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CONTINUITY FOR SOME MULTILINEAR OPERATORS OF INTEGRAL OPERATORS ON TRIEBEL-LIZORKIN SPACES

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The continuity for some multilinear operators related to certain fractional singular integral operators on Triebel-Lizorkin spaces is obtained. The operators include Calderon-Zygmund singular integral operator and fractional integral operator.

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1. Introduction. Let T be a Calderon-Zygmund singular integral operator; a well-known result of Coifman et al. (see [6]) states that the commutator $[b,T]=T(bf)-bTf$ (where $b\in BMO$) is bounded on $L^p(R^n)$ for $1<p<\infty$; Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator; in [10, 11], these results on the Triebel-Lizorkin spaces and the case $b\in \operatorname{Lip}\beta$ (where $\operatorname{Lip}\beta$ is the homogeneous Lipschitz space) are obtained. The main purpose of this paper is to discuss the continuity for some multilinear operators related to certain fractional singular integral operators on the Triebel-Lizorkin spaces. In fact, we will establish the continuity on the Triebel-Lizorkin spaces for the multilinear operators only under certain conditions on the size of the operators. As to the applications, the continuity for the multilinear operators related to the Calderon-Zygmund singular integral operator and fractional integral operator on the Triebel-Lizorkin spaces is obtained.

2. Notations and results. Throughout this paper, Q will denote a cube of R^n with side parallel to the axes, and for a cube Q, let $f_Q=|Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x)=\sup_{x\in Q}|Q|^{-1}\int_Q|f(y)-f_Q|dy$. For $1\le r<\infty$ and $0\le\delta<n$, let

$$M_{\delta,r}(f)(x)=\sup_{x\in Q}\left(\frac{1}{|Q|^{1-r\delta/n}}\int_Q|f(y)|^r dy\right)^{1/r}; \tag{2.1}$$

we denote $M_{\delta,r}(f)=M_r(f)$ if $\delta=0$, which is the Hardy-Littlewood maximal function when $r=1$. For $\beta>0$ and $p>1$, let $\dot F_p^{\beta,\infty}$ be the homogeneous Triebel-Lizorkin space; the Lipschitz space $\dot\wedge_\beta$ is the space of functions f such that

$$\|f\|_{\dot\wedge_\beta}=\sup_{\substack{x,h\in R^n\\ h\neq 0}}\frac{\left|\Delta_h^{[\beta]+1}f(x)\right|}{|h|^\beta}<\infty, \tag{2.2}$$

where Δ_h^k denotes the kth difference operator (see [11]).

We are going to consider the fractional singular integral operator as follows.

Definition 2.1. Let $T : S \to S'$ be a linear operator. T is called a fractional singular integral operator if there exists a locally integrable function $K(x,y)$ on $R^n \times R^n$ such that

$$T(f)(x) = \int K(x,y) f(y)\, dy \tag{2.3}$$

for every bounded and compactly supported function f. Let m be a positive integer and A a function on R^n. Denote that

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha. \tag{2.4}$$

The multilinear operator related to fractional singular integral operator T is defined by

$$T_A(f)(x) = \int \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)\, dy. \tag{2.5}$$

Note that when $m = 0$, T_A is just the commutator of T and A while when $m > 0$, it is nontrivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2, 3, 4, 5, 7, 8, 14]). The main purpose of this paper is to study the continuity for the multilinear operator on Triebel-Lizorkin spaces. We will prove the following theorem in Section 3.

Theorem 2.2. *Let $0 < \beta < 1$ and $D^\alpha A \in \dot\Lambda_\beta$ for $|\alpha| = m$. Suppose T is the fractional singular integral operator such that T is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$, and $1/p - 1/q = \delta/n$. If T satisfies the size condition*

$$|T_A(f)(x) - T_A(f)(x_0)| \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot\Lambda_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x) \tag{2.6}$$

for any cube $Q = Q(x_0,l)$ with $\operatorname{supp} f \subset (2Q)^c$, $x \in Q$, and some $0 \le \delta < n$, then

(a) *T_A maps $L^p(R^n)$ continuously into $\dot F_q^{\beta,\infty}(R^n)$ for $0 \le \delta < n$, $1 < p < n/\delta$, and $1/p - 1/q = \delta/n$;*

(b) *T_A maps $L^p(R^n)$ continuously into $L^q(R^n)$ for $0 \le \delta < n - \beta$, $1 < p < n/(\delta + \beta)$, and $1/p - 1/q = (\delta + \beta)/n$.*

From the theorem, we get the following corollary.

Corollary 2.3. *Fix $\varepsilon > 0$, $0 < \beta < \min(1,\varepsilon)$, $\delta \ge 0$, and $D^\alpha A \in \dot\Lambda_\beta$ for $|\alpha| = m$. Let K be a locally integrable function on $R^n \times R^n$ satisfying*

$$\begin{aligned} |K(x,y)| &\le C|x-y|^{-n+\delta}, \\ |K(y,x) - K(z,x)| &\le C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta} \end{aligned} \tag{2.7}$$

if $2|y-z| \le |x-z|$. *Denote that (2.3) holds and denote the multilinear operator of* T *by (2.5) for every bounded and compactly supported function* f *and* $x \in (\operatorname{supp} f)^c$. *Suppose* T *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $0 \le \delta < n$, $1 < p < n/\delta$, *and* $1/q = 1/p - \delta/n$. *Then*

(a) T_A *maps* $L^p(R^n)$ *continuously into* $\dot{F}_q^{\beta,\infty}(R^n)$ *for* $0 \le \delta < n$, $1 < p < n/\delta$, *and* $1/p - 1/q = \delta/n$;

(b) T_A *maps* $L^p(R^n)$ *continuously into* $L^q(R^n)$ *for* $0 \le \delta < n-\beta$, $1 < p < n/(\delta+\beta)$, *and* $1/p - 1/q = (\delta+\beta)/n$.

3. Proof of Theorem 2.2.

To prove the theorem, we need the following lemmas.

Lemma 3.1 [11]. *For* $0 < \beta < 1$ *and* $1 < p < \infty$,

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| \, dx \right\|_{L^p} \approx \left\| \sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| \, dx \right\|_{L^p}. \tag{3.1}$$

Lemma 3.2 [11]. *For* $0 < \beta < 1$ *and* $1 \le p \le \infty$,

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| \, dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p}. \tag{3.2}$$

Lemma 3.3 [1]. *Suppose that* $1 \le r < p < n/\delta$ *and* $1/q = 1/p - \delta/n$. *Then* $\|M_{\delta,r}(f)\|_{L^q} \le C\|f\|_{L^p}$.

Lemma 3.4 [5]. *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q}, \tag{3.3}$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

Proof of Theorem 2.2. Fix a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} (1/\alpha!)(D^\alpha A)_{\tilde{Q}} x^\alpha$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T_A(f)(x) &= \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y) \, dy \\ &= \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f_2(y) \, dy \\ &\quad + \int_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} K(x,y) f_1(y) \, dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y) \, dy, \end{aligned} \tag{3.4}$$

then

$$\begin{aligned}&\left|T_A(f)(x)-T_A(f_2)(x_0)\right|\\&\quad\le\left|T\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|T\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right|\\&\qquad+\left|T_A(f_2)(x)-T_A(f_2)(x_0)\right|:=I(x)+II(x)+III(x).\end{aligned}\tag{3.5}$$

Thus,

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_A(f)(x)-T_A(f_2)(x_0)\right|dx\\&\quad\le\frac{1}{|Q|^{1+\beta/n}}\int_Q I(x)dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q II(x)dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q III(x)dx\\&\quad:=I+II+III.\end{aligned}\tag{3.6}$$

Now, we estimate I, II, and III, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, using Lemmas 3.4 and 3.2, we get

$$\begin{aligned}R_m(\tilde{A};x,y)&\le C|x-y|^m\sum_{|\alpha|=m}\sup_{x\in\tilde{Q}}\left|D^\alpha A(x)-(D^\alpha A)_{\tilde{Q}}\right|\\&\le C|x-y|^m|Q|^{\beta/n}\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}.\end{aligned}\tag{3.7}$$

Thus, taking r, s such that $1\le r<p$ and $1/s=1/r-\delta/n$, by (L^r,L^s) boundedness of T and Hölder's inequality, we obtain

$$\begin{aligned}I&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}\frac{1}{|Q|}\int_Q\left|T(f_1)(x)\right|dx\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}\|T(f_1)\|_{L^s}|Q|^{-1/s}\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}\|f_1\|_{L^r}|Q|^{-1/s}\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}\left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}}\int_{\tilde{Q}}|f(y)|^r dy\right)^{1/r}\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\dot{\Lambda}_\beta}M_{\delta,r}(f)(\tilde{x}).\end{aligned}\tag{3.8}$$

Second, using the inequality (see [11])

$$\left\|\left(D^\alpha A-(D^\alpha A)_{\tilde{Q}}\right)f\chi_{\tilde{Q}}\right\|_{L^r}\le C|Q|^{1/s+\beta/n}\|D^\alpha A\|_{\dot{\Lambda}_\beta}M_{\delta,r}(f)(x),\tag{3.9}$$

we gain

$$
\begin{aligned}
II &\le \frac{C}{|Q|^{1+\beta/n}} \sum_{|\alpha|=m} \left\| T\left((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{2Q} \right) \right\|_{L^s} |Q|^{1-1/s} \\
&\le C |Q|^{-\beta/n-1/s} \sum_{|\alpha|=m} \| (D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{2Q} \|_{L^r} \\
&\le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} M_{\delta,r}(f)(\tilde{x}).
\end{aligned} \tag{3.10}
$$

For III, using the size condition of T, we have

$$
III \le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \tag{3.11}
$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemmas 3.1 and 3.3, we obtain

$$
\| T_A(f) \|_{\dot{F}_q^{\beta,\infty}} \le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} \| f \|_{L^p}. \tag{3.12}
$$

This completes the proof of (a).

For (b), by the same argument as in the proof of (a), we have

$$
\frac{1}{|Q|} \int_Q |T_A(f)(y) - T_A(f_2)(x_0)| \, dy \le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)). \tag{3.13}
$$

Thus, we get the sharp estimate of T_A as follows:

$$
(T_A(f))^{\#} \le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} (M_{\delta+\beta,r}(f) + M_{\delta+\beta,1}(f)). \tag{3.14}
$$

Now, using Lemma 3.3, we gain

$$
\begin{aligned}
\| T_A(f) \|_{L^q} &\le C \| (T_A(f))^{\#} \|_{L^q} \\
&\le C \sum_{|\alpha|=m} \| D^\alpha A \|_{\dot{\Lambda}_\beta} (\| M_{\delta+\beta,r}(f) \|_{L^q} + \| M_{\delta+\beta,1}(f) \|_{L^q}) \le C \| f \|_{L^p}.
\end{aligned} \tag{3.15}
$$

This completes the proof of (b) and the theorem. □

Proof of Corollary 2.3. It suffices to verify that T satisfies the size condition in Theorem 2.2.

Suppose supp $f \subset \tilde{Q}^c$ and $x \in Q = Q(x_0, l)$. We write

$$
\begin{aligned}
&T_A(f)(x) - T_A(f)(x_0) \\
&\quad = \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right] R_m(\tilde{A};x,y) f(y)\,dy \\
&\qquad + \int_{R^n} \frac{K(x_0,y) f(y)}{|x_0-y|^m} [R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y)]\,dy \\
&\qquad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x_0,y)(x_0-y)^\alpha}{|x_0-y|^m} \right) D^\alpha \tilde{A}(y) f(y)\,dy \\
&\quad := I_1 + I_2 + I_3.
\end{aligned}
\tag{3.16}
$$

By Lemma 3.4 and the following inequality, for $b \in \dot{\Lambda}_\beta$:

$$
|b(x) - b_Q| \le \frac{1}{|Q|} \int_Q \|b\|_{\dot{\Lambda}_\beta} |x-y|^\beta\,dy \le \|b\|_{\dot{\Lambda}_\beta} (|x-x_0| + l)^\beta,
\tag{3.17}
$$

we get

$$
|R_m(\tilde{A};x,y)| \le \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} (|x-y| + l)^{m+\beta}.
\tag{3.18}
$$

Note that by $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of K,

$$
\begin{aligned}
|I_1| &\le C \int_{R^n \setminus \tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) |R_m(\tilde{A};x,y)|\,|f(y)|\,dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-\beta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta-\beta}} \right) |f(y)|\,dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} (2^{k(\beta-1)} + 2^{k(\beta-\varepsilon)}) \frac{1}{|2^k \tilde{Q}|^{1-\delta/n}} \int_{2^k \tilde{Q}} |f(y)|\,dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\dot{\Lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x).
\end{aligned}
\tag{3.19}
$$

For I_2, by the formula (see [3])

$$
R_m(\tilde{A};x,y) - R_m(\tilde{A};x_0,y) = \sum_{|\eta|<m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A};x,x_0)(x-y)^\eta
\tag{3.20}
$$

and Lemma 3.4, we get

$$
\begin{aligned}
|I_2| &\le C\int_{R^n\setminus\tilde{Q}} \frac{|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n-\delta}}|f(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|^{\beta+1}}{|x_0-y|^{n+1-\delta}}|f(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x).
\end{aligned}
\tag{3.21}
$$

For I_3, similar to the estimates of I_1, we obtain

$$
\begin{aligned}
|I_3| &\le C\int_{R^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}}\right)|D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}||f(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\sum_{k=1}^{\infty}\left(2^{k(\beta-1)}+2^{k(\beta-\varepsilon)}\right)\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\dot{\Lambda}_\beta}|Q|^{\beta/n}M_{\delta,1}(f)(x).
\end{aligned}
\tag{3.22}
$$

Thus (2.6) holds. This completes the proof of the corollary. □

4. Applications. In this section, we will apply Theorem 2.2 and Corollary 2.3 to some particular operators such as the Calderon-Zygmund singular integral operator and fractional integral operator.

Application 1 (Calderon-Zygmund singular integral operator)**.** Let T be the Calderon-Zygmund operator defined by (2.3) (see [9, 12, 13]); the multilinear operator related to T is defined by

$$
T_Af(x)=\int\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)dy. \tag{4.1}
$$

Then it is easy to see that T satisfies the conditions in Corollary 2.3 with $\delta=0$; thus T_A is bounded from $L^p(R^n)$ to $\dot{F}_p^{\beta,\infty}(R^n)$ for $D^\alpha A\in\dot{\Lambda}_\beta$, $|\alpha|=m$, $0<\beta<1$, $1<p<\infty$ and from $L^p(R^n)$ to $L^q(R^n)$ for $D^\alpha A\in\dot{\Lambda}_\beta$, $|\alpha|=m$, $0<\beta<1$, $1<p<n/\beta$, and $1/p-1/q=\beta/n$.

Application 2 (fractional integral operator with rough kernel)**.** For $0\le\delta<n$, let T_δ be the fractional integral operator with rough kernel defined by (see [7, 8, 14])

$$
T_\delta f(x)=\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n-\delta}}f(y)dy; \tag{4.2}
$$

the multilinear operator related to T_δ is defined by

$$T_\delta^A f(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y)\,dy, \tag{4.3}$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$, and $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$ for $0 < \gamma \le 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. Then T_δ satisfies the conditions in Corollary 2.3. In fact, for $\operatorname{supp} f \subset (2Q)^c$ and $x \in Q = Q(x_0, l)$, by the condition of Ω, we have (see [14])

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-\delta}} \right| \le C\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta}} \right); \tag{4.4}$$

thus, similar to the proof of Corollary 2.3, we get

$$\begin{aligned}
&|T_\delta^A(f)(x) - T_\delta^A(f)(x_0)| \\
&\quad \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot\Lambda_\beta} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde Q \setminus 2^k \tilde Q} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta-\beta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta-\beta}} \right) |f(y)|\,dy \\
&\quad \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot\Lambda_\beta} |Q|^{\beta/n} \sum_{k=1}^{\infty} \left(2^{k(\beta-1)} + 2^{k(\beta-\gamma)}\right) \frac{1}{|2^k \tilde Q|^{1-\delta/n}} \int_{2^k \tilde Q} |f(y)|\,dy \\
&\quad \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot\Lambda_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(x).
\end{aligned} \tag{4.5}$$

Therefore, T_δ^A is bounded from $L^p(R^n)$ to $\dot F_q^{\beta,\infty}(R^n)$ for $D^\alpha A \in \dot\Lambda_\beta$, $|\alpha| = m$, $0 < \beta < \gamma$, $1 < p < n/\beta$ and from $L^p(R^n)$ to $L^q(R^n)$ for $D^\alpha A \in \dot\Lambda_\beta$, $|\alpha| = m$, $0 < \beta < 1$, $1 < p < n/(\delta+\beta)$, and $1/p - 1/q = (\delta+\beta)/n$.

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