IJMMS 2004:36, 1887–1892
PII. S0161171204310422
http://ijmms.hindawi.com
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PERIODIC RINGS WITH FINITELY GENERATED UNDERLYING GROUP

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Received 21 October 2003

We study periodic rings that are finitely generated as groups. We prove several structure results. We classify periodic rings that are free of rank at most 2, and also periodic rings R such that R is finitely generated as a group and $R/t(R) \simeq \mathbb{Z}$. In this way, we construct new classes of periodic rings. We also ask a question concerning the connection to periodic rings that are finitely generated as rings.

2000 Mathematics Subject Classification: 16R10, 20K15.

1. Introduction. We work with associative rings which do not necessarily have identity. A ring R is called periodic if its multiplicative semigroup is periodic, that is, the semigroup generated by any element is finite. This is equivalent to the fact that for any $x \in R$, there exist positive integers $m < n$ such that $x^m = x^n$. For other characterizations of periodic rings, see [7]. A systematic study of periodic rings was initiated in [5]. Examples of periodic rings are finite rings, nil rings, Boolean rings, and matrix rings over algebraic extensions of finite fields. There have been two main directions of study concerned with periodic rings. The first one was to find sufficient conditions for the commutativity of a periodic ring. The interest in this direction goes back to [6]. For more recent developments, see also [1, 4] and the references indicated therein. The second direction of study was to find structure results for periodic rings; see [2, 3, 7, 8]. The structure of periodic rings is far from being understood, and this is due, of course, to the fact that the class of periodic rings is very large. However, apart from the examples mentioned above and the ones constructed from them (like, for instance, taking finite direct products of periodic rings), there are not many other known classes. We are interested in the following general problem.

Problem 1.1. Describe periodic rings R such that the underlying abelian group $(R,+)$ is finitely generated.

Of course, a solution to this problem is up to knowing finite rings, so when we work on this problem, we assume that somehow we know finite rings. In this paper, we give some partial answer to this question. We will see that the class of such rings is quite complex itself. In particular, we will obtain new examples of periodic rings.

We show that if the periodic ring R has identity and $(R,+)$ is finitely generated, then R is necessarily finite. We obtain some structure results for periodic rings whose underlying additive groups are free; in particular, we classify all such rings where $(R,+)$

is free of rank 1 or 2. For rank at most 2, these rings are all commutative. We give examples of noncommutative periodic R such that $(R,+)$ is free of rank 3. We also investigate some periodic rings R such that $(R,+)$ has nontrivial torsion and torsion-free components. We end by discussing some facts about periodic rings that are finitely generated as rings.

2. Preliminary results.

Our main problem has an easy solution in the case where the periodic ring R has identity.

Proposition 2.1. *Let R be a periodic ring with identity such that $(R,+)$ is finitely generated. Then R is finite.*

Proof. Let 1_R be the identity element of R. Then there exists a positive integer n such that $n1_R = 0$, since otherwise, $\mathbb{Z} \simeq \mathbb{Z}1_R$ is a subring of R; thus it is periodic, a contradiction. Hence, for any $x \in R$, we have $nx = (n1_R)x = 0$, showing that $(R,+)$ is torsion. Since $(R,+)$ is finitely generated, we conclude that R must be finite. □

For the rest of the paper, we deal with rings which do not necessarily have identity.

Proposition 2.2. *Let R be a periodic ring such that $(R,+)$ is finitely generated. Then R fits into an extension $F \to R \to S$, where F is a finite ring and S is a periodic ring such that $(S,+)$ is torsion-free.*

Proof. Write $(R,+) \simeq \mathbb{Z}^n \oplus t(R)$ for some nonnegative integer n, where $t(R)$ is the torsion part of R. Then $t(R)$ is an ideal of R. Indeed, if $x \in t(R)$, let n be a positive integer such that $nx = 0$. Then for any $r \in R$, we have $nrx = nxr = 0$, so $rx, xr \in t(R)$. The proof is finished if we take $F = t(R)$ and $S = R/t(R)$. □

The previous result shows that describing periodic rings with finitely generated underlying abelian group reduces to knowing the finite rings, the periodic rings which are free of finite rank as groups, and computing some ring extensions. Now we focus our attention on periodic rings R such that $(R,+) \simeq \mathbb{Z}^n$ for some n.

Lemma 2.3. *Let R be a periodic ring such that $(R,+)$ is torsion-free. Then any element of R is nilpotent.*

Proof. Let $x \in R$, $x \neq 0$. Since R is periodic, there exist $n_1, p > 0$ such that $x^{n+p} = x^n$ for any $n \geq n_1$. Also, there exist $n_2, q > 0$ such that $(2x)^{n+q} = (2x)^n$ for any $n \geq n_2$. Then for $n \geq \max(n_1, n_2)$, we have that

$$2^{n+pq}x^n = 2^{n+pq}x^{n+pq} = (2x)^{n+pq} = (2x)^n = 2^n x^n, \tag{2.1}$$

so $(2^{n+pq} - 2^n)x^n = 0$, which shows that $x^n = 0$ since R is torsion-free. □

As a first consequence, we obtain the structure of periodic rings that are free of rank 1.

Corollary 2.4. *Let R be a periodic ring such that $(R,+) \simeq \mathbb{Z}$. Then R has trivial multiplication.*

Proof. Let $R = \mathbb{Z}x$ and let $x^2 = ax$ for some $a \in \mathbb{Z}$. Then $x^n = a^{n-1}x$ for any n. Since x is nilpotent, we must have $a = 0$, and then $x^2 = 0$. We conclude that R has trivial multiplication. □

Lemma 2.3 needs no assumption on $(R,+)$ being finitely generated. If $(R,+)$ is free of finite rank, then we have more precise information about the nilpotency index of its elements.

Lemma 2.5. *Let R be a periodic ring such that $(R,+) \simeq \mathbb{Z}^m$. Then $x^{m+1} = 0$ for any $x \in R$.*

Proof. Let $x \in R$, $x \neq 0$, and let $n \geq 2$ be the smallest positive integer for which $x^n = 0$. Then the sum $\mathbb{Z}x + \mathbb{Z}x^2 + \cdots + \mathbb{Z}x^{n-1}$ is direct. Indeed, if $a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = 0$, then multiplying this relation by x^{n-2}, we get $a_1 = 0$. Then multiplying by x^{n-3}, we see that $a_2 = 0$, and, continuing, we see that all coefficients must be 0. Hence $\mathbb{Z}x \oplus \mathbb{Z}x^2 \oplus \cdots \oplus \mathbb{Z}x^{n-1} \simeq \mathbb{Z}^{n-1}$ is a subgroup of $\mathbb{Z}^m$, and this implies that $n-1 \leq m$. In particular, $x^{m+1} = 0$. □

3. Periodic rings R with $(R,+)$ free of rank 2.

In the following result, we classify periodic rings which are free of rank 2 as groups.

Theorem 3.1. *Let R be a periodic ring such that $(R,+) \simeq \mathbb{Z}^2$. Then either R has trivial multiplication or there exist $a,b \in \mathbb{Z}$, $b \neq 0$, such that $b^2 | a^3$ and $R = \mathbb{Z}x \oplus \mathbb{Z}y$, with multiplication defined by*

$$x^2 = ax + by, \qquad y^2 = \frac{a^3}{b^2}x + \frac{a^2}{b}y, \qquad xy = yx = -\frac{a^2}{b}x - ay. \tag{3.1}$$

In particular, any such ring R is commutative.

Proof. Assume that the multiplication of R is not trivial. Let $x,y \in R$ such that $R = \mathbb{Z}x \oplus \mathbb{Z}y$, and let

$$x^2 = ax + by, \qquad y^2 = cx + dy, \qquad xy = mx + ny \tag{3.2}$$

for some integers a, b, c, d, m, n. By Lemma 2.5, we have that $u^3 = 0$ for any $u \in R$.

We first prove that $xy = yx$, for if we assume that $xy \neq yx$, then $xx^2 = x^2x$ implies that $ax^2 + bxy = ax^2 + byx$, so $b = 0$. Then $x^2 = ax$, so $0 = x^3 = ax^2 = a^2x$; therefore $a = 0$ and $x^2 = 0$. Similarly, we get $y^2 = 0$. On the other hand, if we multiply $xy = mx + ny$ by x to the left, we obtain $nxy = 0$; so either $n = 0$ or $xy = 0$. If $xy \neq 0$, then $n = 0$, and hence $xy = mx$ implies that $mxy = 0$; so $m = 0$. But then $xy = 0$, a contradiction. Thus xy must be zero. Since x and y play a symmetric role in the relations, the assumption that $yx \neq xy$ would similarly imply that $yx = 0$. But then $xy = yx = 0$, which is a final contradiction. We conclude that $xy = yx$, so R must be commutative.

If we write the associativity conditions $x(xy) = x^2y$ and $(xy)y = xy^2$ (the other possible combinations follow from these two and the commutativity) and express them in terms of the generators x, y, we get the conditions

$$bc = mn, \qquad n^2 + mb = an + bd, \qquad m^2 + cn = ac + dm. \tag{3.3}$$

On the other hand, $0 = x^3 = ax^2 + bxy = a^2x + aby + bmx + bny$ shows that

$$a^2 + bm = 0, \qquad b(a+n) = 0. \tag{3.4}$$

Similarly, $y^3 = 0$ requires

$$d^2 + cn = 0, \qquad c(m+d) = 0. \tag{3.5}$$

Since $u^3 = 0$ for any $u \in R$, we must have $x^2y = xy^2 = 0$ (to see this, it is enough to look at $(x+y)^3 = (x+2y)^3 = 0$), and these relations require as above the conditions

$$am + bc = 0, \qquad an + bd = 0, \qquad ac + dm = 0, \qquad bc + dn = 0. \tag{3.6}$$

Collecting all the above conditions, we obtain the following system:

$$\begin{array}{cccc} bc = mn, & n^2 + mb = an + bd, & m^2 + cn = ac + dm, & \\ a^2 + bm = 0, & b(a+n) = 0, & d^2 + cn = 0, & c(m+d) = 0, \\ am + bc = 0, & an + bd = 0, & ac + dm = 0, & bc + dn = 0. \end{array} \tag{3.7}$$

If $c = 0$, we successively get that each of d, m, a, n must be 0, and this means that

$$x^2 = by, \qquad y^2 = 0, \qquad xy = yx = 0, \tag{3.8}$$

which is a set of relations as in the statement (we take $a = 0$, and then obviously $b^2 | a^3$).

If $b = 0$, we similarly obtain a set of relations as in the statement (with the role of x and y interchanged).

We consider the case where $b \neq 0$, $c \neq 0$. Then we see that $n = -a$, $m = -d$, $bc = ad$, $a^2 = bd$, and $d^2 = ac$. In particular, $a, d \neq 0$. Hence $d = a^2/b$, and then $c = d^2/a = a^3/b^2$. These require the condition $b^2 | a^3$ (which also implies that $b | a^2$). We obtain a set of relations as in the statement.

Finally, we note that the relations define indeed a periodic ring structure. This follows from the proof above, which shows the associativity and also the fact that $u^3 = 0$ for any u, so $u^3 = u^6$. $\square$

The theorem above shows that any periodic ring R which is free of rank 2 as a group must be commutative. The next example provides a class of noncommutative periodic rings that are free of rank 3 as groups.

Example 3.2. Let $A = \mathbb{Z}\langle X, Y \rangle$ be the free algebra in two indeterminates, and let $B = (X, Y)$ be the ideal of A generated by X and Y. For any $a \in \mathbb{Z}$, the ideal $I = (X^2, Y^2, YX - aXY)$ of A is also an ideal of B, and the factor ring $R = B/I$ is generated by x, y (the classes of X, Y in the factor ring) subject to the relations

$$x^2 = 0, \qquad y^2 = 0, \qquad yx = axy. \tag{3.9}$$

Note that $(R,+) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}xy \simeq \mathbb{Z}^3$, and R is commutative if and only if $a = 1$. The ring R is periodic since obviously $u^3 = 0$ for any $u \in R$.

4. Periodic rings R with $R/t(R) \simeq \mathbb{Z}$. Proposition 2.2 shows that any periodic ring which is finitely generated as a group fits into an extension of a finite ring and a torsion-free ring. Computing such extensions is in general a difficult task, and one can expect quite complicated structure of periodic rings which are neither-finite nor torsion-free. Evidence for this is the following result, where we take a first step into discussing such extensions.

Theorem 4.1. *Let R be a periodic ring which is finitely generated as a group. Then $R/t(R) \simeq \mathbb{Z}$ (as groups) if and only if there exist a finite ring T, elements $x \in R$, $a \in T$, a morphism of right T-modules $\varphi : T \to T$, and a morphism of left T-modules $\psi : T \to T$ such that $R = \mathbb{Z}x \oplus T$, T is a subring of R, the multiplication of R is induced by*

$$x^2 = a, \qquad xt = \varphi(t), \qquad tx = \psi(t), \tag{4.1}$$

and the following conditions are satisfied:

$$\psi(s)t = s\varphi(t), \qquad at = \varphi^2(t), \qquad ta = \psi^2(t), \qquad \psi\varphi = \varphi\psi. \tag{4.2}$$

Proof. Assume that $R/t(R) \simeq \mathbb{Z}$. Let $R = \mathbb{Z}x \oplus T$, where $x \in R$ and T is the torsion part of $(R,+)$. We have seen in Proposition 2.2 that T is an ideal of R; in particular, it is a finite subring. Thus there exist group morphisms $\varphi, \psi : T \to T$ such that $xt = \varphi(t)$ and $tx = \psi(t)$ for any $t \in T$. Since R/T is periodic and $(R/T,+) \simeq \mathbb{Z}$, we have, from Corollary 2.4, that R/T has trivial multiplication, and so $x^2 \in T$. Denote $x^2 = a$.

The associativity condition $(xt)s = x(ts)$ (resp., $(st)x = s(tx)$) for any $s,t \in T$ is equivalent to φ (resp., ψ) being a morphism of right (resp., left) T-modules. The associativity conditions $(sx)t = s(xt)$, $x^2t = x(xt)$, $tx^2 = (tx)x$, and $(xt)x = x(tx)$ are equivalent to $\psi(s)t = s\varphi(t)$, $at = \varphi^2(t)$, $ta = \psi^2(t)$, and $\psi\varphi = \varphi\psi$. Thus R is of the form required in the statement.

Conversely, if $R = \mathbb{Z}x \oplus T$, with the given conditions satisfied, then we see from above that R is indeed a ring (the associativity conditions are satisfied). It remains to show that R is periodic. This follows immediately from the fact that for any $r = mx + t$, with $m \in \mathbb{Z}$ and $t \in T$, we have $r^2 = m^2x^2 + mxt + mtx + t^2 \in T$, so there exist $p < q$ with $(r^2)^p = (r^2)^q$, which means that $r^{2p} = r^{2q}$. $\square$

The structure theorem above shows that describing R effectively depends heavily on the structure of the finite ring T. In the case where T is a ring with identity, we can give a more precise description. Indeed, in this case, the maps φ and ψ must be of the form $\varphi(t) = tb$, $\psi(t) = ct$ for some $b,c \in T$. The relations $\psi(s)t = s\varphi(t)$, $at = \varphi^2(t)$, and $ta = \psi^2(t)$ are equivalent to $cst = stb$, $at = tb^2$, and $ta = c^2t$ for any $s,t \in T$, and these are clearly equivalent to $b = c$, $a = b^2$, and $b \in Z(T)$. Thus we obtain the following class of periodic rings. We take $R = \mathbb{Z}x \oplus T$, where T is a finite ring with identity. We also take an element $b \in Z(T)$. Then we define on R a multiplication such that T is a subring, $x^2 = b^2$, and $xt = tx = bt$ for any $t \in T$. Then R is a periodic ring with $R/t(R) \simeq \mathbb{Z}$. Note that R does not have identity, and R is commutative if and only if so is T.

5. Periodic rings that are finitely generated as rings. It is interesting to discuss another finiteness condition on a periodic ring.

Problem 5.1. Describe periodic rings that are finitely generated as rings.

The following result shows that in the commutative case, Problems 1.1 and 5.1 are equivalent.

Proposition 5.2. *Let R be a commutative periodic ring. Then R is finitely generated as a ring if and only if the group $(R,+)$ is finitely generated.*

Proof. Let $\{r_1, r_2, \dots, r_p\}$ be a finite set generating R as a ring. Therefore, since R is commutative, the group $(R,+)$ is generated by the elements of the form $r_1^{m_1} r_2^{m_2} \cdots r_p^{m_p}$ with nonnegative integers $m_1, m_2, \dots, m_p$ (and not all of them being zero in the case where R does not have identity). Since R is periodic, there are positive integers $n_1, n_2, \dots,$ n_p such that $\{r_i^m \mid m > 0\} = \{r_i^m \mid 0 < m \le n_i\}$ for any i. Therefore, $(R,+)$ is generated by $\{r_1^{m_1} r_2^{m_2} \cdots r_p^{m_p} \mid m_1 \le n_1,\ m_2 \le n_2, \dots, m_p \le n_p\}$, so $(R,+)$ is finitely generated. Conversely, if $(R,+)$ is finitely generated, then obviously R is finitely generated as a ring. $\square$

We do not have any example of a periodic ring which is finitely generated as a ring, but not finitely generated as an abelian group. Therefore, we ask the following.

Question 5.3. If R is a periodic ring which is finitely generated as a ring, is $(R,+)$ finitely generated?

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