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ON ALMOST AUTOMORPHIC SOLUTIONS OF LINEAR OPERATIONAL-DIFFERENTIAL EQUATIONS

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We prove almost periodicity and almost automorphy of bounded solutions of linear differential equations $x'(t) = Ax(t) + f(t)$ for some class of linear operators acting in a Banach space.

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1. Introduction. Throughout this note, X will be a Banach space equipped with the norm topology and X^* its dual space. $\langle \cdot, \cdot \rangle$ will denote the duality between X and X^*.

We recall that a (strongly) continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that

$$\lim_{n\to\infty} f(t+s_n) = g(t), \tag{1.1}$$

$$\lim_{n\to\infty} g(t-s_n) = f(t), \tag{1.2}$$

pointwise on $\mathbb{R}$.

f is said to be weakly almost automorphic if (1.1) and (1.2) are replaced, respectively, by

$$\operatorname*{weak-lim}_{n\to\infty} f(t+s_n) = g(t), \tag{1.3}$$

$$\operatorname*{weak-lim}_{n\to\infty} g(t-s_n) = f(t), \tag{1.4}$$

for each $t \in \mathbb{R}$.

When convergence in (1.1) and (1.2) is uniform in $t \in \mathbb{R}$, f is said to be (Bochner) almost periodic. Almost periodic functions are characterized by the following so-called Bochner's criterion.

$f : \mathbb{R} \to X$ is almost periodic if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(f(t+s_n))$ is uniformly convergent in $t \in \mathbb{R}$.

For more information on almost automorphic and almost periodic functions, see, for instance, [2, 4].

We consider both differential equations

$$x'(t) = Ax(t), \quad t \in \mathbb{R}, \tag{1.5}$$

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}. \tag{1.6}$$

Recently, we proved in [1] that if A is the infinitesimal generator of a C_0-group of bounded linear operators, then every solution of (1.3) with a relatively compact range in X is almost periodic. In Section 2 below, we will investigate the case in which A is a nilpotent operator, that is, there exists a natural number n such that $A^n = \theta$. In Section 3, we deal with (1.4) where A is of simplest type (see definition below). We prove that if f is almost automorphic, then every solution is almost automorphic too. We also give a property of (1.5) when A generates a C_0-group of bounded linear operators.

2. We first recall a result by Zaidman [4].

Proposition 2.1. *If A is nilpotent, then every solution of (1.1) is constant.*

Now we state and prove the following result.

Theorem 2.2. *If A is nilpotent, then every solution of (1.1) with a relatively compact range in X is almost periodic.*

Proof. Let $x(t)$ be a solution of (1.1) with a relatively compact range in X. Then, it is bounded. So, by the result above, it is constant over $\mathbb{R}$, that is,

$$||x(t)|| = ||x(0)||, \quad t \in \mathbb{R}. \tag{2.1}$$

Fix $s \in \mathbb{R}$ and consider $y_s(t) = x(t+s)$, $t \in \mathbb{R}$. Then, $y_s(t)$ is also a bounded solution of (1.1), so $\|y_s(t)\| = \|y_s(0)\|$ for all $t \in \mathbb{R}$. Now fix s_1, s_2 in $\mathbb{R}$. Then, $y_{s_1}(t) - y_{s_2}(t)$ is a bounded solution of (1.1), so,

$$||y_{s_1}(t) - y_{s_2}(t)|| = ||y_{s_1}(0) - y_{s_2}(0)|| \tag{2.2}$$

which gives

$$||x(t+s_1) - x(t+s_2)|| = ||x(s_1) - x(s_2)||. \tag{2.3}$$

Consider a given sequence (s'_n) in $\mathbb{R}$. Since $x(t)$ has relatively compact range in X, then there exists $(s_n) \subset (s'_n)$ such that $(x(s_n))$ is convergent, thus Cauchy. Given $\epsilon > 0$, there exists N such that $\|x(s_n) - x(s_m)\| < \epsilon$, if $n, m > N$ so that for every $t \in \mathbb{R}$,

$$||x(t+s_n) - x(t+s_m)|| < \epsilon \tag{2.4}$$

which proves that $x(t)$ is almost periodic. □

Theorem 2.3. *In a reflexive Banach Space X, if A is nilpotent, then every bounded solution of (1.1) is weakly almost periodic.*

The proof is based on the fact that in a reflexive Banach space, every sequence which is bounded in norm has weakly convergent subsequent (see [3, Theorem 1, page 26]). We apply the second part of the proof of the previous theorem to $\langle \phi, x(t) \rangle$ (arbitrary $\phi \in X^*$) to complete the proof.

3. It is known that if $A \in \pounds(X)$, where X is a finite-dimensional space and f is almost automorphic, then every bounded solution of (1.6) is almost automorphic [2]. We generalize this result here for a uniformly convex Banach space X and A of simplest type, that is, $A \in \pounds(X)$ and

$$A = \sum_{j=1}^{n} \alpha_j P_j, \tag{3.1}$$

where $\alpha_j \in \mathbb{C}$, $j = 1, \ldots, n$, are mutually distinct and P_j form a complete system (i.e., $\sum_{j=1}^{n} P_j = I$) of pairwise disjoint operators in X with $P_i P_j = \delta_{ij} P_i$. We state and prove the following theorem.

Theorem 3.1. *If A is of simplest type, then every bounded solution of (1.6) is almost automorphic.*

Proof. If $x(t)$ is a solution, then $x(t) = \sum_{j=1}^{n} x_j(t)$, where $x_j(t) = P_j(x(t))$, $j = 1, 2, \ldots, n$. We show that each $x_j(t)$ is almost automorphic. In fact,

$$\begin{aligned} x_j(t) &= P_j A x(t) + P_j f(t) \\ &= P_j \left(\sum_{i=1}^{n} \alpha_i P_i \right) x(t) + P_j f(t) \\ &= \alpha_j x_j(t) + P_j f(t), \end{aligned} \tag{3.2}$$

where $P_j f(t)$ is almost automorphic. Therefore, $x_j(t)$ is almost automorphic (see [2, page 69]) and so is $x(t)$. The proof is complete. □

It is known that if A is bounded linear operator on a Banach space X and the function $x(t) = e^{tA} x_0$ is almost automorphic for some $x_0 \in D(A)$, then either $\inf_{t \in \mathbb{R}} \|x(t)\| > 0$ or $x(t) = 0$, for every $t \in \mathbb{R}$. This is presented in [2, Theorem 2.1.9, page 17]. We now state a more general result.

Theorem 3.2. *Suppose $(T(t))_{t \in \mathbb{R}}$ is a C_0-group of bounded linear operators in a Banach space X and the function $x(t) = T(t)x_0 : \mathbb{R} \to X$ is weakly almost automorphic for some $x_0 \in X$. Then, either $\inf_{t \in \mathbb{R}} \|x(t)\| > 0$, or $x(t) = 0$ for every $t \in \mathbb{R}$.*

Proof. We assume that $\inf_{t \in \mathbb{R}} \|x(t)\| = 0$. Then, we find a minimizing sequence of real numbers (s'_n) such that $s'_n \to \infty$ and $\|s'_n\| \to 0$ as $n \to \infty$.

Since $x(t)$ is weakly almost automorphic, there exists a subsequence (s_n) of (s'_n) such that (1.3) and (1.4) are pointwise in $t \in \mathbb{R}$. And so, for every $\phi \in X^*$, we have

$$\begin{aligned} \lim_{n \to \infty} \langle \phi, x(t + s_n) \rangle &= \langle \phi, y(t) \rangle, \\ \lim_{n \to \infty} \langle \phi, y(t - s_n) \rangle &= \langle \phi, x(t) \rangle \end{aligned} \tag{3.3}$$

for each $t \in \mathbb{R}$. It is

$$x(t + s_n) = T(t + s_n) x_0 = T(t) T(s_n) x_0 = T(t) x(s_n), \tag{3.4}$$

$t \in \mathbb{R}$, $n = 1, 2, \ldots$, so that

$$\lim_{n\to\infty} \langle \phi, T(t)x(s_n) \rangle = \langle \phi, y(t) \rangle \tag{3.5}$$

for each $t \in \mathbb{R}$, $\phi \in X^*$. But

$$\left| \langle \phi, T(t)x(s_n) \rangle \right| \le \|\phi\| \|T(t)\| \|x(s_n)\| \to 0 \tag{3.6}$$

as $n \to \infty$. That implies $\langle \phi, y(t) \rangle = 0$, for each $\phi \in X^*$ and $t \in \mathbb{R}$. So, $y(t) = 0$, $t \in \mathbb{R}$, and consequently $x(t) = 0$, $t \in \mathbb{R}$. The proof is now complete. □

Application 3.3. Consider (1.5), where A is the infinitesimal generator of a C_0-group of bounded linear operators $T(t)$, $t \in \mathbb{R}$. Then, every mild weakly almost automorphic solution $x(t)$ satisfies the property: $\inf_{t\in\mathbb{R}} \|x(t)\| > 0$, or $x(t) = 0$ for every $t \in \mathbb{R}$.

Proof. Mild solutions of (1.6) in this case are written as $x(t) = T(t)x(0)$, $t \in \mathbb{R}$. Then apply Theorem 3.2. □

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