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DOUBLE-DUAL n-TYPES OVER BANACH SPACES NOT CONTAINING ℓ_1

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Let E be a Banach space. The concept of *n-type over E* is introduced here, generalizing the concept of *type over E* introduced by Krivine and Maurey. Let E'' be the second dual of E and fix $g_1'',\dots,g_n'' \in E''$. The function $\tau : E\times\mathbb{R}^n \to \mathbb{R}$, defined by letting $\tau(x,a_1,\dots,a_n) = \|x+\sum_{i=1}^n a_i g_i''\|$ for all $x\in E$ and all $a_1,\dots,a_n\in\mathbb{R}$, defines an n-type over E. Types that can be represented in this way are called double-dual n-types; we say that $(g_1'',\dots,g_n'') \in (E'')^n$ realizes τ. Let E be a (not necessarily separable) Banach space that does not contain ℓ_1. We study the set of elements of $(E'')^n$ that realize a given double-dual n-type over E. We show that the set of realizations of this n-type is convex. This generalizes a result of Haydon and Maurey who showed that the set of realizations of a given 1-type over a separable Banach space E is convex. The proof makes use of Henson's language for normed space structures and uses ideas from mathematical logic, most notably the Löwenheim-Skolem theorem.

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1. Introduction. We first give a definition of n-types over Banach spaces and show how this definition generalizes the definition of *type* given by Krivine and Maurey.

For every $\bar{x} = (x_1,\dots,x_n) \in E^n$, define $\tau_{\bar{x}} : E\times\mathbb{R}^n \to \mathbb{R}$ by setting

$$\tau_{\bar{x}}(y,a_1,\dots,a_n) = \left\| y + \sum_{i=1}^n a_i x_i \right\| \tag{1.1}$$

for all $y\in E$ and for all $a_1,\dots,a_n\in\mathbb{R}$.

Definition 1.1. Let E be a Banach space and fix $n\in\mathbb{N}$. For every $\bar{x}\in E^n$, let $\tau_{\bar{x}}$ be defined as above. A function $\tau : E\to\mathbb{R}$ is an *n-type over E* if τ is a function in the closure (with respect to the topology of pointwise convergence) of the set of functions $\{\tau_{\bar{x}} : \bar{x}\in E^n\}$.

Krivine and Maurey [3] defined types over a Banach space E in the following way. For every $x\in E$, let $t_x : E\to\mathbb{R}$ be defined by $t_x(y) = \|y+x\|$ for all $y\in E$. Then a type over E is a function $t : E\to\mathbb{R}$ in the closure (with respect to the topology of pointwise convergence) of the set $\{t_x : x\in E\}$.

The types introduced by Krivine and Maurey coincide with the 1-types introduced in Definition 1.1 above. Indeed, every 1-type over E, τ defines a type (in the sense of Krivine and Maurey) by letting $t(x) = \tau(x,1)$ for all $x\in E$. Conversely, if t is a type (in the sense of Krivine and Maurey), define $\tau(x,a) = |a|t((1/a)x)$ if $a\neq 0$ and $\tau(x,0) = \|x\|$. With this definition, τ is a 1-type over E.

The definition of *n-type over a Banach space E* given above reflects an analyst's view of an n-type as a description of an n-tuple of elements $(u_1,\dots,u_n)$ from a Banach space ultrapower of E. This notion of n-type coincides with the model theorist's notion of quantifier-free n-type over E in the language of Banach spaces. The reader is referred to [2] for more details.

Let E'' denote the second dual of E and let $\bar{g}'' = (g_1'',\dots,g_n'') \in (E'')^n$. Define $\tau_{\bar{g}''} : E\times\mathbb{R}\to\mathbb{R}$ by setting $\tau_{\bar{g}''}(y,a_1,\dots,a_n) = \|y+\sum_{i=1}^n a_i g_i''\|$ for all $y\in E$ and all $a_1,\dots,a_n\in\mathbb{R}$. By the principle of local reflexivity, the function $\tau_{\bar{g}''}$ is an n-type over E. Types that can be realized in this way are called *double-dual* n-types over E.

Suppose that $A\subseteq E$ and $\bar{g}'' = (g_1'',\dots,g_n'') \in (E'')^n$. We let $\mathrm{tp}(\bar{g}''/A)$ denote the function $\tau : A\times\mathbb{R}^n\to\mathbb{R}$ defined by setting $\tau(x,a_1,\dots,a_n) = \|x+\sum_{i=1}^n a_i g_i''\|$ for all $x\in A$ and all $a_1,\dots,a_n\in\mathbb{R}$.

Let τ be a double-dual n-type over E. Following the notation introduced in [1], we let

$$\mathrm{Rep}[\tau] = \{\bar{g}''\in(E'')^n : \tau = \tau_{\bar{g}''}\} \tag{1.2}$$

be the set of elements of $(E'')^n$ that realize τ.

2. Statement of the main theorem. The purpose of this paper is to prove the following theorem.

Theorem 2.1. *Let E be a Banach space that does not contain ℓ_1. Let τ be a double-dual n-type over E. Then* $\mathrm{Rep}[\tau]$ *is convex.*

If we take $n=1$, we obtain the following proposition.

Proposition 2.2. *Let E be a Banach space that does not contain ℓ_1. Let τ be a double-dual* 1*-type over E. Then* $\mathrm{Rep}[\tau]$ *is convex.*

The previous proposition is a generalization of a result of Haydon and Maurey [1, Theorem 3.2].

Theorem 2.3 (Haydon and Maurey)**.** *Let E be a separable Banach space that does not contain ℓ_1. Let τ be a double-dual* 1*-type over E. Then* $\mathrm{Rep}[\tau]$ *is convex.*

The proof provided in [1] requires the hypothesis that E is separable. The purpose of this paper is to show how methods from model theory can be used to remove this hypothesis.

The proof of Proposition 2.2 will be provided in Section 3. The following proposition shows that Theorem 2.1 is an immediate consequence of Proposition 2.2.

Proposition 2.4. *Let E be a Banach space. Suppose that* $\mathrm{Rep}[\tau]$ *is convex whenever τ is a double-dual* 1*-type over E. Then* $\mathrm{Rep}[\tau]$ *is convex whenever τ is a double-dual n-type over E.*

Proof. Let τ be a double-dual n-type over E and suppose that τ is realized in $(E'')^n$ by $(g_1'',\dots,g_n'')$. For any given n-tuple $(a_1,\dots,a_n)\in\mathbb{R}^n$, consider the function

$$D_{(a_1,\dots,a_n)} : (E'')^n \longrightarrow E'', \quad (u_1'',\dots,u_n'') \longmapsto \sum_{i=1}^n a_i u_i''. \tag{2.1}$$

This map is $\sigma(E'',E')$-continuous and linear.

Therefore, $D^{-1}_{(a_1,\dots,a_n)}(\mathrm{Rep}[\mathrm{tp}(\sum_{i=1}^n a_i g_i''/E)])$ is convex for all $a_1,\dots,a_n \in \mathbb{R}$. Thus

$$\mathrm{Rep}\left[\mathrm{tp}\left(g_1'',\dots,g_n''\right)\right] = \bigcap_{(a_1,\dots,a_n)\in\mathbb{R}^n} D^{-1}_{(a_1,\dots,a_n)}\left(\mathrm{Rep}\left[\mathrm{tp}\left(\sum_{i=1}^n a_i g_i''/E\right)\right]\right) \tag{2.2}$$

is convex. □

3. Lemmas. The proof of Proposition 2.2 requires a sequence of lemmas, which will be discussed in this section. The proof of the proposition will be provided at the end of this section.

Throughout this section, we assume that E is a Banach space that does not contain ℓ_1. We denote by E' its dual and by E'' its second dual. If F is any Banach space and $G \subseteq F'$, we denote by $\sigma(F,G)$ the topology on F induced by open sets of the form $\{x \in F \mid \langle x, g_i\rangle \le \varepsilon \text{ for all } i = 1,\dots,n\}$, where $n \in \mathbb{N}$, $g_1,\dots,g_n \in G$, and $\varepsilon > 0$. The closure of a set $V \subseteq F$ with respect to this topology is denoted by $\sigma(F,G)\,\mathrm{cl}(V)$. The weak*-topology on E'' is therefore denoted by $\sigma(E'',E')$. For brevity, we write $\widetilde{V} = \sigma(E'',E')\,\mathrm{cl}(V)$ for any $V \subseteq E''$.

If F is a normed space and $M > 0$, we let $\mathcal{B}_M(F) = \{x \in F \mid \|x\| \le M\}$ denote the closed M-ball in F.

Let τ be a double-dual 1-type over E. Let $S = \mathrm{Rep}[\tau]$ and $g_1'', g_2'' \in S$. In order to prove that S is convex, we need to show that the line segment joining g_1'' and g_2'' is contained in S. Set $M = \|g_1''\|$.

We will use Henson's language for normed space structures. See [2] for more details. It is assumed that the reader is familiar with the concepts of normed space structures [2, Sections 2 and 3], positive bounded formulas (Section 5), approximate satisfaction and approximate elementary substructures (Section 6), and the Löwenheim-Skolem theorem (Section 9).

Consider the $\mathscr{L}$-structure (E,E',E'') whose sorts are $\mathbb{R}$, E, E', and E'' and whose functions are addition, scalar multiplication and the norm for each sort, the absolute value function for real numbers, the constants g_1'' and g_2'', and the following additional functions:

$$\begin{aligned}
&E \longrightarrow E'', \quad x \longmapsto I(x) := \text{the canonical image of } x \text{ in } E'',\\
&E'' \longrightarrow \mathbb{R}, \quad x'' \longmapsto d(x'') := \inf\{\|x''-s''\| : s'' \in S\},\\
&E \times E' \longrightarrow \mathbb{R}, \quad (x,x') \longmapsto \langle x,x'\rangle,\\
&E' \times E'' \longrightarrow \mathbb{R}, \quad (x',x'') \longmapsto \langle x',x''\rangle.
\end{aligned} \tag{3.1}$$

Lemma 3.1. *Let A_c be a separable subset of E. There exists a separable approximate elementary substructure of (E,E',E''), $(A,B,C) \preceq_A (E,E',E'')$, such that $A_c \subseteq A$ and for all $c \in C$ and all $\delta > 0$, the following condition holds:*

(i) *$(A,B,C) \vDash_A d(c) \le \delta$ implies that there exists a $c_0 \in C$ such that $\|c_0 - c\| \le 2\delta$ and $d(c_0) = 0$.*

Furthermore, for any approximate elementary substructure of (E,E',E''), the following holds:

(ii) *if $d(c) > 0$, then there exists $a \in A$ with $\|a + g_1''\| \ne \|a + c\|$.*

Proof. Using the downward Löwenheim-Skolem theorem [2, Theorem 9.14], choose a separable approximate elementary substructure of (E,E',E''),

$$(A_0,B_0,C_0) \preceq_A (E,E',E''), \tag{3.2}$$

such that $A_c \subseteq A$. Let $c_1,c_2,\ldots$ enumerate a dense subset of C_0. There exist $e_1'',e_2'',\ldots \in E''$ such that

$$\|c_j - e_j''\| \leq 2d(c_j), \quad d(e_j'') = 0, \tag{3.3}$$

for all $j \in \mathbb{N}$.

There exists another separable approximate elementary substructure of (E,E',E''),

$$(A_1,B_1,C_1) \preceq_A (E,E',E''), \tag{3.4}$$

which contains $(A_0,B_0,C_0 \cup \{e_1'',e_2'',\ldots\})$. We continue in this fashion through countably many steps and then take

$$(A,B,C) = \mathrm{cl}\left(\bigcup_{k=1}^{\infty}(A_k,B_k,C_k)\right). \tag{3.5}$$

This structure is an approximate elementary substructure of (E,E',E'') in which (i) holds.

Condition (ii) holds in every approximate elementary substructure of (E,E',E''). Suppose (A,B,C) is an approximate elementary substructure of (E,E',E'') and $c \in C$ such that $d(c) = \delta_1 > 0$. Then $c \notin S = \mathrm{Rep}[\mathrm{tp}(g_1''/E)]$. Therefore, there exist $e \in E$ and $\delta_2 > 0$ such that $|\|e+c\| - \|e+g_1''\|| \geq \delta_2$. Let $m = \|e\|+1$. Hence

$$(E,E',E'') \vDash_A \exists_m x\big(\big|\|x+c\| - \|x+g_1''\|\big| \leq \delta_2\big). \tag{3.6}$$

Here, the variable x ranges over the sort associated with E. Condition (ii) follows because the same formula is approximately true in (A,B,C). □

Lemma 3.2. *Let $(A,B,C) \preceq_A (E,E',E'')$ as in Lemma 3.1. There exists an isometric embedding $P : B \to A'$ such that $\langle a,Pb\rangle = \langle a,b\rangle$ for all $a \in A$ and all $b \in B$.*

Proof. Define $P : B \to A'$ by setting $\langle a,Pb\rangle = \langle a,b\rangle$ for all $b \in B$ and all $a \in A$. The function P is linear; we need to show that it is an isometry. Let $1 > \varepsilon > 0$. Observe that

$$(A,B,C) \vDash_A \forall_1 b\big(\|b\| \leq (1-\varepsilon)^3 \vee \exists_1 a\big(\langle a,b\rangle \geq (1-\varepsilon)^4\big)\big) \tag{3.7}$$

because the same sentence is approximately true in the structure (E,E',E'') by the definition of the norm of a linear functional. Here, the variable b ranges over B and a ranges over A. Now, fix $b \in B$ of norm 1 and set $b_0 = (1-\varepsilon)b$. Consider the sentence

$$\forall_{1-\varepsilon} b\big(\|b\| \leq (1-\varepsilon)^2 \vee \exists_{(1-\varepsilon)^{-1}} a\big(\langle a,b\rangle \geq (1-\varepsilon)^2\big)\big). \tag{3.8}$$

Because this sentence is true in (A,B,C), there exists $a \in A$ such that $\|a\| \leq (1-\varepsilon)^{-1}$ and $\langle a, b_0\rangle \geq (1-\varepsilon)^2$. Then

$$\|Pb\| = (1-\varepsilon)^{-1}\|Pb_0\| \geq \frac{\langle a, Pb_0\rangle}{(1-\varepsilon)\|a\|} \geq \frac{(1-\varepsilon)^2}{(1-\varepsilon)(1-\varepsilon)^{-1}} = (1-\varepsilon)^2. \tag{3.9}$$

Thus, $\|P\| \geq 1$. For each $b \in B$, Pb is the restriction of b to A and we obtain $\|Pb\| \leq 1$. □

The following lemma is not needed but is of its own interest.

LEMMA 3.3. *Let $(A,B,C) \preceq_A (E,E',E'')$ as in Lemma 3.1. Let U denote the unit ball of A' and let V denote the unit ball of PB. Then V is $\sigma(A',A)$-dense in U.*

PROOF. Let $\widetilde{V} = \sigma(A',A)\,\mathrm{cl}(V)$. Lemma 3.2 and the weak*-lower semicontinuity of the norm yield $\widetilde{V} \subseteq U$. Suppose $\widetilde{V} \neq U$. Then there exists $b_0 \in U \setminus \widetilde{V}$. Since $\widetilde{V}$ is convex and weak*-closed (i.e., $\sigma(A',A)$-closed) and $\{b_0\}$ is $\sigma(A',A)$-compact, there exist a weak*-continuous linear functional a of norm at most 1 and real numbers $r < s$ such that for all $b \in \widetilde{V}$,

$$\langle a, b\rangle \leq r < s < \langle a, b_0\rangle \leq 1. \tag{3.10}$$

Since $\widetilde{V}$ is symmetric about the origin, we get

$$|\langle a, b\rangle| \leq r < s < \langle a, b_0\rangle \leq 1 \tag{3.11}$$

for all $b \in \widetilde{V}$. Because a is a weak*-continuous linear functional on A', we see that $a \in \mathcal{B}_1(A)$. But then

$$(A,B,C) \vDash_A \exists_1 a\big((\|a\| \geq s) \wedge \forall_1 b(-r \leq \langle a, b\rangle \leq r)\big), \tag{3.12}$$

where $r < s \leq 1$ are as before. Here, the variable a ranges over the sort associated with A and b ranges over the sort associated with B. We may then choose $\varepsilon > 0$ such that $(1+\varepsilon)^3 r < s$. Set $\lambda = (1+\varepsilon)$. Because

$$(E,E',E'') \vDash_A \exists_1 a\big((\|a\| \geq s) \wedge \forall_1 b(-r \leq \langle a, b\rangle \leq r)\big), \tag{3.13}$$

we obtain

$$(E,E',E'') \vDash \exists_\lambda a\big((\|a\| \geq \lambda^{-1}s) \wedge \forall_{1/\lambda} b(-\lambda r \leq \langle a, b\rangle \leq \lambda r)\big). \tag{3.14}$$

Fix such an element $a \in E$. We obtain

$$\begin{aligned} \|a\| &= \sup\{\langle a, b\rangle : b \in \mathcal{B}_1(E')\} \\ &= \sup\{(1+\varepsilon)\langle a, b\rangle : b \in \mathcal{B}_{(1+\varepsilon)^{-1}}(E')\} \\ &\leq (1+\varepsilon)^2 r < (1+\varepsilon)^{-1} s \leq \|a\|. \end{aligned} \tag{3.15}$$

This is a contradiction. □

Let $(x_k)_{k\in\mathbb{N}}$ enumerate a dense set in A and $\{\lambda_1,\lambda_2,\dots\}$ enumerate a dense set in $\mathbb{R}$. Without loss of generality, $x_0 = 0$, $\lambda_0 = 0$, and $\lambda_1 = 1$.

Lemma 3.4. *Let $(A,B,C) \preceq_A (E,E',E'')$ as in Lemma 3.1 and let P be as given by Lemma 3.2. For $i = 1,2$, there exists a bounded sequence $(a_{i,j})_{j\in\mathbb{N}}$ in A such that $\lim_{j\to\infty}\langle a_{i,j} - g_i'', b\rangle = 0$ for all $b \in B$ and $\|x_{k_1} + \lambda_{k_2}a_{1,j} + \lambda_{k_3}a_{2,j}\| \le \|x_{k_1} + \lambda_{k_2}g_1'' + \lambda_{k_3}g_2''\| + 2^{-j+1}$ for all $1 \le k_1,k_2,k_3 \le j \in \mathbb{N}$.*

Proof. Let $b_1,b_2,\dots$ enumerate a dense set in B. Fix $j \in \mathbb{N}$. Let $k_1,k_2,k_3 \le j$ and consider the set

$$U_{k_1,k_2,k_3,j} = \{(e_1,e_2) \in E^2 : ||x_{k_1} + \lambda_{k_2}e_1 + \lambda_{k_3}e_2|| < ||x_{k_1} + \lambda_{k_2}g_1'' + \lambda_{k_3}g_2''|| + 2^{-j}\}. \tag{3.16}$$

Observe that $U_{k_1,k_2,k_3,j}$ is an open convex set. Furthermore, $(g_1'',g_2'') \in {}^\circ(\widetilde{U}_{k_1,k_2,k_3,j})$ for all $k_1,k_2,k_3 \le j$. Thus

$$(g_1'',g_2'') \in \bigcap_{k_1,k_2,k_3\le j} {}^\circ\big(\widetilde{U}_{k_1,k_2,k_3,j}\big). \tag{3.17}$$

A consequence of the Hahn-Banach theorem (see [5, Section 15, Lemma II.E., pages 76–79]) yields that $\bigcap_{k_1,k_2,k_3\le j} U_{k_1,k_2,k_3,j}$ is not empty and

$$(g_1'',g_2'') \in {}^\circ\left(\widetilde{\left(\bigcap_{k_1,k_2,k_3\le j} U_{k_1,k_2,k_3,j}\right)}\right). \tag{3.18}$$

Therefore, there exists $(e_{1,j},e_{2,j}) \in \bigcap_{k_1,k_2,k_3\le j} U_{k_1,k_2,k_3,j}$ such that $|\langle e_{i,j} - g_i'', b_k\rangle| \le 2^{-j}$ for $i = 1,2$ and all $k \le j$.

Let y_1,y_2 be variables that range over the sort associated with E. For all $k_1,k_2,k_3 \le j$, let $\phi_{k_1,k_2,k_3,j}(y_1,y_2)$ be the positive bounded $\mathcal{L}(A,B,C)$-formula

$$||x_{k_1} + \lambda_{k_2}y_1 + \lambda_{k_3}y_2|| - ||x_{k_1} + \lambda_{k_2}g_1'' + \lambda_{k_3}g_2''|| \le 2^{-j}. \tag{3.19}$$

For all $k \le j$, let $\psi_{k,j}(y_1,y_2)$ be the positive bounded $\mathcal{L}(A,B,C)$-formula

$$|\langle y_1 - g_1'', b_k\rangle| \le 2^{-j} \wedge |\langle y_2 - g_2'', b_k\rangle| \le 2^{-j}. \tag{3.20}$$

Let $\rho_j(y_1,y_2)$ be the positive bounded $\mathcal{L}(A,B,C)$-formula

$$\bigwedge_{k_1,k_2,k_3\le j} \phi_{k_1,k_2,k_3,j}(y_1,y_2) \wedge \bigwedge_{k\le j} \psi_{k,j}(y_1,y_2). \tag{3.21}$$

Recall that $M = \|g_1\|$. By the initial observation, we have

$$(E,E',E'') \vDash_A \exists_{M+1}y_1\,\exists_{M+1}y_2\,\rho_j(y_1,y_2). \tag{3.22}$$

Because $(A,B,C) \preceq_A (E,E',E'')$, we have

$$(A,B,C) \vDash_A \exists_{M+1}y_1\,\exists_{M+1}y_2\,\rho_j(y_1,y_2). \tag{3.23}$$

We may therefore choose $a_{1,j}$ and $a_{2,j}$ in A such that

$$\|x_{k_1}+\lambda_{k_2}a_{1,j}+\lambda_{k_3}a_{2,j}\|<\|x_{k_1}+\lambda_{k_2}g_1''+\lambda_{k_3}g_2''\|+2^{-j+1} \tag{3.24}$$

for all $k_1,k_2,k_3\le j$ and $|\langle a_{i,j}-g_i'',b_k\rangle|\le 2^{-j+1}$ for $i=1,2$ and all $k\le j$. Further, $\|a_{i,j}\|\le M+1$. Thus, the sequences $(a_{i,j})_{j\in\mathbb{N}}$ are bounded for $i=1,2$. The statement of the lemma follows because $\{b_1,b_2,\dots\}$ is dense in B and $(a_{i,j})_{j\in\mathbb{N}}$ is bounded for $i=1,2$. □

The hypothesis that E does not contain ℓ_1 has not been used so far. In the following two lemmas, we make use of this hypothesis.

Lemma 3.5. *Let $(A,B,C)\preceq_A(E,E',E'')$ as in Lemma 3.1. There exists an isometric embedding $Q:\operatorname{span}(A\cup\{g_1'',g_2''\})\to A''$ such that $\langle Pb,Q(Ia+\lambda g_1''+\mu g_2'')\rangle=\langle b,Ia+\lambda g_1''+\mu g_2''\rangle$ for all $a\in A$, $b\in B$, and $\lambda,\mu\in\mathbb{R}$.*

Proof. Using Lemma 3.4, there exists, for each $i=1,2$, a bounded sequence $(a_{i,j})_{j\in\mathbb{N}}$ in A such that $\lim_{j\to\infty}\langle a_{i,j}-g_i'',b\rangle=0$ for all $b\in B$ and $\|x_{k_1}+\lambda_{k_2}a_{1,j}+\lambda_{k_3}a_{2,j}\|\le\|x_{k_1}+\lambda_{k_2}g_1''+\lambda_{k_3}g_2''\|+2^{-j+1}$ for all $1\le k_1,k_2,k_3\le j\in\mathbb{N}$. Because A does not contain ℓ_1 and the sequence $(a_{i,j})_{j\in\mathbb{N}}$ is bounded, we may apply Rosenthal's theorem [4]. We obtain a function $j:\mathbb{N}\to\mathbb{N}$ with $m\le j(m)<j(m+1)$ for all $m\in\mathbb{N}$ such that the subsequence $(a_{i,j(m)})_{m\in\mathbb{N}}$ is $\sigma(A,A')$-Cauchy for each $i=1,2$. We then define, for every $i=1,2$, a linear functional $\Psi_i\in A''$ by setting $\Psi_i(a')=\lim_{m\to\infty}\langle a_{i,j(m)},a'\rangle$ for all $a'\in A'$. We then define a linear operator $Q:\operatorname{span}(A\cup\{g_1'',g_2''\})\to A''$ by setting $Qg_i''=\Psi_i$ for $i=1,2$ and $Qa=Ja$ for all $a\in A$. (Here, $J:A\to A''$ denotes the canonical embedding from A into A''.)

It is immediate from the definition that Q fixes IA pointwise and $\langle Pb,Q(Ia+\lambda g_1''+\mu g_2'')\rangle=\langle b,Ia+\lambda g_1''+\mu g_2''\rangle$ for all $a\in A$, $b\in B$, and $\lambda,\mu\in\mathbb{R}$.

We show that Q is an isometry. Let $\lambda,\mu\in\mathbb{R}$ and $a\in A$. Because $P(B)\subseteq A'$, we have

$$\begin{aligned}\|Q(\lambda g_1''+\mu g_2''+a)\|&=\sup\{\langle Q(\lambda g_1''+\mu g_2''+a),a'\rangle:a'\in\mathcal{B}_1(A')\}\\&\ge\sup\{\langle Q(\lambda g_1''+\mu g_2''+a),a'\rangle:a'\in\mathcal{B}_1(P(B))\}\\&=\sup\{\langle\lambda g_1''+\mu g_2''+a,b\rangle:b\in\mathcal{B}_1(B)\}\\&=\|\lambda g_1''+\mu g_2''+a\|.\end{aligned} \tag{3.25}$$

On the other hand, the norm is $\sigma(A'',A')$-lower semicontinuous. Thus, for all integers $k_1,k_2,k_3\in\mathbb{N}$, we have

$$\begin{aligned}\|Q(x_{k_1}+\lambda_{k_2}g_1''+\lambda_{k_3}g_2'')\|&\le\liminf_{m\to\infty}\|Q(x_{k_1}+\lambda_{k_2}a_{1,j(m)}+\lambda_{k_3}a_{2,j(m)})\|\\&\le\liminf_{m\to\infty}\left(\|x_{k_1}+\lambda_{k_2}a_{1,j(m)}+\lambda_{k_3}a_{2,j(m)}\|+2^{-j(m)+1}\right)\\&\le\|x_{k_1}+\lambda_{k_2}g_1''+\lambda_{k_3}g_2''\|.\end{aligned} \tag{3.26}$$

Therefore, Q is an isometry. □

Let $C_0=\operatorname{span}(A\cup\{g_1'',g_2''\})$. Since every element $g''\in S$ realizes $\operatorname{tp}(g_1''/E)$, every element $c\in Q(S\cap C_0)$ realizes $\operatorname{tp}(g_1''/A)$. Indeed, if $c\in Q(S\cap C_0)$, then $c=Q(g'')$ for

some $g'' \in S \cap C_0$. Then, for every $a \in A$, we have

$$\|a+c\| = \|a+g''\| = \|a+g_1''\|. \tag{3.27}$$

So, $Q(S \cap C_0) \subseteq \mathrm{Rep}[\mathrm{tp}(g_1''/A)] \cap Q(C_0)$.

Conversely, $Q(S \cap C_0) \supseteq \mathrm{Rep}[\mathrm{tp}(g_1''/A)] \cap Q(C_0)$. Indeed, if $c \in \mathrm{Rep}[\mathrm{tp}(g_1''/A)] \cap Q(C_0)$, there exists $g'' \in C_0$ such that $Q(g'') = c$. We show that $g'' \in S$: suppose that $g'' \notin S$. Then $d(g'') > 0$, and by Lemma 3.1(ii), there exists $a \in A$ such that $\|a+g''\| \neq \|a+g_1''\|$. But then $\|a+c\| = \|a+Qg''\| = \|a+g''\| \neq \|a+g_1''\|$, which contradicts the assumption that $c \in \mathrm{Rep}[\mathrm{tp}(g_1''/A)]$. Therefore, $g'' \in S \cap C_0$, and so $c = Q(g'') \in Q(S \cap C_0)$.

We obtain

$$Q(S \cap C_0) = \mathrm{Rep}\left[\mathrm{tp}\left(g_1''/A\right)\right] \cap Q(C_0). \tag{3.28}$$

The following lemma shows that S contains all convex combinations of g_1'' and g_2''. Because g_1'' and g_2'' are arbitrary elements of $S = \mathrm{Rep}[\tau]$, this shows that $\mathrm{Rep}[\tau]$ is convex.

Lemma 3.6. *Let $(A,B,C) \preceq_A (E,E',E'')$ as in Lemma 3.1 and let Q be as given by Lemma 3.5. Then $Q(C \cap S)$ contains all convex combinations of Qg_1'' and Qg_2'', and S contains all convex combinations of g_1'' and g_2''.*

Proof. Let $\lambda \in [0,1]$. By construction, the signature has constants g_1'' and g_2'' of the sort associated with E''. Therefore, $g_1'', g_2'' \in C$. Since $g_1'', g_2'' \in S$, we obtain $g_1'', g_2'' \in S \cap C$. By the previous remark, $Q(g_1'')$ and $Q(g_2'')$ are elements of $\mathrm{Rep}[\mathrm{tp}(g_1''/A)]$. Since A is separable, Theorem 2.3 yields that $\mathrm{Rep}[\mathrm{tp}(g_1''/A)]$ is convex. Therefore

$$Q(\lambda g_1'' + (1-\lambda)g_2'') = \lambda Q g_1'' + (1-\lambda) Q g_2'' \in \mathrm{Rep}\left[\mathrm{tp}\left(g_1''/A\right)\right] \cap Q(C_0). \tag{3.29}$$

By (3.28),

$$\lambda g_1'' + (1-\lambda)g_2'' \in S \cap C_0 \subseteq \mathrm{Rep}\left[\mathrm{tp}\left(g_1''/E\right)\right]. \tag{3.30}$$

□

We are now ready to prove Proposition 2.2.

Proof. Let E be a Banach space that does not contain ℓ_1, and let τ be a double-dual 1-type over E. Let $g_1'', g_2'' \in S = \mathrm{Rep}[\tau]$. Let $A_0 \subseteq E$ be any separable set. Then choose an approximate elementary substructure $(A,B,C) \preceq (E,E',E'')$ as in Lemma 3.1. Choose the isometric embedding Q as in Lemma 3.5. By Lemma 3.6, $S = \mathrm{Rep}[\tau]$ contains all linear combinations of the form $\lambda g_1'' + (1-\lambda)g_2''$. Because $g_1'', g_2'' \in \mathrm{Rep}[\tau]$ were arbitrary, we obtain that $\mathrm{Rep}[\tau]$ is convex. □

4. Remarks and questions. We conclude this paper by remarking that the hypothesis that E does not contain ℓ_1 in Proposition 2.2 cannot be removed. Indeed, let $E = \ell_1$. Choose $g'' \in E''$ in the band $\ell_1^{\perp\perp}$ with $\|g''\| = 1$. Then $\tau_{g''}(x) = \|x+g''\| = \|x\| + \|g''\| = \|x\| + 1$ and $\tau_{-g''}(x) = \|x-g''\| = \|x\| + \|g''\| = \|x\| + 1$ for all $x \in E$. Therefore, $-g'' \in \mathrm{Rep}[\tau_{g''}]$, which shows that $\mathrm{Rep}[\tau_{g''}]$ is not convex.

Haydon and Maurey also proved that $\mathrm{Rep}[\tau]$ is compact with respect to the weak*-topology if E is separable and τ is a double-dual 1-type over E.

This poses the following question.

Question 4.1. Let E be a (not necessarily separable) Banach space that does not contain ℓ_1. Let τ be a double-dual 1-type over E. Is then $\mathrm{Rep}[\tau]$ compact with respect to the weak*-topology on E''?

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