# INCLUSION RESULTS FOR CONVOLUTION SUBMETHODS

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If $B$ is a summability matrix, then the submethod $B_\lambda$ is the matrix obtained by deleting a set of rows from the matrix $B$. Comparisons between Euler-Knopp submethods and the Borel summability method are made. Also, an equivalence result for convolution submethods is established. This result will necessarily apply to the submethods of the Euler-Knopp, Taylor, Meyer-König, and Borel matrix summability methods.

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**1. Introduction and notation.** Let $E$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and consider $E$ as the range of a strictly increasing sequence of nonnegative integers, say $E := \{\lambda(n)\}_{n=0}^{\infty}$. If $B := (b_{n,k})$ is a summability matrix, then the submethod $B_\lambda$ is the matrix whose $nk$th entry is $B_\lambda[n,k] := b_{\lambda(n),k}$. Thus, for a given sequence $x$, the $B_\lambda$-transform of $x$ is the sequence $B_\lambda x$ with

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} := \sum_{k=0}^{\infty} b_{\lambda(n),k} x_k. \tag{1.1}$$

Since $B_\lambda$ is a row submatrix of $B$, it is regular (i.e., limit preserving) whenever $B$ is regular.

Row submatrices have appeared throughout the literature [5, 6, 8, 12], but they were first studied as a class unto themselves by Goffman and Petersen [7], and later by Steele [14]. The class of Cesàro submethods has been studied by Armitage and Maddox [1] and Osikiewicz [11].

Let $A$ and $B$ be two summability matrices. If every sequence which is $A$-summable is also $B$-summable to the same limit, then $B$ includes $A$, denoted by $A \subseteq B$. Also, $B$ is called a triangle if $b_{n,k} = 0$ for all $k > n$ and $b_{n,n} \neq 0$ for all $n$. The following lemma extends [1, Theorem 1].

**LEMMA 1.1.** *Let $B$ be a summability matrix and let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$.*

*(1) If $F \setminus E$ is finite, then $B_\lambda \subseteq B_\rho$.*

*(2) If $B$ is a triangle and $B_\lambda \subseteq B_\rho$, then $F \setminus E$ is finite.*

*(3) If $B$ is a triangle, then $B_\lambda$ is equivalent to $B_\rho$ if and only if the symmetric difference $E \triangle F$ is finite.*

*In particular, $B \subseteq B_\lambda$ for any $\lambda$.*

**PROOF.** Assume $F \setminus E$ is finite and let $x$ be a sequence that is $B_\lambda$-summable to $L$. Then there exists an $N$ such that $\{\rho(n) : n \geq N\} \subseteq E$. That is, $\{\rho(n) : n \geq N\}$ is a

subsequence of $\{\lambda(n)\}$. Since $\lim_n (B_\lambda x)_n = \lim_n (Bx)_{\lambda(n)} = L$, we have $\lim_n (B_\rho x)_n = \lim_n (Bx)_{\rho(n)} = L$.

Now assume $B$ is a triangle, and hence invertible, and $F \setminus E$ is infinite. Let $F \setminus E := \{\rho(n(j))\}_{j=0}^\infty$ with $\rho(n(j)) < \rho(n(j+1))$. Consider the sequence $y$ defined by

$$y_k := \begin{cases} (-1)^j, & \text{if } k = \rho(n(j)) \text{ for some } j, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

and let $x$ be the sequence $B^{-1}y$. Then, for every $n$,

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} = \left(B(B^{-1}y)\right)_{\lambda(n)} = y_{\lambda(n)} = 0. \tag{1.3}$$

Hence, $\lim_n (B_\lambda x)_n = 0$. However, for every $j$,

$$(B_\rho x)_{n(j)} = (Bx)_{\rho(n(j))} = \left(B(B^{-1}y)\right)_{\rho(n(j))} = y_{\rho(n(j))} = (-1)^j. \tag{1.4}$$

Thus $x$ is not $B_\rho$-summable. Therefore $B_\rho$ does not include $B_\lambda$, which completes the contrapositive of assertion (2). Lastly, assertion (3) follows from (1) and (2) since $E \triangle F := (E \setminus F) \cup (F \setminus E)$. □

To show the reason for the necessity of $B$ being a triangle in assertion (2) of Lemma 1.1, consider the matrix $B$ whose $nk$th entry is

$$B[n,k] := \begin{cases} 0, & \text{if } n \text{ even and } k \neq \dfrac{n}{2}, \\ 1, & \text{if } n \text{ even and } k = \dfrac{n}{2}, \\ 0, & \text{if } n \text{ odd and } n \neq k, \\ 1, & \text{if } n \text{ odd and } n = k. \end{cases} \tag{1.5}$$

Then if $\lambda(n) := 2n$ and $\rho(n) := 2n+1$, $F \setminus E$ is infinite and $B_\lambda \subseteq B_\rho$.

**2. Inclusion results for Euler-Knopp submethods.** For $r \in \mathbb{C} \setminus \{0,1\}$, the Euler-Knopp method of order $r$ is given by the matrix $E_r$ whose $nk$th entry is

$$E_r[n,k] := \begin{cases} \dbinom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \tag{2.1}$$

For the case $r = 1$, $E_1$ is the identity matrix, and $E_0$ is the matrix whose $nk$th entry is

$$E_0[n,k] := \begin{cases} 1, & \text{if } k = 0, n = 0,1,2,\ldots, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

It is well known that $E_r$ is regular if and only if $0 < r \leq 1$ (see [4]).

Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r \in \mathbb{C} \setminus \{0, 1\}$. The submethod $E_{r,\lambda}$ is the matrix whose $nk$th entry is

$$E_{r,\lambda}[n,k] := \begin{cases} \dbinom{\lambda(n)}{k} r^k (1-r)^{\lambda(n)-k}, & \text{if } k \leq \lambda(n), \\ 0, & \text{if } k > \lambda(n). \end{cases} \tag{2.3}$$

Then $E_{r,\lambda}$ is regular if and only if $E_r$ is regular.

By a direct application of Lemma 1.1, we have the following inclusion result for the $E_{r,\lambda}$ methods.

**LEMMA 2.1.** *Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$ and $r \neq 0$.*
*(1) The method $E_{r,\lambda} \subseteq E_{r,\rho}$ if and only if $F \setminus E$ is finite.*
*(2) The method $E_{r,\lambda}$ is equivalent to $E_{r,\rho}$ if and only if the symmetric difference $E \bigtriangleup F$ is finite.*

We now examine the relationship between $E_{r,\lambda}$ and the Borel summability method. Recall that a sequence $x$ is Borel summable to $L$ if

$$\lim_{t \to \infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = L. \tag{2.4}$$

**THEOREM 2.2.** *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r > 0$. Then the Borel summability method includes $E_{r,\lambda}$ if and only if $S := (\mathbb{N} \cup \{0\}) \setminus E$ is finite.*

**PROOF.** If $S$ is finite, then by Lemma 2.1, $E_r$ and $E_{r,\lambda}$ are equivalent. But the Borel summability method includes $E_r$ for $r > 0$ (see [4]). Hence, it also includes $E_{r,\lambda}$. If $S$ is infinite, then it may be written as a strictly increasing sequence of nonnegative integers, say $S := \{\rho(m)\}_{m=0}^{\infty}$. If $M_n := \max_{0 \leq k \leq n} |E_r[n,k]|$, consider the sequence $y$ defined by

$$y_n := \begin{cases} (\rho(m)!)^2 (\rho(m)+1) M_{\rho(m)}, & \text{if } n = \rho(m), \\ 0, & \text{otherwise,} \end{cases} \tag{2.5}$$

and let $x$ be the sequence $E_r^{-1} y$; that is, $y = E_r x$ and

$$\lim_{n \to \infty} (E_{r,\lambda} x)_n = \lim_{n \to \infty} (E_r x)_{\lambda(n)} = \lim_{n \to \infty} y_{\lambda(n)} = 0. \tag{2.6}$$

Hence, $x$ is $E_{r,\lambda}$-summable to $0$. Now observe that for a given $n$,

$$|y_n| = |(E_r x)_n| \leq \sum_{k=0}^{n} |E_r[n,k]| \, |x_k| \leq M_n \sum_{k=0}^{n} |x_k|. \tag{2.7}$$

Thus, for $n = \rho(m)$, we have

$$
\begin{aligned}
(\rho(m)!)^{1/\rho(m)} &= \left( \frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \cdot \frac{|y_{\rho(m)}|}{M_{\rho(m)}} \right)^{1/\rho(m)} \\
&\leq \left( \frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \sum_{k=0}^{\rho(m)} |x_k| \right)^{1/\rho(m)}.
\end{aligned}
\tag{2.8}
$$

Since $\limsup_m (\rho(m)!)^{1/\rho(m)} = \infty$,

$$
\limsup_{m \to \infty} \left( \frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \sum_{k=0}^{\rho(m)} |x_k| \right)^{1/\rho(m)} = \infty,
\tag{2.9}
$$

and it follows that $\limsup_n (|x_n|/n!)^{1/n} = \infty$. Thus, $\sum_{k=0}^{\infty} (x_k/k!) t^k$ diverges for all nonzero $t$ and hence $x$ is not Borel summable. $\qquad\square$

**THEOREM 2.3.** *There exists a sequence which is Borel summable but not $E_{r,\lambda}$-summable for any $\lambda$ and $r > 0$.*

**PROOF.** Let $r > 0$ and consider the sequence $x$ defined by

$$
x_n := n\left( -\frac{1}{r} \right)\left( 1 - \frac{2}{r} \right)^{n-1}.
\tag{2.10}
$$

Then it can be shown that $(E_{r,\lambda} x)_n = (-1)^{\lambda(n)} \lambda(n)$. Hence $x$ is not $E_{r,\lambda}$-summable for any $\lambda$. However,

$$
\begin{aligned}
e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} &= e^{-t} \sum_{k=1}^{\infty} \left[ k\left( -\frac{1}{r} \right)\left( 1 - \frac{2}{r} \right)^{k-1} \right] \frac{t^k}{k!} \\
&= \left( -\frac{1}{r} \right) e^{-t} \sum_{k=1}^{\infty} \left( 1 - \frac{2}{r} \right)^{k-1} \frac{t^k}{(k-1)!} \\
&= \left( -\frac{1}{r} \right) t e^{-t} \sum_{k=0}^{\infty} \left( 1 - \frac{2}{r} \right)^k \frac{t^k}{k!} \\
&= \left( -\frac{1}{r} \right) t e^{-t} e^{(1-2/r)t} \\
&= \left( -\frac{1}{r} \right) t e^{-(2/r)t}.
\end{aligned}
\tag{2.11}
$$

Since $r > 0$,

$$
\lim_{t \to \infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = \lim_{t \to \infty} \left( -\frac{1}{r} \right) t e^{-(2/r)t} = 0,
\tag{2.12}
$$

and hence $x$ is Borel summable to 0. $\qquad\square$

**3. Convolution methods.** Let $p$ and $q$ be sequences of real numbers with $p_k \geq 0$, $q_k \geq 0$, $\sum_{k=0}^{\infty} p_k = 1$, and $\sum_{k=0}^{\infty} q_k = 1$. The convolution summability method is given by the matrix $C^* := (c_{n,k})$ whose $nk$th entry is

$$c_{n,k} := \begin{cases} q_k, & \text{if } n = 0, \\ \displaystyle\sum_{j=0}^{k} c_{n-1,j} p_{k-j}, & \text{if } n \geq 1. \end{cases} \tag{3.1}$$

It is clear that $C^*$ is a nonnegative matrix such that for every $n$, $\sum_{k=0}^{\infty} c_{n,k} = 1$. Some classical summability matrices are examples of the matrix $C^*$. If $0 \leq r \leq 1$, $p := \{1 - r, r, 0, 0, \ldots\}$, and $q := \{1, 0, 0, \ldots\}$, then $C^*$ is the Euler-Knopp method of order $r$. If $0 \leq r < 1$, $p := \{0, (1-r), (1-r)r, (1-r)r^2, \ldots\}$, and $q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\}$, then $C^*$ is the Taylor method of order $r$, denoted by $T_r$. If $0 < r < 1$ and $p := q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\}$, then $C^*$ is the Meyer-König method of order $r$, denoted by $S_r$. If $p := q := \{1/k!e\}$, then $C^*$ is the Borel matrix method $B^*$. Similar forms of the convolution method are known by different names, such as the random-walk method and Sonnenschein method. (Further information on all of these methods may be found in [3, 4, 13].)

If $C^*$ is the convolution method formed from the sequences $p$ and $q$, then let

$$\mu := \sum_{j=0}^{\infty} j p_j, \qquad \nu := \sum_{j=0}^{\infty} j q_j. \tag{3.2}$$

We note here that for the remainder of this work, $p$ and $q$ are nonnegative sequences whose sums are 1, and $\mu$ and $\nu$ represent the sums in (3.2). Also, $c_{n,k} := 0$ whenever $k < 0$.

We next present some preliminary results concerning the convolution method.

**LEMMA 3.1.** *The convolution method $C^*$ is regular if and only if $p_0 < 1$.*

**PROOF.** See [9]. $\square$

**LEMMA 3.2.** *If $\mu < \infty$ and $\nu < \infty$, then for every $n$,*

$$\sum_{k=0}^{\infty} k c_{n,k} = n\mu + \nu. \tag{3.3}$$

**PROOF.** Note that for $n = 0$, the result holds. So assume the result holds for some integer $n > 0$. Then

$$\sum_{k=0}^{\infty} k c_{n+1,k} = \sum_{k=0}^{\infty} k \left( \sum_{j=0}^{k} c_{n,j} p_{k-j} \right) = \sum_{j=0}^{\infty} c_{n,j} \sum_{k=j}^{\infty} k p_{k-j}$$
$$= \sum_{j=0}^{\infty} c_{n,j} \left( \sum_{i=0}^{\infty} i p_i + j \sum_{i=0}^{\infty} p_i \right) = \sum_{j=0}^{\infty} \mu c_{n,j} + \sum_{j=0}^{\infty} j c_{n,j} = (n+1)\mu + \nu. \tag{3.4}$$

By induction, the result follows. $\square$

**LEMMA 3.3.** *Let $C^*$ be the convolution method formed from the sequences $p$ and $q$ and $D^* := (d_{n,k})$ the convolution method formed from the sequences $p$ and $\tilde{q} := \{1, 0, 0, \ldots\}$. Then for nonnegative integers $n$, $k$, and $j$,*

$$c_{n+j,k} = \sum_{i=0}^{k} c_{n,k-i} d_{j,i}. \tag{3.5}$$

The proof of this lemma is a straightforward induction argument left to the reader.

**LEMMA 3.4.** *Let $C^*$ be the convolution method formed from the sequences $p$ and $q$. If $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$, then*

$$\sum_{k=0}^{\infty} |c_{n,k+1} - c_{n,k}| = O\left(\frac{1}{\sqrt{n}}\right). \tag{3.6}$$

**PROOF.** Let $D^* := (d_{n,k})$ be the convolution method formed from the sequences $p$ and $\tilde{q} := \{1, 0, 0, \ldots\}$. We first prove that the result holds for $D^*$.

Let $\phi(t) := (\sqrt{2\pi} e^{t^2/2})^{-1}$ and $x_{n,k} := (k - n\mu)/\sigma \sqrt{n}$, where $\sigma^2 := \sum_{j=0}^{\infty} (j - \mu)^2 p_j$. Then

$$\sqrt{n} \sum_{k=0}^{\infty} |d_{n,k+1} - d_{n,k}| \leq \sqrt{n} \sum_{k=0}^{\infty} \left| d_{n,k+1} - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) \right|$$

$$+ \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) \right| \tag{3.7}$$

$$+ \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) - d_{n,k} \right|.$$

The first and the third terms on the right-hand side of the inequality are bounded by a result of Bikjalis and Jasjunas [2]. For the middle term, the mean value theorem yields

$$\sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) \right| = \frac{1}{\sigma} \sum_{k=0}^{\infty} |\phi'(\xi_{n,k})| (x_{n,k+1} - x_{n,k})$$

$$< \frac{K}{\sigma} \int_{\mathbb{R}} |\phi'(t)| \, dt < \infty, \tag{3.8}$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$ and $K > 0$ is some constant. Thus, the result holds for the convolution method $D^*$. Then, by Lemma 3.3,

$$\sum_{k=0}^{\infty} |c_{n,k+1} - c_{n,k}| = \sum_{k=0}^{\infty} \left| \sum_{i=0}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^{k} q_{k-i} d_{n,i} \right|$$

$$= \sum_{k=0}^{\infty} \left| q_{k+1} d_{n,0} + \sum_{i=1}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^{k} q_{k-i} d_{n,i} \right|$$

$$\leq p_0^n \sum_{k=0}^{\infty} q_{k+1} + \sum_{k=0}^{\infty} \sum_{i=0}^{k} q_{k-i} |d_{n,i+1} - d_{n,i}|$$

$$\leq p_0^n + \sum_{i=0}^{\infty} |d_{n,i+1} - d_{n,i}| \sum_{k=i}^{\infty} q_{k-i}$$

$$= p_0^n + \sum_{i=0}^{\infty} |d_{n,i+1} - d_{n,i}| = O\left(\frac{1}{\sqrt{n}}\right).$$

(3.9)

$\square$

**4. Equivalence results for convolution submethods.** Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$. The convolution submethod $C_\lambda^*$ is the matrix whose $nk$th entry is

$$C_\lambda^*[n,k] := C^*[\lambda(n),k]. \tag{4.1}$$

**LEMMA 4.1.** *The convolution submethod $C_\lambda^*$ is regular if and only if $p_0 < 1$.*

**PROOF.** If $p_0 < 1$, then $C^*$ is regular and hence $C_\lambda^*$ is also regular. Conversely, if $C_\lambda^*$ is regular and $p_0 = 1$, then $C_\lambda^*[n,k] = q_k$ for all $n$ and $k$. Since $\sum_{k=0}^{\infty} q_k = 1$, there exists a $\hat{k}$ such that $q_{\hat{k}} \neq 0$. Then $\lim_n C_\lambda^*[n,\hat{k}] = q_{\hat{k}} \neq 0$, which contradicts the regularity of $C_\lambda^*$. $\square$

The following theorem compares $C_\lambda^*$ with $C^*$ for bounded sequences.

**THEOREM 4.2.** *Let $C^*$ be the convolution method formed from the sequences $p$ and $q$ with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty}(j-\mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$. If*

$$\lim_{n \to \infty} \frac{\lambda(n+1) - \lambda(n)}{\sqrt{\lambda(n)}} = 0, \tag{4.2}$$

*then $C^*$ and $C_\lambda^*$ are equivalent for bounded sequences.*

**PROOF.** By Lemma 1.1, $C^* \subseteq C_\lambda^*$ for any $\lambda$. So assume $\lim_n (\lambda(n+1) - \lambda(n))/\sqrt{\lambda(n)} = 0$ and let $x$ be a bounded sequence that is $C_\lambda^*$-summable to $L$. Consider the set $S := \{\rho(n)\} := (\mathbb{N} \cup \{0\}) \setminus E$. If $S$ is finite, then Lemma 1.1 shows that $C_\lambda^*$ and $C^*$ are equivalent for all sequences. So assume $S$ is infinite. Then there exists an $N$ such that for $n \geq N$, $\rho(n) > \lambda(0)$. Since $E$ and $S$ are disjoint, for $n \geq N$, there exists an integer $m$ such that $\lambda(m) < \rho(n) < \lambda(m+1)$. We write $\rho(n) := \lambda(m) + j$, where $0 < j < \lambda(m+1) - \lambda(m)$. Then, for $n \geq N$,

$$|(C_\rho^* x)_n - (C_\lambda^* x)_m| = \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m),k} x_k \right|$$

$$= \left| \sum_{k=0}^{\infty} c_{\lambda(m)+j,k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m),k} x_k \right|. \tag{4.3}$$

By Lemma 3.3, this becomes

$$
\begin{aligned}
\left| (C_\rho^* x)_n - (C_\lambda^* x)_m \right| &= \left| \sum_{k=0}^\infty \left( \sum_{i=0}^\infty c_{\lambda(m),k-i} d_{j,i} \right) x_k - \sum_{k=0}^\infty c_{\lambda(m),k} x_k \right| \\
&= \left| \sum_{k=0}^\infty x_k \left[ \left( \sum_{i=0}^\infty c_{\lambda(m),k-i} d_{j,i} \right) - \left( \sum_{i=0}^\infty c_{\lambda(m),k} d_{j,i} \right) \right] \right| \\
&\le \|x\|_\infty \sum_{k=0}^\infty \sum_{i=0}^\infty d_{j,i} \left| c_{\lambda(m),k-i} - c_{\lambda(m),k} \right| \\
&= \|x\|_\infty \sum_{i=0}^\infty d_{j,i} \sum_{k=0}^\infty \left| \sum_{l=0}^{i-1} c_{\lambda(m),k-l} - c_{\lambda(m),k-l-1} \right| \\
&\le \|x\|_\infty \sum_{i=0}^\infty d_{j,i} \sum_{k=0}^\infty \sum_{l=0}^{i-1} \left| c_{\lambda(m),k-l} - c_{\lambda(m),k-l-1} \right| \\
&= \frac{\|x\|_\infty}{\sqrt{\lambda(m)}} \sum_{i=0}^\infty d_{j,i} \sum_{l=0}^{i-1} \sqrt{\lambda(m)} \sum_{k=0}^\infty \left| c_{\lambda(m),k-l} - c_{\lambda(m),k-l-1} \right|.
\end{aligned}
\tag{4.4}
$$

By Lemma 3.4, there exists an $M > 0$ such that

$$
\sqrt{\lambda(m)} \sum_{k=0}^\infty \left| c_{\lambda(m),k-l} - c_{\lambda(m),k-l-1} \right| < M.
\tag{4.5}
$$

Then, by Lemma 3.2,

$$
\left| (C_\rho^* x)_n - (C_\lambda^* x)_m \right| \le \frac{\|x\|_\infty}{\sqrt{\lambda(m)}} \sum_{i=0}^\infty d_{j,i} \sum_{l=0}^{i-1} M = \frac{\|x\|_\infty M}{\sqrt{\lambda(m)}} \sum_{i=0}^\infty i d_{j,i} \le \frac{\|x\|_\infty M}{\sqrt{\lambda(m)}} \cdot j\mu.
\tag{4.6}
$$

Since $0 < j < \lambda(m+1) - \lambda(m)$,

$$
\left| (C_\rho^* x)_n - (C_\lambda^* x)_m \right| < \|x\|_\infty M\mu \cdot \frac{\lambda(m+1) - \lambda(m)}{\sqrt{\lambda(m)}} = o(1).
\tag{4.7}
$$

Thus,

$$
0 \le \left| (C_\rho^* x)_n - L \right| \le \left| (C_\rho^* x)_n - (C_\lambda^* x)_m \right| + \left| (C_\lambda^* x)_m - L \right| = o(1) + o(1) = o(1).
\tag{4.8}
$$

Therefore, the sequence $C^* x$ may be partitioned into two disjoint subsequences, namely $(C_\lambda^* x)_n = (C^* x)_{\lambda(n)}$ and $(C_\rho^* x)_n = (C^* x)_{\rho(n)}$, each having the common limit $L$. Thus, $x$ must be $C^*$-summable to $L$, and hence $C^*$ and $C_\lambda^*$ are equivalent for bounded sequences.                                                                         □

The following theorem is a well-known result due to Meyer-König (see [10, Theorem 25]).

**THEOREM 4.3.** *The methods $E_r$ $(0 < r < 1)$, $S_r$ $(0 < r < 1)$, $T_r$ $(0 < r < 1)$, and the Borel method are equivalent for bounded sequences.*

Since the Euler-Knopp methods of order $0 < r < 1$, Taylor methods of order $0 < r < 1$, Meyer-König methods of order $0 < r < 1$, and the Borel matrix method all have generating sequences satisfying the conditions in Theorem 4.2, the following corollary is immediate.

**COROLLARY 4.4.** *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $0 < r < 1$. If $\lambda$ satisfies condition (4.6), then $E_{r,\lambda}$, $E_r$, $T_{r,\lambda}$, $T_r$, $S_{r,\lambda}$, $S_r$, $B_\lambda^*$, $B^*$, and the Borel method are all equivalent for bounded sequences.*

The next theorem presents an equivalence relationship between the $C_\lambda^*$ submethods.

**THEOREM 4.5.** *Let $C^*$ be the convolution method formed from the sequences $p$ and $q$ with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$. If*

$$\lim_{n \to \infty} \frac{\rho(n) - \lambda(n)}{\sqrt{\lambda(n)}} = 0, \tag{4.9}$$

*then $C_\lambda^*$ and $C_\rho^*$ are equivalent for bounded sequences.*

**PROOF.** Let $x$ be a bounded sequence and consider the sequences $M(n) := \max\{\lambda(n), \rho(n)\}$ and $m(n) := \min\{\lambda(n), \rho(n)\}$. We write $M(n) := m(n) + j$, where $j := M(n) - m(n)$. For $n \geq 1$, we have

$$
\begin{aligned}
\left| (C_\rho^* x)_n - (C_\lambda^* x)_n \right| &= \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(n),k} x_k \right| \\
&= \left| \sum_{k=0}^{\infty} c_{M(n),k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right| \\
&= \left| \sum_{k=0}^{\infty} c_{m(n)+j,k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right|.
\end{aligned}
\tag{4.10}
$$

Then, as in the proof of Theorem 4.2, we have

$$
\begin{aligned}
\left| (C_\rho^* x)_n - (C_\lambda^* x)_n \right| &\leq O(1) \frac{j}{\sqrt{m(n)}} = O(1) \frac{M(n) - m(n)}{\sqrt{m(n)}} \\
&= O(1) \sqrt{\frac{\lambda(n)}{m(n)}} \frac{|\rho(n) - \lambda(n)|}{\sqrt{\lambda(n)}} \\
&= O(1) \cdot O(1) \cdot o(1) = o(1).
\end{aligned}
\tag{4.11}
$$

Then if $x$ is $C_\lambda^*$-summable to $L$,

$$
\begin{aligned}
0 \leq \left| (C_\rho^* x)_n - L \right| &\leq \left| (C_\rho^* x)_n - (C_\lambda^* x)_n \right| + \left| (C_\lambda^* x)_n - L \right| \\
&= o(1) + o(1) = o(1).
\end{aligned}
\tag{4.12}
$$

Similarly, if $x$ is $C_\rho^*$-summable to $L$, then

$$0 \le \left| (C_\lambda^* x)_n - L \right| \le \left| (C_\rho^* x)_n - (C_\lambda^* x)_n \right| + \left| (C_\rho^* x)_n - L \right|$$
$$= o(1) + o(1) = o(1). \tag{4.13}$$

Thus, $C_\lambda^*$ and $C_\rho^*$ are equivalent for bounded sequences.         □

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