proof of the modified Sobolev inequality (4.5) and estimate the constant involved in this inequality. This result has been used to prove Proposition 4.1 and will be used to prove Theorem 4.10.

To derive the modified Sobolev inequality (4.5), we first formulate the following result which is a special case of the so-called Friedrich inequality.

Lemma 4.8 (cf. [18, Theorem 6.5.7]). *Suppose E is a Caccioppoli set with piecewise Lipschitz continuous boundary. Then for any $f \in BV(E)$, the inequality*

$$\|f\|_{L^{n*}(E)} \le \frac{n}{\omega_n}\left(\int_E |Df|\,dx + \int_{\partial E} |f|\,d\mathcal{H}_{n-1}\right) \tag{4.49}$$

is valid, where ω_n is the Lebesgue measure of the unit n-dimensional ball.

Inserting (4.25) and (4.42) into (4.49), we obtain the following.

Proposition 4.9. *If inequality (4.42) holds, and given $f \in BV(\Omega)$, the boundary strip E_ε adjacent to $\{x : x \in \partial E : f(x) \ne 0\}$ and with width ε is included in E (cf. (4.40)), then* *the inequality*

$$\|f\|_{L^{n*}(E)} \le \frac{n(\mu+1)}{\omega_n}\int_E |Df|\,dx + \frac{n\cdot C_E}{\omega_n}\int_E |f|\,dx \tag{4.50}$$

is valid with

$$\mu = 1, \qquad C_E = 2(n-1)\mathcal{K}_{\partial E} + 2\varepsilon^{-1} \tag{4.51}$$

if ∂E is piecewise C^2, and

$$\mu = \sqrt{1+L^2}, \qquad C_E = \sum_j \sup|D\phi_j| + 2\varepsilon^{-1} \tag{4.52}$$

if ∂E is piecewise Lipschitz continuous with Lipschitz constant L such that there exists a partition of unity for E_ε satisfying the conditions indicated in Lemma 4.6, and

$$\mu = N_1\sqrt{1+L^2}, \qquad C_E = 2N_1\varepsilon^{-1} \tag{4.53}$$

if ∂E is piecewise Lipschitz continuous such that, with decomposition of ∂E into graphs of Lipschitz continuous functions, each of the associated boundary strips cannot intersect more than N_1 others, as indicated in Lemma 4.6.

If $\{x : x \in E, w(x) = 0\}$ is of positive $(n-1)$-dimensional Hausdorff measure and divides $\partial\Omega$ into two connected portions intersecting with each other at their endpoints, then inequality (4.50) holds with

$$C_E = 2(n-1)\mathcal{K}_{\partial E}, \qquad \sum_j \sup|D\phi_j|, \qquad 0, \tag{4.54}$$

respectively in the three cases indicated above.

4.2.4. Consider the capillarity problem (1.8) such that (4.21) holds for some constant a.

Let A be a set with a portion of the boundary $\partial^* A$ included in $\partial^*\Omega_{\delta_0}$ such that $\Omega\cap A$ satisfies (3.63), (B), and

$$\begin{aligned} &\operatorname{diam}\partial\Omega\cap A \le (\delta_0)^{1+\varepsilon}, \qquad &&\operatorname{diam}\partial^* A \le (\delta_0)^{1+\varepsilon},\\ &|\partial\Omega\cap A| \ge \left(\frac{\delta_0}{2}\right)^{(1+\varepsilon)(n-1)}, &&|\partial^* A| \ge \left(\frac{\delta_0}{2}\right)^{(1+\varepsilon)(n-1)}, \end{aligned} \tag{4.55}$$

for a constant δ_0, with $\delta_0 \le (\mathcal{K}_{\partial\Omega\cap A})^{-1}$ in case $\partial\Omega\cap\overline{A} \in C^2$ and δ_0 being so small that the boundary strip $\mathcal{S}_{\partial\Omega\cap A,\delta_0}$ defined in (4.34) is in $\Omega\cap\overline{A}$ in case $\partial\Omega\cap A$ can be represented as a graph of a Lipschitz continuous function. Furthermore, let $(\Omega\cap\partial A)\setminus\partial^* A$ be made up of gradient trajectories of u; that is,

$$Du\cdot\nu_{\Omega\cap A}|_{(\Omega\cap\partial A)\setminus\partial^* A} \le 0. \tag{4.56}$$

Choosing δ_0 to be so small that each component of $(\Omega\cap\partial A)\setminus\partial^* A$ can be represented as the graph of a Lipschitz function with Lipschitz constant L_{**} and that

$$\operatorname{diam}\partial^*\Omega_t\cap A \le (\delta_0)^{1+\varepsilon} \quad \text{for each } t,\ 0 < t < \delta_0, \tag{4.57}$$

and that there holds

$$\begin{aligned}(1-a)\mu<1,\\ \lambda(1-(1-a)\mu)\cdot\left(\frac{(n/\omega_n)-\hat{C}_{\partial\Omega\cap\bar{A}}\cdot\delta_0}{1+\mu+\sqrt{1+(L_{**})^2}}\right)-(1-a)\hat{C}_{\partial\Omega\cap\bar{A}}\delta_0\\ -\|H_{t_0}\|_{L^p(\Omega)}\cdot(\delta_0)^{(p-n)/p}\ge 0,\end{aligned} \tag{4.58}$$

where $\mu=1$, $\hat{C}_{\partial\Omega\cap\bar{A}}=2(n-1)\mathcal{K}_{\partial\Omega\cap\bar{A}}$ if $\partial\Omega\cap\overline{A}$ is C^2 and $\mu=\sqrt{1+L^2}$, $\hat{C}_{\partial\Omega\cap\bar{A}}=0$ if $\partial\Omega\cap\overline{A}$ can be represented as the graph of a Lipschitz continuous function with Lipschitz constant L. The reason for the choice of such a constant $\hat{C}_{\partial\Omega\cap\bar{A}}$ will be made clear in Section 4.5, where we will show the following.

Theorem 4.10. *Let u be a solution to the variational problem (1.8) which is of class $C^2(\Omega)$. Suppose H satisfies (3.38) and (3.27) for some $t_0\in\mathbb{R}$. Let A be a set with $\partial\Omega\cap\overline{A}$ being either C^2 or the graph of a Lipschitz continuous function. Suppose that (4.21) is satisfied for all $x\in\partial\Omega\cap\overline{A}$ in which $1>a>0$ is a constant and (4.55), (4.56), (4.57), and (4.58) are satisfied with the constant δ_0. Suppose that $\beta(x)$ is continuous in $\partial\Omega\cap\overline{A}$. If $\beta(x)>0$ for all $x\in\partial\Omega\cap\overline{A}$, then the inequality*

$$\sup_{\partial\Omega\cap A}u\le C_A^*\cdot\max\left(\inf_{\partial\Omega\cap A}u,t_0,0\right)+C_A^*\cdot C_A^{**}\cdot|\Omega\cap A|^{1/n} \tag{4.59}$$

holds true, and if $\beta(x)<0$ for all $x\in\partial\Omega\cap\overline{A}$, then the inequality

$$\inf_{\partial\Omega\cap A}u\ge C_A^*\cdot\min\left(\sup_{\partial\Omega\cap A}u,t_0,0\right)-C_A^*\cdot C_A^{**}\cdot|\Omega\cap A|^{1/n} \tag{4.60}$$

holds true; thus, if $\beta(x)>0$ for all $x\in\partial\Omega\cap\overline{A}$ and if $\inf_{\partial\Omega}u\ge\max(t_0,0)$,

$$\sup_{\partial\Omega\cap A}u-\inf_{\partial\Omega\cap A}u\le(C_A^*-1)\cdot\inf_{\partial\Omega\cap A}u+C_A^*\cdot C_A^{**}\cdot|\Omega\cap A|^{1/n}. \tag{4.61}$$

If $\beta(x)<0$ for all $x\in\partial\Omega\cap\overline{A}$ and if $\sup_{\partial\Omega}u\le\min(t_0,0)$, then

$$\sup_{\partial\Omega\cap A}u-\inf_{\partial\Omega\cap A}u\le-(C_A^*-1)\cdot\sup_{\partial\Omega\cap A}u+C_A^*\cdot C_A^{**}\cdot|\Omega\cap A|^{1/n} \tag{4.62}$$

holds true. Here

$$C_A^*=\left(1-2\sqrt{2}\cdot(\delta_0)^{\varepsilon(n-1)/n}\cdot C_A^{**}\right)^{-1}, \tag{4.63}$$

$$C_A^{**}=2^{n+1}\cdot(1-\lambda)^{-1}\cdot(1-(1-a)\mu)^{-1}\cdot\hat{\hat{C}}_{\partial\Omega\cap A}, \tag{4.64}$$

with

$$\hat{\hat{C}}_{\partial\Omega\cap A}=\frac{(n/\omega_n)-\hat{C}_{\partial\Omega\cap\bar{A}}\delta_0}{1+\mu+\sqrt{1+(L_{**})^2}}. \tag{4.65}$$

4.3. Global estimates for the oscillation of $|u|$ for capillary surfaces with $|\cos\theta|$ bounded away from 1. We pay some special attention to the case where (4.22) or (4.34) holds locally on $\partial\Omega$, that is, a sufficiently small tubular neighborhood Ω_{δ_0} of the boundary $\partial\Omega$ can be covered by sets $\Omega\cap A_\alpha$, $\alpha\in\mathbb{N}$, such that each set A_α is of the type indicated above, which satisfies (4.55) and (4.57) and for which (4.22) or (4.34) is valid with $\Gamma = \partial\Omega\cap\overline{A}$. We note that we here allow elements in this covering with distinct indices to intersect at a set of positive n-dimensional Hausdorff measure. If condition (4.21) holds for all $x\in\partial\Omega$ and δ_0 is so small that (4.29) is satisfied for sets $\Omega\cap A_\alpha$ in such a covering of Ω_{δ_0}, we obtain estimates of $\sup_{\partial\Omega} u$, $\inf_{\partial\Omega} u$, and $\sup_{\partial\Omega} u-\inf_{\partial\Omega} u$ from (4.59), (4.60), and (4.61). Combining with Theorem 3.8, we obtain estimates of $\sup_\Omega u-\inf_\Omega u$. Thus, we have the following.

Theorem 4.11. *Suppose that (4.22) or (4.34) holds locally in the sense indicated above; in particular, $\partial\Omega$ is piecewise Lipschitz continuous without outward cusps. Suppose that (3.27) holds for some $t_0\in\mathbb{R}$. If u is a solution of (1.8) such that $\cos\theta(x)=\beta(x)$ satisfies condition (4.21) for all $x\in\partial\Omega$ and is a piecewise continuous function on $\partial\Omega$, and if $H(x,t)$ is bounded in $\overline{\Omega}\times\mathbb{R}$, then*

$$\sup_\Omega u-\inf_\Omega u \tag{4.66}$$

can be estimated in terms of t_0, n, $\|H_{t_0}\|_{L^p(\Omega)}$, $\int_\Omega H(x,0)dx$, $|\Omega|$, a, and the geometry of Ω.

4.4. Boundary regularity for capillary surfaces. From Theorem 4.10, we obtain the following for solutions to the capillarity problem with $|\cos\theta|$ being bounded away from 1 and 0.

Theorem 4.12. *Let u be a bounded solution to (1.1) and (1.3). Suppose for $x_0\in\partial\Omega$, and for positive constants $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, $\tilde{\beta},\tilde{\tilde{\beta}}\le 1$ and a ball $B_R(x_0)$ intersecting the interior of Ω, that assumptions (A1) and (A2) hold. Furthermore, assumption (1.14) on H holds. Assume that (3.38) holds. Then the trace of u on $\partial\Omega$ is Lipschitz continuous locally in $\partial\Omega\cap B_R(x_0)$.*

The Lipschitz norm of u near x_0 depends only on H, $\tilde{\beta}$, $\tilde{\tilde{\beta}}$, and a constant $\tilde{C}_{\partial\Omega\cap\bar{A}}$ depending on the geometry of Ω, where $\tilde{C}_{\partial\Omega\cap\bar{A}}=\mathcal{K}_{\partial\Omega\cap B_R(x_0)}$ in the case where the portion of $\partial\Omega\cap B_R(x_0)$ is C^2 and $\tilde{C}_{\partial\Omega\cap\bar{A}}=\sqrt{1+L^2}$ in the case where $\partial\Omega\cap B_R(x_0)$ is Lipschitz continuous with Lipschitz constant L; here $\mathcal{K}_{\partial\Omega\cap B_R(x_0)}$ is an upper bound of the absolute value of the principal curvatures of $\partial\Omega\cap B_R(x_0)$ in the case where $\partial\Omega\cap B_R(x_0)$ is C^2.

To see that Theorem 4.10 implies Theorem 4.12, we notice that C_A^* and L_{**} in Theorem 4.10 approach the respective values 1 and 0 as $\delta_0\to 0$. Thus, from letting $\delta_0\to 0$ and letting $\varepsilon\to 0$, we obtain Theorem 4.12 after a possible renormalization which makes $\inf_{\partial\Omega} u\le 0$ or $\sup_{\partial\Omega} u\ge 0$.

We emphasize again that Theorems 4.12 and 2.6 yield the *Hölder continuity with exponent* $1/2$ *up to the boundary* locally in $\partial\Omega\cap B_R(x_0)$, under the assumptions on $\cos\theta$, $\partial\Omega\cap B_R(x_0)$ indicated in Theorem 4.12, the assumption that $\inf_{\partial\Omega} u\le 0$ or $\sup_{\partial\Omega} u\ge 0$, and the assumption that H *is nonnegative and bounded above.*

4.5. Proof of Theorem 4.10. We set η to be a smooth function such that

$$0 \le \eta \le 1, \quad \eta\,|_{\partial\Omega\cap\overline{A}} = 1, \tag{4.67}$$
$$\operatorname{supp}\eta \subseteq (\overline{\Omega\cap A}) \setminus \partial^* A. \tag{4.68}$$

Let k be a number greater than $\max(\inf_{\partial\Omega\cap\bar{A}} u, t_0, 0)$. We set

$$u_k = (1-\eta)u + \min(\eta u, k). \tag{4.69}$$

Then, u_k belongs to $BV(\Omega)$, and using the notation

$$A(k,\eta) = \{x \in \Omega : \eta u > k\}, \tag{4.70}$$

we obtain from the minimizing property of u,

$$J_k(u) \le J_k(u_k), \tag{4.71}$$

where

$$J_k(v) = \int_{A(k,\eta)} \sqrt{1+|Dv|^2}\,dx + \int_{\partial A(k,\eta)} \beta_{A(k,\eta)}\, v\, d\mathcal{H}_{n-1}. \tag{4.72}$$

Assume for a moment that u is smooth. We obtain from this

$$\begin{aligned}
&\int_{A(k,\eta)} \sqrt{1+|Du|^2}\,dx + \int_\Omega \int_{u_k}^{u} H(x,t)\,dt\,dx - \int_{\partial\Omega\cap\overline{A(k,\eta)}} \beta\cdot(\eta u - k)\,d\mathcal{H}_{n-1} \\
&\quad\le \int_{A(k,\eta)} \sqrt{1+\left|D[(1-\eta)u]^2\right|}\,dx + \int_{\partial^{**}A\cap\overline{A(k,\eta)}} \beta_{\Omega\cap A}\cdot(\eta u - k)\,d\mathcal{H}_{n-1} \\
&\quad\le \int_{A(k,\eta)} \sqrt{1+(1-\eta)^2|Du|^2}\,dx + \int_{A(k,\eta)} \sqrt{1+u^2|D\eta|^2}\,dx,
\end{aligned} \tag{4.73}$$

where we set

$$\partial^{**}A = (\Omega\cap A) \setminus \partial^* A \tag{4.74}$$

and where the last inequality is obtained from (4.56) and the inequality $\sqrt{1+|a+b|^2} \le \sqrt{1+|a|^2} + \sqrt{1+|b|^2}$. Taking into account the inequality

$$\sqrt{1+t^2} - \sqrt{1+(1-\eta)^2 t^2} \ge t - [1+(1-\eta)t] = \eta t - 1, \tag{4.75}$$

we obtain from (4.73)

$$\begin{aligned}
&\int_{A(k,\eta)} |D(\eta u)|\,dx + \int_\Omega \int_{u_k}^{u} H(x,t)\,dt\,dx \\
&\quad\le 2|A(k,\eta)| + 2\left(\sup_\Omega |D\eta|\right)\cdot \int_{A(k,\eta)} u\,dx + (1-a)\cdot \int_{\partial\Omega\cap\overline{A(k,\eta)}} (\eta u - k)\,d\mathcal{H}_{n-1}.
\end{aligned} \tag{4.76}$$

We set $w = \max(\eta u - k, 0)$. The monotonicity condition (1.5) on $H(x,\cdot)$ yields

$$\int_{u_k}^{u} H(x,t)dt \geq H(x,t_0)\cdot(u-u_k) = H_{t_0} w. \tag{4.77}$$

Inserting this into (4.76), we obtain

$$\begin{aligned}&\int_{A(k,\eta)} |Dw|\,dx + \int_{A(k,\eta)} H_{t_0} w\,dx\\ &\quad\leq 2|A(k,\eta)| + 2\Big(\sup_{\Omega}|D\eta|\Big)\cdot\int_{A(k,\eta)} u\,dx + (1-a)\cdot\int_{\partial\Omega\cap\overline{A(k,\eta)}} w\,d\mathcal{H}_{n-1},\end{aligned} \tag{4.78}$$

which will also be valid for $u \in BV(\Omega)$ using an approximation argument.

By the modified Sobolev inequality (4.49) with constants concerned in (4.51), we have

$$\begin{aligned}&\frac{n}{\omega_n}\|w\|_{L^{n*}(A(k,\eta))}\\ &\quad\leq \int_{A(k,\eta)} |Dw|\,dx + \int_{\partial\Omega\cap\overline{A(k,\eta)}} w\,d\mathcal{H}_{n-1} + \int_{\partial^{**}A\cap\overline{A(k,\eta)}} w\,d\mathcal{H}_{n-1},\end{aligned} \tag{4.79}$$

in which, from (4.68), Lemmas 4.2 and 4.5, we obtain

$$\int_{\partial^{**}A\cap\overline{A(k,\eta)}} w\,d\mathcal{H}_{n-1} \leq \sqrt{1+(L_{**})^2}\cdot\int_{A(k,\eta)} |Dw|\,dx, \tag{4.80}$$

$$\int_{\partial\Omega\cap\overline{A(k,\eta)}} w\,d\mathcal{H}_{n-1} \leq \mu\cdot\int_{A(k,\eta)} |Dw|\,dx + \hat{C}_{\partial\Omega\cap\bar{A}}\cdot\int_{A(k,\eta)} w\,dx, \tag{4.81}$$

where the constant $\hat{C}_{\partial\Omega\cap\bar{A}}$ takes the value indicated immediately below (4.58) in Section 4.3. Inserting (4.80) and (4.81) into (4.79), we obtain

$$\begin{aligned}&\frac{n}{\omega_n}\|w\|_{L^{n*}(A(k,\eta))}\\ &\quad\leq \Big(1+\mu+\sqrt{1+(L_{**})^2}\Big)\cdot\int_{A(k,\eta)} |Dw|\,dx + \hat{C}_{\partial\Omega\cap\bar{A}}\cdot\int_{A(k,\eta)} w\,dx,\end{aligned} \tag{4.82}$$

from which and from Hölder inequality, we obtain

$$\int_{A(k,\eta)} |Dw|\,dx \geq \left(\frac{(n/\omega_n) - \hat{C}_{\partial\Omega\cap\bar{A}}|A(k,\eta)|^{1/n}}{1+\mu+\sqrt{1+(L_{**})^2}}\right)\cdot\|w\|_{L^{n*}(A(k,\eta))}. \tag{4.83}$$

Assuming that (3.27) holds, we can derive, by the reasoning leading to (3.30),

$$\left|\int_{A(k,\eta)} H_{t_0} w\,dx\right| \leq \|w\|_{L^{n*}(A(k,\eta))}\cdot\|H_{t_0}\|_p\cdot|A(k,\eta)|^{(p-n)/(np)}. \tag{4.84}$$

Inserting (4.81), (4.83), and (4.84) into (4.79), we obtain

$$\begin{aligned}&\left[(1-(1-a)\cdot\mu)\cdot\left(\frac{(n/\omega_n)-\hat{C}_{\partial\Omega\cap\bar{A}}\cdot|A(k,\eta)|^{1/n}}{1+\mu+\sqrt{1+(L_{**})^2}}\right)\right.\\&\quad\left.-(1-a)\hat{C}_{\partial\Omega\cap\bar{A}}|A(k,\eta)|^{1/n}-\|H_{t_0}\|_p\cdot|A(k,\eta)|^{(p-n)/(np)}\right]\cdot\|w\|_{L^{n*}(A(k,\eta))}\\&\qquad\le 2|A(k,\eta)|+2\Big(\sup_{\Omega}|D\eta|\Big)\cdot\int_{A(k,\eta)}u\,dx.\end{aligned}\tag{4.85}$$

From (4.58) and (4.85), we obtain

$$\begin{aligned}&(1-\lambda)\cdot[1-(1-a)\cdot\mu]\cdot\big(\hat{\hat{C}}_{\partial\Omega\cap A}\big)^{-1}\|w\|_{n^*}\\&\quad\le 2|A(k,\eta)|+2\Big(\sup_{\Omega}|D\eta|\Big)\cdot\int_{A(k,\eta)}u\,dx,\end{aligned}\tag{4.86}$$

with $\hat{\hat{C}}_{\partial\Omega\cap A}$ being given in the statement of Theorem 4.10. This and Hölder's inequality then yield

$$\begin{aligned}(h-k)\cdot|A(h,\eta)|&\le 2(1-\lambda)^{-1}[1-(1-a)\cdot\mu]^{-1}\cdot\hat{\hat{C}}_{\partial\Omega\cap A}\\&\quad\cdot\left[|A(k,\eta)|^{1+1/n}+\Big(\sup_{\Omega}|D\eta|\Big)\cdot|A(k,\eta)|^{1/n}\cdot\int_{A(k,\eta)}u\,dx\right]\end{aligned}\tag{4.87}$$

for each $h>k>\max(\inf_{\partial\Omega\cap\bar{A}}u,t_0,0)$.

We have

$$\int_{A(k,\eta)}u\,dx\le\Big(\sup_{\Omega\cap A}u\Big)\cdot|A(k,\eta)|;\tag{4.88}$$

since $\beta(x)>0$ for all $x\in\partial\Omega\cap\bar{A}$, we have

$$\sup_{\Omega\cap A}u=\sup_{\partial\Omega\cap\bar{A}}u,\tag{4.89}$$

and hence, by the identity in (4.67), we have

$$\sup_{\Omega\cap A}u=\sup_{\partial\Omega\cap\bar{A}}\eta u.\tag{4.90}$$

Inserting this into (4.88), we obtain

$$\int_{\Omega\cap A}u\,dx\le\Big(\sup_{\partial\Omega\cap\bar{A}}\eta u\Big)\cdot|A(k,\eta)|.\tag{4.91}$$

From (4.87), (4.91), and (3.14) in Lemma 3.2, we obtain

$$\sup_{\Omega\cap A}\eta u\le\max\Big(\inf_{\partial\Omega\cap A}u,t_0,0\Big)+C_A^{**}\cdot\Big(\sup_{\Omega}|D\eta|\Big)\cdot\Big(\sup_{\partial\Omega\cap\overline{A}}\eta u\Big)\cdot|\Omega\cap A|^{1/n}\tag{4.92}$$

with C_A^{**} being given by (4.64), which yields (4.59) with

$$C_A^* \leq \left[1 - C_A^{**} \cdot \left(\sup_\Omega |D\eta|\right) \cdot |\Omega \cap A|^{1/n}\right]^{-1}, \tag{4.93}$$

which is less than the right-hand side of (4.63) by (4.55) and (4.57).

Analogously, we can obtain (4.60) in case that $\beta(x) < 0$ for all $x \in \partial\Omega \cap \overline{A}$.

4.6. The capillarity problem possibly with $\cos\theta$ being 0 or π. Cases where $H(x,t)$ satisfies the growth conditions (3.10) and (3.11). For the capillarity problem with boundary contact angle not bounded away from 0 and/or π, we will treat only the cases where $H(x,t)$ satisfies the growth conditions (3.10) and (3.11). We will prove the following.

Theorem 4.13. *Let u be a variational solution to (3.3) for which $H(x,t)$ satisfies the growth conditions (3.10) and (3.11). Suppose $\partial\Omega$ is piecewise Lipschitz continuous without outward cusps. Then, with the constant C_Ω given in (4.5), there exist two numbers $\hat{t}_0$ and $\hat{\hat{t}}_0$ satisfying the respective conditions*

$$\inf_{x\in\Omega} H(x,\hat{t}_0) > -C_\Omega, \tag{4.94}$$

$$\sup_{x\in\Omega} H(x,\hat{\hat{t}}_0) < C_\Omega, \tag{4.95}$$

for which there hold

$$\sup_\Omega u \leq \max\left(\inf_\Omega u, \hat{t}_0\right) + 2^{n+1} \cdot \left(\inf_{x\in\Omega} H(x,\hat{t}_0) - C_\Omega\right)^{-1} \cdot |\Omega|^{1/n}, \tag{4.96}$$

$$\inf_\Omega u \geq \min\left(\sup_\Omega u, \hat{\hat{t}}_0\right) - 2^{n+1} \cdot \left(C_\Omega + \sup_{x\in\Omega} H(x,\hat{\hat{t}}_0)\right)^{-1} \cdot |\Omega|^{1/n}. \tag{4.97}$$

Proof of Theorem 4.13. Assume that $H(x,t)$ satisfies the growth conditions (3.10) and (3.11). Then, assuming that $\partial\Omega$ is of class C^2, we have $\mu = 1$ in (4.4). Allowing $a = 0$ in (4.1), we obtain from (4.16)

$$\int_\Omega H_t \cdot w\,dx - C_\Omega \cdot \int_{A(k)} w\,dx \leq |A(k)|, \tag{4.98}$$

where

$$w = \max(u-k,0), \qquad A(k) = \{x : x \in \Omega,\ u(x) \geq k\}, \tag{4.99}$$

for $k > \max(\inf_\Omega u, t)$. Under the assumption of (3.10), there exists a number $\hat{t}_0$ such that (4.59) holds. We obtain

$$\left(\inf_{x\in\Omega} H(x,\hat{t}_0) - C_\Omega\right) \cdot \int_\Omega w\,dx \leq |A(k)| \tag{4.100}$$

for $k > \max(\inf_\Omega u, \hat{t}_0)$. This and Lemma 3.2 yield (4.96).

Analogously, the growth condition (3.11) gives us a number $\hat{\hat{t}}_0$ satisfying (4.95), which yields analogously

$$\Big(-C_\Omega - \sup_{x\in\Omega} H(x,\hat{\hat{t}}_0)\Big)\cdot\int_\Omega (u+k)dx \le |\{x : x\in\Omega,\ u(x)\le -k\}| \tag{4.101}$$

for $k > \min(-\sup_\Omega u, -\hat{\hat{t}}_0)$, and hence (4.97) follows from Lemma 3.2. □

Acknowledgment. I wish to express my thanks to the referees for assisting me in revising this paper. Without their help, this paper reads as a rough draft rather than a finished paper.

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