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CALCULATIONS ON SOME SEQUENCE SPACES

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We deal with space of sequences generalizing the well-known spaces $w_\infty^p(\lambda)$, $c_\infty(\lambda,\mu)$, replacing the operators $C(\lambda)$ and $\Delta(\mu)$ by their transposes. We get generalizations of results concerning the strong matrix domain of an infinite matrix A.

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1. Notations and preliminary results. For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$, the operators A_n are defined, for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m, \tag{1.1}$$

where $X = (x_n)_{n\geq 1}$, the series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n, \quad n = 1,2,\ldots, \tag{1.2}$$

where $B = (b_n)_{n\geq 1}$ is a one-column matrix and X the unknown, see [1, 2, 3, 4, 5, 6, 7, 8, 10]. Equation (1.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper, we will also consider A an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\| \|_E$ is a BK space if each projection $P_nX = x_n$ is continuous for all $X \in E$. A BK space E is said to have AK, (see [12, 13]), if $B = \sum_{m=1}^{\infty} b_m e_m$, for every $B = (b_n)_{n\geq 1} \in E$, (with $e_n = (0,\ldots,1,\ldots)$, 1 being in the nth position), that is,

$$\left\| \sum_{m=N+1}^{\infty} b_m e_m \right\|_E \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{1.3}$$

We will write s for the set of all complex sequences, l_∞, c, c_0 for the sets of bounded, convergent, and null sequences, respectively. We will denote by cs and l_1 the sets of convergent and absolutely convergent series, respectively.

In all that follows we will use the set

$$U^{+*} = \{(u_n)_{n\geq 1} \in s \mid u_n > 0 \ \forall n\}. \tag{1.4}$$

From Wilansky's notations [15], we define for any sequence

$$\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}, \tag{1.5}$$

and for any set of sequences E, the set

$$\left(\frac{1}{\alpha}\right)^{-1} * E = \left\{(x_n)_{n\geq 1} \in s \;\middle|\; \left(\frac{x_n}{\alpha_n}\right)_n \in E\right\}. \tag{1.6}$$

We will write $\alpha * E$ instead of $(1/\alpha)^{-1} * E$ for short. So we get

$$\alpha * E = \begin{cases} s_\alpha^\circ & \text{if } E = c_0, \\ s_\alpha^{(c)} & \text{if } E = c, \\ s_\alpha & \text{if } E = l_\infty. \end{cases} \tag{1.7}$$

We have for instance

$$\alpha * c_0 = s_\alpha^\circ = \{(x_n)_{n\geq 1} \in s \mid x_n = o(\alpha_n)\ n \longrightarrow \infty\}. \tag{1.8}$$

Each of the spaces $\alpha * E$, where $E \in \{c_0, c, l_\infty\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1}\left(\frac{|x_n|}{\alpha_n}\right), \tag{1.9}$$

and s_α° has AK.

Now let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$. $S_{\alpha,\beta}$ is the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$(a_{nm}\alpha_m)_{m\geq 1} \in l^1 \quad \forall n \geq 1, \qquad \sum_{m=1}^{\infty} (|a_{nm}|\alpha_m) = O(\beta_n) \quad (n \longrightarrow \infty). \tag{1.10}$$

$S_{\alpha,\beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha,\beta}} = \sup_{n\geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}|\frac{\alpha_m}{\beta_n}\right). \tag{1.11}$$

Let E and F be any subsets of s. When A maps E into F, we will write $A \in (E,F)$, see [11]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$). It has been proved in [9] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So we can write that $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$, we obtain the unital Banach algebra $S_{\alpha,\beta} = S_\alpha$, (see [1, 2, 3, 5, 6, 10]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we will say that $A \in \Gamma_\alpha$. Since the set S_α is a unital algebra, we have the useful result that if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n\geq 1}$, Γ_α, S_α, s_α, s_α°, and $s_\alpha^{(c)}$ are replaced by Γ_r, S_r, s_r, s_r°, and $s_r^{(c)}$, respectively, (see [1, 2, 3, 5, 6, 10]). When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^\circ = c_0$, and $s_1^{(c)} = c$, and putting $e = (1,1,\dots)$, we have $S_1 = S_e$. It is well known, see [11], that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1. \tag{1.12}$$

For any subset E of s, we put

$$AE = \{Y \in s \mid \exists X \in E,\ Y = AX\}. \tag{1.13}$$

If F is a subset of s, we will denote

$$F(A) = F_A = \{X \in s \mid Y = AX \in F\}. \tag{1.14}$$

We can see that $F(A) = A^{-1}F$.

2. Some properties of the operators Δ^+ and Σ^+. Here we will deal with the operators represented by $C^+(\lambda)$ and $\Delta^+(\lambda)$.

Let

$$U = \{(u_n)_{n\geq 1} \in s \mid u_n \neq 0\ \forall n\}. \tag{2.1}$$

We define $C(\lambda) = (c_{nm})_{n,m\geq 1}$, for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

So, we put $C^+(\lambda) = C(\lambda)^t$. It can be proved that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1,\ n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

is the inverse of $C(\lambda)$, see [12, 14]. Similarly, we put $\Delta^+(\lambda) = \Delta(\lambda)^t$. If $\lambda = e$, we get the well-known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written as $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Σ belong to any given space S_R with $R > 1$. Writing $D_\lambda = (\lambda_n \delta_{nm})_{n,m\geq 1}$, (where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$ otherwise), we have $\Delta^+(\lambda) = D_\lambda \Delta^+$. So for any given $\alpha \in U^{+*}$, we see that if $(\alpha_{n-1}/\alpha_n)|\lambda_n/\lambda_{n-1}| = O(1)$, then $\Delta^+(\lambda) \in (s_{(\alpha/|\lambda|)}, s_\alpha)$. Since $\operatorname{Ker}\Delta^+(\lambda) \neq 0$, we are lead to define the set

$$s_\alpha^*(\Delta^+(\lambda)) = s_\alpha(\Delta^+(\lambda)) \bigcap s_{(\alpha/|\lambda|)} = \{X = (x_n)_{n\geq 1} \in s_{(\alpha/|\lambda|)} \mid \Delta^+(\lambda)X \in s_\alpha\}. \tag{2.4}$$

It can easily be seen that

$$s^*_{(\alpha/|\lambda|)}(\Delta^+(e)) = s^*_{(\alpha/|\lambda|)}(\Delta^+) = s^*_{\alpha}(\Delta^+(\lambda)). \tag{2.5}$$

2.1. Properties of the sequence $C(\alpha)\alpha$. We will use the following sets:

$$\begin{aligned}
\widehat{C_1} &= \left\{\alpha \in U^{+*} \;\middle|\; \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1)\ (n \longrightarrow \infty)\right\},\\
\widehat{C} &= \left\{\alpha \in U^{+*} \;\middle|\; \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) \in c\right\},\\
\widehat{C_1^+} &= \left\{\alpha \in U^{+*} \bigcap cs \;\middle|\; \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = O(1)\ (n \longrightarrow \infty)\right\},\\
\Gamma &= \left\{\alpha \in U^{+*} \;\middle|\; \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\},\\
\Gamma^+ &= \left\{\alpha \in U^{+*} \;\middle|\; \overline{\lim_{n\to\infty}}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.
\end{aligned} \tag{2.6}$$

Note that $\alpha \in \Gamma^+$ if and only if $1/\alpha \in \Gamma$. We will see in Proposition 2.1 that if $\alpha \in \widehat{C_1}$, α tends to infinity. On the other hand, we see that $\Delta \in \Gamma_\alpha$ implies $\alpha \in \Gamma$ and $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n \geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1. \tag{2.7}$$

We obtain the following results in which we put $[C(\alpha)\alpha]_n = (\sum_{k=1}^{n}\alpha_k)/\alpha_n$.

Proposition 2.1. *Let $\alpha \in U^{+*}$. Then*

(i) $\alpha_{n-1}/\alpha_n \to 0$ *if and only if* $[C(\alpha)\alpha]_n \to 1$,

(ii) (a) $\alpha \in \widehat{C}$ *implies that* $(\alpha_{n-1}/\alpha_n)_{n\geq 1} \in c$,

(b) $[C(\alpha)\alpha]_n \to l$ *implies that* $\alpha_{n-1}/\alpha_n \to 1 - 1/l$,

(iii) *if* $\alpha \in \widehat{C_1}$, *there are* $K > 0$ *and* $\gamma > 1$ *such that*

$$\alpha_n \geq K\gamma^n \quad \forall n, \tag{2.8}$$

(iv) *the condition* $\alpha \in \Gamma$ *implies that* $\alpha \in \widehat{C_1}$ *and there exists a real* $b > 0$ *such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \text{for } n \geq q+1,\ \chi = \gamma_q(\alpha) \in\,]0,1[, \tag{2.9}$$

(v) *the condition* $\alpha \in \Gamma^+$ *implies that* $\alpha \in \widehat{C_1^+}$.

Proof. Assume that $\alpha_{n-1}/\alpha_n \to 0$. Then there is an integer N such that

$$n \geq N+1 \Longrightarrow \frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{2}. \tag{2.10}$$

So there exists a real $K > 0$ such that $\alpha_n \geq K2^n$ for all n and

$$\frac{\alpha_k}{\alpha_n} = \frac{\alpha_k}{\alpha_{k+1}} \cdots \frac{\alpha_{n-1}}{\alpha_n} \leq \left(\frac{1}{2}\right)^{n-k} \quad \text{for } N \leq k \leq n-1. \tag{2.11}$$

Then

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) = \frac{1}{\alpha_n}\left(\sum_{k=1}^{N-1}\alpha_k\right) + \sum_{k=N}^{n-1}\frac{\alpha_k}{\alpha_n} \leq \frac{1}{K2^n}\left(\sum_{k=1}^{N-1}\alpha_k\right) + \sum_{k=N}^{n-1}\left(\frac{1}{2}\right)^{n-k}, \tag{2.12}$$

and since $\sum_{k=N}^{n-1}(1/2)^{n-k} = 1-(1/2)^{n-N} \to 1$, $(n \to \infty)$, we deduce that

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) = O(1) \tag{2.13}$$

and $([C(\alpha)\alpha]_n) \in l_\infty$. Using the identity

$$[C(\alpha)\alpha]_n = \frac{\alpha_1+\cdots+\alpha_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} + 1 = [C(\alpha)\alpha]_{n-1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) + 1, \tag{2.14}$$

we get $[C(\alpha)\alpha]_n \to 1$. This proves the necessity.

Conversely, if $[C(\alpha)\alpha]_n \to 1$, then

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{[C(\alpha)\alpha]_n - 1}{[C(\alpha)\alpha]_{n-1}} \longrightarrow 0. \tag{2.15}$$

(ii) is a direct consequence of the identity (2.14).

(iii) We put $\Sigma_n = \sum_{k=1}^{n}\alpha_k$. Then for a real $M > 1$,

$$[C(\alpha)\alpha]_n = \frac{\Sigma_n}{\Sigma_n - \Sigma_{n-1}} \leq M \quad \forall n. \tag{2.16}$$

So $\Sigma_n \geq (M/(M-1))\Sigma_{n-1}$ and $\Sigma_n \geq (M/(M-1))^{n-1}\alpha_1 \,\forall n$. Therefore, from

$$\frac{\alpha_1}{\alpha_n}\left(\frac{M}{M-1}\right)^{n-1} \leq [C(\alpha)\alpha]_n = \frac{\Sigma_n}{\alpha_n} \leq M, \tag{2.17}$$

we conclude that $\alpha_n \geq K\gamma^n$ for all n, with $K = (M-1)\alpha_1/M^2$ and $\gamma = M/(M-1) > 1$.

(iv) If $\alpha \in \Gamma$, there is an integer $q \geq 1$ for which

$$k \geq q+1 \text{ implies } \frac{\alpha_{k-1}}{\alpha_k} \leq \chi < 1, \text{ with } \chi = \gamma_q(\alpha). \tag{2.18}$$

So there is a real $M' > 0$ for which

$$\alpha_n \geq \frac{M'}{\chi^n} \quad \forall n \geq q+1. \tag{2.19}$$

Writing $\sigma_{nq} = (1/\alpha_n)(\sum_{k=1}^{q} \alpha_k)$ and $d_n = [C(\alpha)\alpha]_n - \sigma_{nq}$, we get

$$d_n = \frac{1}{\alpha_n}\left(\sum_{k=q+1}^{n} \alpha_k\right) = 1 + \sum_{j=q+1}^{n-1}\left(\prod_{k=1}^{n-j} \frac{\alpha_{n-k}}{\alpha_{n-k+1}}\right) \leq \sum_{j=q+1}^{n} \chi^{n-j} \leq \frac{1}{1-\chi}. \tag{2.20}$$

Using (2.19), we get $\sigma_{nq} \leq (1/M')\chi^n(\sum_{k=1}^{q} \alpha_k)$. So

$$[C(\alpha)\alpha]_n \leq a + b\chi^n \tag{2.21}$$

with $a = 1/(1-\chi)$ and $b = (1/M')(\sum_{k=1}^{q} \alpha_k)$.

(v) If $\alpha \in \Gamma^+$, there are $\chi' \in]0,1[$ and an integer $q' \geq 1$ such that

$$\frac{\alpha_k}{\alpha_{k-1}} \leq \chi' \quad \text{for } k \geq q'. \tag{2.22}$$

Then for every $n \geq q'$, we have

$$\frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty} \alpha_k\right) = \sum_{k=n}^{\infty}\left(\frac{\alpha_k}{\alpha_n}\right) \leq 1 + \sum_{k=n+1}^{\infty} \prod_{i=0}^{k-n-1}\left(\frac{\alpha_{k-i}}{\alpha_{k-i-1}}\right) \leq \sum_{k=n}^{\infty} \chi'^{k-n} = O(1). \tag{2.23}$$

This gives the conclusion. □

Remark 2.2. Note that as a direct consequence of Proposition 2.1, we have $\widehat{C_1} \bigcap \widehat{C_1^+} = \Gamma \bigcap \Gamma^+ = \phi$.

Remark 2.3. The condition $\alpha \in \widehat{C_1}$ does not imply that $\alpha \in \Gamma$, see [8].

2.2. Some new properties of the operators Δ and Δ^+. In the following we will use some lemmas, the next one is well known, see [15].

Lemma 2.4. *The condition $A \in (c_0, c_0)$ is equivalent to*

$$\begin{gathered} A \in S_1, \\ \lim_n a_{nm} = 0 \quad \text{for each } m \geq 1. \end{gathered} \tag{2.24}$$

Lemma 2.5. *If Δ^+ is bijective from s_α into itself, then $\alpha \in cs$.*

Proof. Assume that $\alpha \notin cs$, that is, $\sum_n \alpha_n = \infty$. Two cases are possible.

(1) $e \in \operatorname{Ker}\Delta^+ \bigcap s_\alpha$. Then Δ^+ cannot be bijective from s_α into itself.

(2) $e \notin \operatorname{Ker}\Delta^+ \bigcap s_\alpha$. Then $1/\alpha \notin s_1$ and there is a sequence of integers $(n_i)_i$ strictly increasing such that $1/\alpha_{n_i} \to \infty$. Assume that the equation $\Delta^+ X = \alpha$ has a solution

$X = (x_{n,0})_{n\geq 1}$ in s_α. Then there is a unique scalar x_1 such that

$$x_{n,0} = x_1 - \sum_{k=1}^{n-1} \alpha_k. \tag{2.25}$$

So

$$\frac{|x_{n_i,0}|}{\alpha_{n_i}} = \left| \frac{1}{\alpha_{n_i}} \left(x_1 - \sum_{k=1}^{n_i-1} \alpha_k \right) \right| \longrightarrow \infty \quad \text{as } i \longrightarrow \infty, \tag{2.26}$$

and $X \notin s_\alpha$, which is contradictory.

We conclude that each of the properties $e \in \operatorname{Ker}\Delta^+ \bigcap s_\alpha$ and $e \notin \operatorname{Ker}\Delta^+ \bigcap s_\alpha$ is impossible and Δ^+ is not bijective from s_α into itself. This proves the lemma. □

Lemma 2.6. *For every $X \in c_0$, $\Sigma^+(\Delta^+ X) = X$ and for every $X \in cs$, $\Delta^+(\Sigma^+ X) = X$.*

Proof. It can easily be seen that

$$\begin{aligned} \left[\Sigma^+(\Delta^+ X)\right]_n &= \sum_{m=n}^{\infty} (x_m - x_{m+1}) = x_n \quad \forall X \in c_0, \\ \left[\Delta^+(\Sigma^+ X)\right]_n &= \sum_{m=n}^{\infty} x_m - \sum_{m=n+1}^{\infty} x_m = x_n \quad \forall X \in cs. \end{aligned} \tag{2.27}$$

We can assert the following result, in which we put $\alpha^+ = (\alpha_{n+1})_{n\geq 1}$ and $s_\alpha^{\circ *}(\Delta^+) = s_\alpha^\circ(\Delta^+) \bigcap s_\alpha^\circ$. Note that from (2.5) we have

$$s_\alpha^*(\Delta^+(e)) = s_\alpha^*(\Delta^+) = s_\alpha(\Delta^+) \bigcap s_\alpha. \tag{2.28}$$

□

Theorem 2.7. (i) (a) *$s_\alpha(\Delta) = s_\alpha$ if and only if $\alpha \in \widehat{C_1}$,*

(b) *$s_\alpha^\circ(\Delta) = s_\alpha^\circ$ if and only if $\alpha \in \widehat{C_1}$,*

(c) *$s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ if and only if $\alpha \in \hat{C}$.*

(ii) (a) *$\alpha \in \widehat{C_1}$ if and only if $s_{\alpha^+}(\Delta^+) = s_\alpha$ and Δ^+ is surjective from s_α into s_{α^+},*

(b) *$\alpha \in \widehat{C_1^+}$ if and only if $s_\alpha^*(\Delta^+) = s_\alpha$ and Δ^+ is bijective from s_α into s_α,*

(c) *$\alpha \in \widehat{C_1^+}$ implies that $s_\alpha^{\circ *}(\Delta^+) = s_\alpha^\circ$ and Δ^+ is bijective from s_α° into s_α°.*

(iii) *$\alpha \in \widehat{C_1^+}$ if and only if $s_\alpha(\Sigma^+) = s_\alpha$ and $s_\alpha(\Sigma^+) = s_\alpha$ implies $s_\alpha^\circ(\Sigma^+) = s_\alpha^\circ$.*

Proof. (i) has been proved in [8].

(ii)(a) Sufficiency. If Δ^+ is surjective from s_α into s_{α^+}, then for every $B \in s_{\alpha^+}$ the solutions of $\Delta^+ X = B$ in s_α are given by

$$x_{n+1} = x_1 - \sum_{k=1}^{n} b_k \quad n = 1, 2, \ldots, \tag{2.29}$$

where x_1 is arbitrary. If we take $B = \alpha^+$, we get $x_n = x_1 - \sum_{k=2}^{n} \alpha_k$. So

$$\frac{x_n}{\alpha_n} = \frac{x_1}{\alpha_n} - \frac{1}{\alpha_n}\left(\sum_{k=2}^{n} \alpha_k\right) = O(1). \tag{2.30}$$

Taking $x_1 = -\alpha_1$, we conclude that $(\sum_{k=1}^{n-1} \alpha_k)/\alpha_n = O(1)$ and $\alpha \in \widehat{C_1}$.

Conversely, assume that $\alpha \in \widehat{C_1}$. From the inequality

$$\frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{\alpha_n}\left(\sum_{k=1}^{n} \alpha_k\right) = O(1), \tag{2.31}$$

we deduce that $\alpha_{n-1}/\alpha_n = O(1)$ and $\Delta^+ \in (s_\alpha, s_{\alpha^+})$. Then for any given $B \in s_{\alpha^+}$, the solutions of the equation $\Delta^+ X = B$ are given by $x_1 = -u$ and

$$-x_n = u + \sum_{k=1}^{n-1} b_k \quad \text{for } n \geq 2, \tag{2.32}$$

where u is an arbitrary scalar. So there exists a real $K > 0$ such that

$$\frac{|x_n|}{\alpha_n} = \frac{\left|u + \sum_{k=1}^{n-1} b_k\right|}{\alpha_n} \leq \frac{|u| + K\left(\sum_{k=2}^{n} \alpha_k\right)}{\alpha_n} = O(1) \tag{2.33}$$

and $X \in s_\alpha$. We conclude that Δ^+ is surjective from s_α into s_{α^+}.

(ii)(b) Necessity. Assume that $\alpha \in \widehat{C_1^+}$. Then $\Delta^+ \in (s_\alpha, s_\alpha)$, since

$$\frac{\alpha_{n+1}}{\alpha_n} \leq \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty} \alpha_k\right) = O(1) \quad (n \longrightarrow \infty). \tag{2.34}$$

Further, from $s_\alpha \subset cs$, we deduce, using Lemma 2.4, that for any given $B \in s_\alpha$,

$$\Delta^+(\Sigma^+ B) = B. \tag{2.35}$$

On the other hand, $\Sigma^+ B = (\sum_{k=n}^{\infty} b_k)_{n\geq 1} \in s_\alpha$, since $\alpha \in \widehat{C_1^+}$. So Δ^+ is surjective from s_α into s_α. Finally, Δ^+ is injective because the equation

$$\Delta^+ X = O \tag{2.36}$$

admits the unique solution $X = O$ in s_α, since

$$\operatorname{Ker}\Delta^+ = \{ue^t \mid u \in C\} \tag{2.37}$$

and $e^t \notin s_\alpha$.

Sufficiency. For every $B \in s_\alpha$, the equation $\Delta^+ X = B$ admits a unique solution in s_α. Then from Lemma 2.5, $\alpha \in cs$ and since $s_\alpha \subset cs$, we deduce from Lemma 2.6 that $X = \Sigma^+ B \in s_\alpha$ is the unique solution of $\Delta^+ X = B$. Taking $B = \alpha$, we get $\Sigma^+ \alpha \in s_\alpha$, that is, $\alpha \in \widehat{C_1^+}$.

(ii)(c) If $\alpha \in \widehat{C_1^+}$, Δ^+ is bijective from s_α° into itself. Indeed, we have $D_{1/\alpha}\Delta^+ D_\alpha \in (c_0, c_0)$ from (2.34) and Lemma 2.4. Furthermore, since $\alpha \in \widehat{C_1^+}$ we have $s_\alpha^\circ \subset cs$ and for every $B \in s_\alpha^\circ$,

$$\Delta^+(\Sigma^+ B) = B. \tag{2.38}$$

From Lemma 2.4, we have $\Sigma^+ \in (s_\alpha^\circ, s_\alpha^\circ)$, so the equation $\Delta^+ X = B$ admits the solution $X_0 = \Sigma^+ B$ in s_α° and we have proved that Δ^+ is surjective from s_α° into itself. Finally, $\alpha \in \widehat{C_1^+}$ implies that $e^t \notin s_\alpha^\circ$, so $\operatorname{Ker}\Delta^+ \bigcap s_\alpha^\circ = \{0\}$ and we conclude that Δ^+ is bijective from s_α° into itself.

(iii) comes from (ii), since $\alpha \in \widehat{C_1^+}$ if and only if Δ^+ is bijective from s_α into itself and

$$\Sigma^+(\Delta^+ X) = \Delta^+(\Sigma^+ X) = X \quad \forall X \in s_\alpha. \tag{2.39}$$

□

As a direct consequence of Theorem 2.7 we obtain the following results.

Corollary 2.8. *Let R be any real > 0. Then*

$$R > 1 \iff s_R(\Delta) = s_R \iff s_R^\circ(\Delta) = s_R^\circ \iff s_R(\Delta^+) = s_R. \tag{2.40}$$

Proof. From (i) and (ii) in Theorem 2.7, we see that it is enough to prove that $\alpha = (R^n)_{n\geq 1} \in \widehat{C_1}$ if and only if $R > 1$. We have $(R^n)_{n\geq 1} \in \widehat{C_1}$ if and only if $R \neq 1$ and

$$R^{-n}\left(\sum_{k=1}^{n} R^k\right) = \frac{1}{1-R}R^{-n+1} - \frac{R}{1-R} = O(1) \quad \text{as } n \longrightarrow \infty. \tag{2.41}$$

This means that $R > 1$ and the corollary is proved. □

Using the notation $\alpha^- = (1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \dots)$ we get the next result.

Corollary 2.9. *Let $\alpha \in U^{+*}$ and $\mu \in U$. Then*

(i) $\alpha/|\mu| \in \widehat{C_1}$ *if and only if*

$$s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}, \tag{2.42}$$

(ii) $\alpha/|\mu| \in \widehat{C_1^+}$ *if and only if*

$$s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}. \tag{2.43}$$

Proof. First we have

$$s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}(\Delta^+). \tag{2.44}$$

Indeed,

$$X \in s_\alpha(\Delta^+(\mu)) \Longleftrightarrow D_\mu \Delta^+ X \in s_\alpha \Longleftrightarrow \Delta^+ X \in s_{(\alpha/|\mu|)} \Longleftrightarrow X \in s_{(\alpha/|\mu|)}(\Delta^+). \tag{2.45}$$

Now, if $\alpha/|\mu| \in \widehat{C_1}$, from (i) in Theorem 2.7, we have $s_{(\alpha/|\mu|)}(\Delta^+) = s_{(\alpha/|\mu|)^-}$ and $s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}$. Conversely, assume $s_\alpha(\Delta^+(\mu)) = s_{(\alpha/|\mu|)^-}$. Reasoning as above, we get $s_{(\alpha/|\mu|)}(\Delta^+) = s_{(\alpha/|\mu|)^-}$, and using (i) in Theorem 2.7 we conclude that $\alpha/|\mu| \in \widehat{C_1}$ and (i) holds.

(ii) $\alpha/|\mu| \in \widehat{C_1^+}$ implies that Δ^+ is bijective from $s_{(\alpha/|\mu|)}$ into itself. Thus

$$s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}^*(\Delta^+) = s_{(\alpha/|\mu|)}. \tag{2.46}$$

This proves the necessity. Conversely, assume that $s_\alpha^*(\Delta^+(\mu)) = s_{(\alpha/|\mu|)}$. Then $s_{(\alpha/|\mu|)}^*(\Delta^+) = s_{(\alpha/|\mu|)}$ and from Theorem 2.7(ii)(b), $\alpha/|\mu| \in \widehat{C_1^+}$ and (ii) holds. □

2.3. Spaces $w_\alpha^p(\lambda)$ and $w_\alpha^{+p}(\lambda)$ for given real $p > 0$. Here we will define sets generalizing the well-known sets

$$\begin{aligned} w_\infty^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in l_\infty\}, \\ w_0^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in c_0\}, \end{aligned} \tag{2.47}$$

see [9, 12, 13, 14, 15]. It is proved that each of the sets $w_0^p = w_0^p((n)_n)$ and $w_\infty^p = w_\infty^p((n)_n)$ is a p-normed FK space for $0 < p < 1$ (i.e., a complete linear metric space for which each projection P_n is continuous) and a BK space for $1 \le p < \infty$ with respect to the norm

$$\|X\| = \begin{cases} \sup_{\nu \ge 1} \left(\dfrac{1}{2^\nu} \left(\displaystyle\sum_{n=2^\nu}^{2^{\nu+1}-1} |x_n|^p \right) \right) & \text{if } 0 < p < 1, \\ \sup_{\nu \ge 1} \left(\dfrac{1}{2^\nu} \left(\displaystyle\sum_{n=2^\nu}^{2^{\nu+1}-1} |x_n|^p \right) \right)^{1/p} & \text{if } 1 \le p < \infty. \end{cases} \tag{2.48}$$

The set w_0^p has the property AK, (i.e., every $X = (x_n)_{n \ge 1} \in w_0^p$ has a unique representation $X = \sum_{n=1}^\infty x_n e_n^t$) and every sequence $X = (x_n)_{n \ge 1} \in w^p$ has a unique representation

$$X = le^t + \sum_{n=1}^\infty (x_n - l) e_n^t, \tag{2.49}$$

where $l \in C$ is such that $X - le^t \in w_0^p$, (see [4]). Now, let $\alpha \in U^{+*}$ and $\lambda \in U^{+*}$. We have

$$\begin{aligned} w_\alpha^p(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in s_\alpha\}, \\ w_\alpha^{+p}(\lambda) &= \{X \in s \mid C^+(\lambda)(|X|^p) \in s_\alpha\}, \\ w_\alpha^{\circ p}(\lambda) &= \{X \in s \mid C(\lambda)(|X|^p) \in s_\alpha^\circ\}, \\ w_\alpha^{\circ +p}(\lambda) &= \{X \in s \mid C^+(\lambda)(|X|^p) \in s_\alpha^\circ\}. \end{aligned} \tag{2.50}$$

We deduce from the previous section the following theorem.

Theorem 2.10. (i) (a) *The condition* $\alpha \in \widehat{C_1^+}$ *is equivalent to*

$$w_\alpha^{+p}(\lambda) = s_{(\alpha\lambda)^{1/p}}. \tag{2.51}$$

(b) *If* $\alpha \in \widehat{C_1^+}$, *then*

$$w_\alpha^{\circ p}(\lambda) = s^\circ_{(\alpha\lambda)^{1/p}}. \tag{2.52}$$

(ii) (a) *The condition* $\alpha\lambda \in \widehat{C_1}$ *is equivalent to*

$$w_\alpha^{p}(\lambda) = s_{(\alpha\lambda)^{1/p}}. \tag{2.53}$$

(b) *If* $\alpha\lambda \in \widehat{C_1}$, *then*

$$w_\alpha^{\circ +p}(\lambda) = s^\circ_{(\alpha\lambda)^{1/p}}. \tag{2.54}$$

Proof. Assume that $\alpha \in \widehat{C_1^+}$. Since $C^+(\lambda) = \Sigma^+ D_{1/\lambda}$, we have

$$w_\alpha^{+p}(\lambda) = \{X \mid (\Sigma^+ D_{1/\lambda})(|X|^p) \in s_\alpha\} = \{X \mid D_{1/\lambda}(|X|^p) \in s_\alpha(\Sigma^+)\}, \tag{2.55}$$

and since $\alpha \in \widehat{C_1^+}$ implies $s_\alpha(\Sigma^+) = s_\alpha$, we conclude that

$$w_\alpha^{+p}(\lambda) = \{X \mid |X|^p \in D_\lambda s_\alpha = s_{\alpha\lambda}\} = s_{(\alpha\lambda)^{1/p}}. \tag{2.56}$$

Conversely, we have $(\alpha\lambda)^{1/p} \in s_{(\alpha\lambda)^{1/p}} = w_\alpha^{+p}(\lambda)$. So

$$C^+(\lambda)\left[(\alpha\lambda)^{1/p}\right]^p = \left(\sum_{k=n}^{\infty} \frac{\alpha_k\lambda_k}{\lambda_k}\right)_{n\ge 1} \in s_\alpha, \tag{2.57}$$

that is, $\alpha \in \widehat{C_1^+}$ and we have proved (i). We obtain (i)(b) by reasoning as above.

(ii) Assume that $\alpha\lambda \in \widehat{C_1}$. Then

$$w_\alpha^{p}(\lambda) = \{X \mid |X|^p \in \Delta(\lambda) s_\alpha\}. \tag{2.58}$$

Since $\Delta(\lambda) = \Delta D_\lambda$, we get $\Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda}$. Now, from $\alpha\lambda \in \widehat{C_1}$ we deduce that Δ is bijective from $s_{\alpha\lambda}$ into itself and $w_\alpha^p(\lambda) = s_{(\alpha\lambda)^{1/p}}$. Conversely, assume that $w_\alpha^p(\lambda) = s_{(\alpha\lambda)^{1/p}}$. Then $(\alpha\lambda)^{1/p} \in s_{(\alpha\lambda)^{1/p}}$ implies that

$$C(\lambda)(\alpha\lambda) \in s_\alpha, \tag{2.59}$$

and since $D_{1/\alpha}C(\lambda)(\alpha\lambda) \in s_1 = l_\infty$, we conclude that $C(\alpha\lambda)(\alpha\lambda) \in l_\infty$. The proof of (ii)(b) follows the same lines as in the proof of the necessity in (ii) replacing $s_{\alpha\lambda}$ by $s^\circ_{\alpha\lambda}$. □

3. New sets of sequences of the form $[A_1, A_2]$. In this section, we will deal with the sets

$$[A_1(\lambda), A_2(\mu)] = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\}, \tag{3.1}$$

where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and we give necessary conditions to get $[A_1(\lambda), A_2(\mu)]$ in the form s_γ.

Let λ and $\mu \in U^{+*}$. For simplification, we will write throughout this section

$$[A_1, A_2] = [A_1(\lambda), A_2(\mu)] = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s_\alpha\} \tag{3.2}$$

for any matrices

$$\begin{aligned} &A_1(\lambda) \in \{\Delta(\lambda), \Delta^+(\lambda), C(\lambda), C^+(\lambda)\},\\ &A_2(\mu) \in \{\Delta(\mu), \Delta^+(\mu), C(\mu), C^+(\mu)\}. \end{aligned} \tag{3.3}$$

So we have for instance

$$[C, \Delta] = \{X \in s \mid C(\lambda)(|\Delta(\mu)X|) \in s_\alpha\} = (w_\alpha(\lambda))_{\Delta(\mu)}, \dots. \tag{3.4}$$

In all that follows, the conditions $\xi \in \Gamma$, or $1/\eta \in \Gamma$ for any given sequences ξ and η can be replaced by the conditions $\xi \in \widehat{C_1}$ and $\eta \in \widehat{C_1^+}$.

3.1. Spaces $[C, C]$, $[C, \Delta]$, $[\Delta, C]$, and $[\Delta, \Delta]$. For the convenience of the reader we will write the following identities, where $A_1(\lambda)$ and $A_2(\mu)$ are lower triangles and we will use the convention $\mu_0 = 0$:

$$\begin{aligned} &[C,C] = \left\{X \in s \;\middle|\; \frac{1}{\lambda_n}\left(\sum_{m=1}^{n}\left|\frac{1}{\mu_m}\left(\sum_{k=1}^{m} x_k\right)\right|\right) = \alpha_n O(1)\right\},\\ &[C,\Delta] = \left\{X \in s \;\middle|\; \frac{1}{\lambda_n}\left(\sum_{k=1}^{n}\left|\mu_k x_k - \mu_{k-1}x_{k-1}\right|\right) = \alpha_n O(1)\right\},\\ &[\Delta,C] = \left\{X \in s \;\middle|\; -\lambda_{n-1}\left|\frac{1}{\mu_{n-1}}\left(\sum_{k=1}^{n-1} x_i\right)\right| + \lambda_n\left|\frac{1}{\mu_n}\left(\sum_{k=1}^{n} x_i\right)\right| = \alpha_n O(1)\right\},\\ &[\Delta,\Delta] = \{X \in s \mid -\lambda_{n-1}|\mu_{n-1}x_{n-1} - \mu_{n-2}x_{n-2}| + \lambda_n|\mu_n x_n - \mu_{n-1}x_{n-1}| = \alpha_n O(1)\}. \end{aligned} \tag{3.5}$$

Note that for $\alpha = e$ and $\lambda = \mu$, $[C,\Delta]$ is the well-known set of sequences that are strongly bounded, denoted by $c_\infty(\lambda)$, see [9, 12, 13, 14, 15]. We get the following result.

Theorem 3.1. (i) *If $\alpha\lambda$ and $\alpha\lambda\mu \in \Gamma$, then*

$$[C,C] = s_{(\alpha\lambda\mu)}, \tag{3.6}$$

(ii) *if $\alpha\lambda \in \Gamma$, then*

$$[C,\Delta] = s_{(\alpha(\lambda/\mu))}, \tag{3.7}$$

(iii) *if α and $\alpha\mu/\lambda \in \Gamma$, then*

$$[\Delta,C] = s_{(\alpha(\mu/\lambda))}, \tag{3.8}$$

(iv) *if α and $\alpha/\lambda \in \Gamma$, then*

$$[\Delta,\Delta] = s_{(\alpha(\mu/\lambda))}. \tag{3.9}$$

Proof. We have for any given X

$$C(\lambda)(|C(\mu)X|) \in s_\alpha \tag{3.10}$$

if and only if $C(\mu)X \in s_\alpha(C(\lambda)) = s_{(\alpha\lambda)}$, since $\alpha\lambda \in \Gamma$. So we get

$$X \in \Delta(\mu)s_{\alpha\lambda} \tag{3.11}$$

and the condition $\alpha\lambda\mu \in \Gamma$ implies $\Delta(\mu)s_{\alpha\lambda} = s_{(\alpha\lambda\mu)}$, which permits us to conclude (i).

(ii) Now, for any given X, the condition $C(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ is equivalent to

$$|\Delta(\mu)X| \in \Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda} = s_{\alpha\lambda}, \tag{3.12}$$

since $\alpha\lambda \in \Gamma$. Thus

$$X \in C(\mu)s_{\alpha\lambda} = D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{(\alpha(\lambda/\mu))}. \tag{3.13}$$

(iii) Similarly, $\Delta(\lambda)(|C(\mu)X|) \in s_\alpha$ if and only if

$$|C(\mu)X| \in s_\alpha(\Delta(\lambda)) = C(\lambda)s_\alpha = D_{1/\lambda}\Sigma s_\alpha = s_{(\alpha/\lambda)}, \tag{3.14}$$

since $\alpha \in \Gamma$. So

$$X \in \Delta(\mu)s_{(\alpha/\lambda)} = \Delta s_{(\alpha\mu/\lambda)}. \tag{3.15}$$

We conclude since $\alpha\mu/\lambda \in \Gamma$ implies that $\Delta s_{(\alpha\mu/\lambda)} = s_{(\alpha\mu/\lambda)}$. (iv) Here,

$$\Delta(\lambda)(|\Delta(\mu)X|) \in s_\alpha \quad \text{if and only if } \Delta(\mu)X \in C(\lambda)s_\alpha = s_{(\alpha/\lambda)}, \tag{3.16}$$

if $\alpha \in \Gamma$. Thus we have

$$X \in C(\mu)s_{(\alpha/\lambda)} = s_{(\alpha/\lambda\mu)} \tag{3.17}$$

since $\alpha/\lambda \in \Gamma$. So (iv) holds. □

Remark 3.2. If we define

$$[A_1,A_2]_0 = \{X \in s \mid A_1(\lambda)(|A_2(\mu)X|) \in s^\circ_\alpha\}, \tag{3.18}$$

we get the same results as in Theorem 3.1, replacing in each case (i), (ii), (iii), and (iv) s_ξ by s°_ξ.

3.2. Sets $[\Delta,\Delta^+]$, $[\Delta,C^+]$, $[C,\Delta^+]$, $[\Delta^+\Delta]$, $[\Delta^+,C]$, $[\Delta^+\Delta^+]$, $[C^+,C]$, $[C^+,\Delta]$, $[C^+,\Delta^+]$, and $[C^+,C^+]$. We get immediately from the definitions of the operators $\Delta(\xi)$, $\Delta^+(\eta)$, $C(\xi)$, and $C^+(\eta)$, the following:

$$\begin{aligned}
[\Delta,\Delta^+] &= \{X \mid \lambda_n|\mu_n x_n - \mu_{n+1}x_{n+1}| - \lambda_{n-1}|\mu_{n-1}x_{n-1} - \mu_n x_n| = \alpha_n O(1)\},\\
[\Delta,C^+] &= \left\{X \mid \lambda_n\left|\sum_{i=n}^{\infty}\frac{x_i}{\mu_i}\right| - \lambda_{n-1}\left|\sum_{i=n-1}^{\infty}\frac{x_i}{\mu_i}\right| = \alpha_n O(1)\right\},\\
[C,\Delta^+] &= \left\{X \;\middle|\; \frac{1}{\lambda_n}\left(\sum_{k=1}^{n}|\mu_k x_k - \mu_{k+1}x_{k+1}|\right) = \alpha_n O(1)\right\},\\
[\Delta^+,\Delta] &= \{X \mid \lambda_n|\mu_n x_n - \mu_{n-1}x_{n-1}| - \lambda_{n+1}|\mu_{n+1}x_{n+1} - \mu_n x_n| = \alpha_n O(1)\},\\
[\Delta^+,C] &= \left\{X \;\middle|\; \frac{\lambda_n}{\mu_n}\left|\sum_{i=1}^{n}x_i\right| - \frac{\lambda_{n+1}}{\mu_{n+1}}\left|\sum_{i=1}^{n+1}x_i\right| = \alpha_n O(1)\right\},\\
[\Delta^+,\Delta^+] &= \{X \mid \lambda_n|\mu_n x_n - \mu_{n+1}x_{n+1}| - \lambda_{n+1}|\mu_{n+1}x_{n+1} - \mu_{n+2}x_{n+2}| = \alpha_n O(1)\},\\
[C^+,C] &= \left\{X \;\middle|\; \sum_{k=n}^{\infty}\left(\frac{1}{\lambda_k}\left|\frac{1}{\mu_k}\sum_{i=1}^{k}x_i\right|\right) = \alpha_n O(1)\right\},\\
[C^+,\Delta] &= \left\{X \;\middle|\; \sum_{k=n}^{\infty}\left(\frac{1}{\lambda_k}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right) = \alpha_n O(1)\right\},\\
[C^+,\Delta^+] &= \left\{X \;\middle|\; \sum_{k=n}^{\infty}\left(\frac{1}{\lambda_k}|\mu_k x_k - \mu_{k+1}x_{k+1}|\right) = \alpha_n O(1)\right\},\\
[C^+,C^+] &= \left\{X \;\middle|\; \sum_{k=n}^{\infty}\left(\frac{1}{\lambda_k}\left|\sum_{i=k}^{\infty}\frac{x_i}{\mu_i}\right|\right) = \alpha_n O(1)\right\}.
\end{aligned} \tag{3.19}$$

We can assert the following result, in which we do the convention $\alpha_n = 1$ for $n \le 0$.

Theorem 3.3. (i) *Assume that $\alpha \in \Gamma$. Then*

$$\begin{aligned}
[\Delta,\Delta^+] &= s_{(\alpha/\lambda\mu)^-} \quad \textit{if } \frac{\alpha}{\lambda\mu} \in \Gamma,\\
[\Delta,C^+] &= s_{(\alpha(\mu/\lambda))} \quad \textit{if } \frac{\lambda}{\alpha} \in \Gamma.
\end{aligned} \tag{3.20}$$

(ii) *The conditions $\alpha\lambda \in \Gamma$ and $\alpha\lambda/\mu \in \Gamma$ together imply*

$$[C,\Delta^+] = s_{(\alpha(\lambda/\mu))^-}. \tag{3.21}$$

(iii) *The condition* $\alpha/\lambda \in \Gamma$ *implies*

$$[\Delta^+, \Delta] = s_{(\alpha_{n-1}/\mu_n\lambda_{n-1})_n} = s_{(1/\mu(\alpha/\lambda)^-)}. \tag{3.22}$$

(iv) *If* α/λ *and* $\mu(\alpha/\lambda)^- = (\mu_n(\alpha_{n-1}/\lambda_{n-1}))_n \in \Gamma$, *then*

$$[\Delta^+, C] = s_{\mu(\alpha/\lambda)^-}. \tag{3.23}$$

(v) *If* α/λ *and* $1/\mu(\alpha/\lambda)^- = (\alpha_{n-1}/\mu_n\lambda_{n-1})_n \in \Gamma$, *then*

$$[\Delta^+, \Delta^+] = s_{((\alpha/\lambda)^-/\mu)^-} = s_{(\alpha_{n-2}/\lambda_{n-2}\mu_{n-1})_n}. \tag{3.24}$$

(vi) *If* $1/\alpha$ *and* $\alpha\lambda\mu \in \Gamma$, *then*

$$[C^+, C] = s_{(\alpha\lambda\mu)}. \tag{3.25}$$

(vii) *If* $1/\alpha$ *and* $\alpha\lambda \in \Gamma$, *then*

$$[C^+, \Delta] = s_{(\alpha(\lambda/\mu))}. \tag{3.26}$$

(viii) *If* $1/\alpha$ *and* $\alpha(\lambda/\mu) \in \Gamma$, *then*

$$[C^+, \Delta^+] = s_{(\alpha(\lambda/\mu))^-}. \tag{3.27}$$

(ix) *If* $1/\alpha$ *and* $1/\alpha\lambda \in \Gamma$, *then*

$$[C^+, C^+] = s_{(\alpha\lambda\mu)}. \tag{3.28}$$

Proof. (i) First, for any given X, the condition $\Delta(\lambda)(|\Delta^+(\mu)X|) \in s_\alpha$ is equivalent to

$$|\Delta^+(\mu)X| \in s_\alpha(\Delta(\lambda)) = s_{(\alpha/\lambda)}, \tag{3.29}$$

since $\alpha \in \Gamma$. So $X \in s_{\alpha\lambda}(\Delta^+(\mu))$ and applying Corollary 2.9, we conclude the first part of the proof of (i).

We have $\Delta(\lambda)(|C^+(\mu)X|) \in s_\alpha$ if and only if

$$|C^+(\mu)X| \in C(\lambda)s_\alpha = D_{1/\lambda}\Sigma s_\alpha. \tag{3.30}$$

Since $\alpha \in \Gamma$, we have $\Sigma s_\alpha = s_\alpha$ and $D_{1/\lambda}\Sigma s_\alpha = s_{(\alpha/\lambda)}$. Then, for $\alpha/\lambda \in \Gamma^+$, $X \in [\Delta, C^+]$ if and only if

$$X \in w^{+1}_{(\alpha/\lambda)}(\mu) = s_{(\alpha(\mu/\lambda))}. \tag{3.31}$$

(ii) For any given X, $C(\lambda)(|\Delta^+(\mu)X|) \in s_\alpha$ is equivalent to

$$\Delta^+(\mu)X \in w^1_\alpha(\lambda), \tag{3.32}$$

and since $\alpha\lambda \in \Gamma$ we have $w^1_\alpha(\lambda) = s_{\alpha\lambda}$. So

$$X \in s_{\alpha\lambda}(\Delta^+(\mu)) = s_{(\alpha(\lambda/\mu))^-} \tag{3.33}$$

if $\alpha\lambda/\mu \in \Gamma$. Then (ii) is proved.

(iii) Here, $\Delta^+(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ if and only if

$$|\Delta(\mu)X| \in s_\alpha(\Delta^+(\lambda)) = s_{(\alpha/\lambda)^-}, \tag{3.34}$$

since $\alpha/\lambda \in \Gamma$. Thus

$$X \in C(\mu)s_{(\alpha/\lambda)^-} = D_{1/\mu}\Sigma s_{(\alpha/\lambda)^-} = s_{(\alpha_{n-1}/\lambda_{n-1}\mu_n)} \tag{3.35}$$

if $(\alpha/\lambda)^- \in \Gamma$, that is, $\alpha/\lambda \in \Gamma$.

(iv) If $\alpha/\lambda \in \Gamma$, we get

$$\begin{aligned}\Delta^+(\lambda)(|C(\mu)X|) \in s_\alpha &\Longleftrightarrow |C(\mu)X| \in s_\alpha(\Delta^+(\lambda)) \\ &= s_{(\alpha/\lambda)^-} \Longleftrightarrow X \in \Delta(\mu)s_{(\alpha/\lambda)^-}.\end{aligned} \tag{3.36}$$

Since $\mu(\alpha/\lambda)^- \in \Gamma$, we conclude that $[\Delta^+, C] = s_{(\mu(\alpha/\lambda)^-)}$.

(v) One has

$$[\Delta^+,\Delta^+] = \{X \mid \Delta^+(\mu)X \in s_\alpha(\Delta^+(\lambda))\}, \tag{3.37}$$

and since $\alpha/\lambda \in \Gamma$, we get

$$s_\alpha(\Delta^+(\lambda)) = s_{(\alpha/\lambda)^-}. \tag{3.38}$$

We deduce that if $\alpha/\lambda \in \Gamma$,

$$[\Delta^+,\Delta^+] = s_{(\alpha/\lambda)^-}(\Delta^+(\mu)). \tag{3.39}$$

Then, from Corollary 2.9, if $\alpha/\lambda \in \Gamma$ and $(\alpha/\lambda)^-/\mu = (\alpha_{n-1}/\lambda_{n-1}\mu_n)_n \in \Gamma$,

$$s_{(\alpha/\lambda)^-}(\Delta^+(\mu)) = s_{((\alpha/\lambda)^-/\mu)^-} = s_{(\alpha_{n-2}/\lambda_{n-2}\mu_{n-1})_n}. \tag{3.40}$$

(vi) We have

$$C^+(\lambda)(|C(\mu)X|) \in s_\alpha \Longleftrightarrow C(\mu)X \in w_\alpha^{+1}(\lambda), \tag{3.41}$$

and since $\alpha \in \Gamma^+$, we have $w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}$. Then for $\alpha\lambda\mu \in \Gamma$, $X \in [C^+, C]$ if and only if

$$X \in \Delta(\mu)s_{\alpha\lambda} = s_{(\alpha\lambda\mu)}. \tag{3.42}$$

(vii) The condition $C^+(\lambda)(|\Delta(\mu)X|) \in s_\alpha$ is equivalent to

$$\Delta(\mu)X \in w_\alpha^{+1}(\lambda), \tag{3.43}$$

and since $\alpha \in \Gamma^+$, we have $w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}$. Thus

$$X \in s_{\alpha\lambda}(\Delta(\mu)) = D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{(\alpha(\lambda/\mu))}, \tag{3.44}$$

since $\alpha\lambda \in \Gamma$. So (vii) holds.

(viii) First, we have

$$[C^+,\Delta^+] = \{X \mid \Delta^+(\mu)X \in w_\alpha^{+1}(\lambda)\}, \tag{3.45}$$

and the condition $\alpha \in \Gamma^+$ implies that $w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}$. Thus

$$[C^+,\Delta^+] = \{X \mid \Delta^+(\mu)X \in s_{\alpha\lambda}\} = s_{\alpha\lambda}(\Delta^+(\mu)), \tag{3.46}$$

and we conclude since

$$s_{\alpha\lambda}(\Delta^+(\mu)) = s_{(\alpha\lambda/\mu)^-} \quad \text{for } \frac{\alpha\lambda}{\mu} \in \Gamma. \tag{3.47}$$

(ix) If $\alpha \in \Gamma^+$,

$$[C^+,C^+] = \{X \mid C^+(\mu)X \in w_\alpha^{+1}(\lambda) = s_{\alpha\lambda}\} = w_{\alpha\lambda}^{+1}(\mu). \tag{3.48}$$

We conclude that $w_{\alpha\lambda}^{+1}(\mu) = s_{(\alpha\lambda\mu)}$, since $\alpha\lambda \in \Gamma^+$. □

Remark 3.4. Note that in Theorem 3.3, we have $[A_1,A_2] = s_\alpha(A_1A_2) = (s_\alpha(A_1))_{A_2}$ for $A_1 \in \{\Delta(\lambda),\Delta^+(\lambda),C(\lambda),C^+(\lambda)\}$ and $A_2 \in \{\Delta(\mu),\Delta^+(\mu),C(\mu),C^+(\mu)\}$. For instance, we have

$$[\Delta,C] = \left\{X \;\middle|\; \left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n-1}}{\mu_{n-1}}\right)\sum_{i=1}^{n-1} x_i + \frac{\lambda_n}{\mu_n}x_n = \alpha_n O(1)\right\} \quad \text{for } \frac{\alpha\mu}{\lambda} \in \Gamma. \tag{3.49}$$

Similarly, under the corresponding conditions given in Theorems 3.1 and 3.3, we get

$$\begin{aligned}
[\Delta,\Delta] &= \{X \mid -\lambda_{n-1}\mu_{n-2}x_{n-2} + \mu_{n-1}(\lambda_n + \lambda_{n-1})x_{n-1} - \lambda_n\mu_n x_n = \alpha_n O(1)\}, \\
[\Delta,C^+] &= \left\{X \;\middle|\; \frac{\lambda_n}{\mu_n}x_n + (\lambda_n - \lambda_{n-1})\sum_{m=n-1}^{\infty}\frac{x_m}{\mu_m} = \alpha_n O(1)\right\}, \\
[\Delta,\Delta^+] &= \{X \mid -\lambda_{n-1}\mu_{n-1}x_{n-1} + \mu_n(\lambda_n + \lambda_{n-1})x_n - \lambda_n\mu_{n+1}x_{n+1} = \alpha_n O(1)\}, \\
[\Delta^+,\Delta] &= \{X \mid -\lambda_n\mu_{n-1}x_{n-1} + (\lambda_n + \lambda_{n+1})\mu_n x_n - \lambda_{n+1}\mu_{n+1}x_{n+1} = \alpha_n O(1)\}.
\end{aligned} \tag{3.50}$$

References

[1] B. de Malafosse, *Systèmes linéaires infinis admettant une infinité de solutions* [*Infinite linear systems admitting an infinity of solutions*], Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur. **65** (1988), 49–59 (French).

[2] ______, *Some properties of the Cesàro operator in the space s_r*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **48** (1999), no. 1-2, 53–71.

[3] ______, *Bases in sequence spaces and expansion of a function in a series of power series*, Mat. Vesnik **52** (2000), no. 3-4, 99–112.

[4] ______, *Application of the sum of operators in the commutative case to the infinite matrix theory*, Soochow J. Math. **27** (2001), no. 4, 405–421.

[5] ______, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002), no. 2, 283–299.

[6] ______, *Recent results in the infinite matrix theory, and application to Hill equation*, Demonstratio Math. **35** (2002), no. 1, 11–26.

[7] ______, *Variation of an element in the matrix of the first difference operator and matrix transformations*, Novi Sad J. Math. **32** (2002), no. 1, 141–158.

[8] ______, *On some BK spaces*, Int. J. Math. Math. Sci. **2003** (2003), no. 28, 1783–1801.

[9] B. de Malafosse and E. Malkowsky, *Sequence spaces and inverse of an infinite matrix*, Rend. Circ. Mat. Palermo (2) **51** (2002), no. 2, 277–294.

[10] R. Labbas and B. de Malafosse, *On some Banach algebra of infinite matrices and applications*, Demonstratio Math. **31** (1998), no. 1, 153–168.

[11] I. J. Maddox, *Infinite Matrices of Operators*, Lecture Notes in Mathematics, vol. 786, Springer-Verlag, Berlin, 1980.

[12] E. Malkowsky, *The continuous duals of the spaces* $c_0(\Lambda)$ *and* $c(\Lambda)$ *for exponentially bounded sequences* Λ, Acta Sci. Math. (Szeged) **61** (1995), no. 1–4, 241–250.

[13] ______, *Linear operators in certain BK spaces*, Approximation Theory and Function Series (Budapest, 1995), Bolyai Soc. Math. Stud., vol. 5, János Bolyai Mathematical Society, Budapest, 1996, pp. 259–273.

[14] F. Móricz, *On* Λ*-strong convergence of numerical sequences and Fourier series*, Acta Math. Hungar. **54** (1989), no. 3-4, 319–327.

[15] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, North-Holland Publishing, Amsterdam, 1984.

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