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SOME REFINEMENTS AND GENERALIZATIONS OF CARLEMAN'S INEQUALITY

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We give some refinements and generalizations of Carleman's inequality with weaker condition for weight coefficient.

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1. Introduction. The following Carleman's inequality (see [6, Theorem 334]) is well known, unless (a_n) is null:

$$\sum_{n=1}^{\infty}\left(a_1a_2\cdots a_n\right)^{1/n}<e\sum_{n=1}^{\infty}a_n. \tag{1.1}$$

The constant is the best possible.

There is a vast literature which deals with alternative proofs, various generalizations and extensions, and numerous variants and applications in analysis of inequality (1.1); see [1, 2, 3, 5, 7, 9, 8, 10, 13, 14, 15, 16, 17, 18, 19] and the references given therein. According to Hardy (see [6, Theorem 349]), Carleman's inequality was generalized as follows. If $a_n \geq 0$, $\lambda_n \geq 0$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m$ $(n\in\mathbb{N})$, and $0<\sum_{n=1}^{\infty}\lambda_na_n<\infty$, then

$$\sum_{n=1}^{\infty}\lambda_n\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n}<e\sum_{n=1}^{\infty}\lambda_na_n. \tag{1.2}$$

In [19], Yuan obtained the refined Carleman's inequality as follows. If $a_n \geq 0$, $n = 1,2,\ldots$, and $0<\sum_{n=1}^{\infty}a_n<\infty$, then

$$\sum_{n=1}^{\infty}\left(a_1a_2\cdots a_n\right)^{1/n}<e\sum_{n=1}^{\infty}\frac{(1-\beta/n)}{(1+1/n)^{\alpha}}a_n, \tag{1.3}$$

where α, β satisfy $0\leq\alpha\leq 1/\ln 2-1$, $0\leq\beta\leq 1-2/e$, and $e\beta+2^{1+\alpha}=e$.

Recently, Kim [10] established the following new extension of the refined Hardy's inequality in the spirit of the property of the power mean of n distinct positive numbers.

Theorem 1.1. *If* $0 < \lambda_{n+1} \le \lambda_n$, $a_n \ge 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $\Lambda_n \ge 1$, $0 < p \le 1$, *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots \lambda_n^{\lambda_n}\right)^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha}(\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^{\alpha}}\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^{n} \lambda_k c_k^p a_k^p\right)^{(1-p)/p}, \tag{1.4}$$

where $c_k^{\lambda k} = (\Lambda_{k+1})^{\Lambda_k} / (\Lambda_k)^{\Lambda_{k-1}}$, *and* α, β *satisfy* $0 \le \alpha \le 1/\ln 2 - 1$, $0 \le \beta \le 1 - 2/e$, *and* $e\beta + 2^{1+\alpha} = e$.

When $p = 1$ *in inequality (1.4), there exists the following class of new refined Hardy's inequality:*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha}(\Lambda_n - \lambda_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n)^{\alpha}}\right) \lambda_n a_n. \tag{1.5}$$

In this paper, we will establish some new refinements and generalizations of Carleman's inequality. Also, our results correspond to Theorem 1.1 with a weaker condition for weight coefficient.

2. Results.

The following results are new generalizations of Carleman's inequality.

Theorem 2.1. *Let* $\lambda_n > 0$, $\nu_n > 0$, $a_n \ge 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m \nu_m$ $(n \in \mathbb{N})$, $0 < p \le 1$, *and* $0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty$. *If*

$$\frac{\Lambda_{n+1}\Lambda_n}{\Lambda_n + \lambda_n \nu_n} \le \Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{1/p} \nu_n^{-(\Lambda_{n-1}/\Lambda_n)} \nu_{n+1}, \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n}\right)^{1/\Lambda_n} \le \frac{e^p}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha}(\Lambda_n - \lambda_n \nu_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n \nu_n)^{\alpha}}\right)^p \lambda_n \nu_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p\right)^{(1-p)/p}, \tag{2.2}$$

where

$$c_k = \left[\frac{\left(\Lambda_k^{(1-1/p)} \Lambda_{k+1}\right)^{\Lambda_k} \nu_k^{\Lambda_{k-1}}}{\left(\Lambda_{k-1}^{(1-1/p)} \Lambda_k\right)^{\Lambda_{k-1}} \nu_{k+1}^{\Lambda_k}}\right]^{1/\lambda_k \nu_k}, \tag{2.3}$$

and α, β *satisfy* $0 \le \alpha \le 1/\ln 2 - 1$, $0 \le \beta \le 1 - 2/e$, *and* $e\beta + 2^{1+\alpha} = e$.

Proof. By the power mean inequality [11, page 15], we have

$$\alpha_1^{q_1}\alpha_2^{q_2}\cdots\alpha_n^{q_n} \le \left(\sum_{m=1}^{n} q_m \alpha_m^p\right)^{1/p} \tag{2.4}$$

for $\alpha_m \ge 0$, $p > 0$, and $q_m > 0$ $(m \in \mathbb{N})$ with $\sum_{m=1}^{n} q_m = 1$. Setting $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m \nu_m/\Lambda_n$, we obtain

$$(c_1a_1)^{\lambda_1\nu_1/\Lambda_n}(c_2a_2)^{\lambda_2\nu_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n\nu_n/\Lambda_n} \le \left(\sum_{m=1}^{n}\frac{\lambda_m\nu_m}{\Lambda_n}(c_ma_m)^p\right)^{1/p}. \tag{2.5}$$

Using the above inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}\\
&\quad=\sum_{n=1}^{\infty}\lambda_{n+1}\frac{(c_1a_1)^{\lambda_1\nu_1/\Lambda_n}(c_2a_2)^{\lambda_2\nu_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n\nu_n/\Lambda_n}}{\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}}\\
&\quad\le\sum_{n=1}^{\infty}\left(\frac{\lambda_{n+1}}{\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}}\right)\left(\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\right)^{1/p}.
\end{aligned} \tag{2.6}$$

By using the inequality (see [4, 12])

$$\left(\sum_{m=1}^{n} z_m\right)^{t} \le t\sum_{m=1}^{n} z_m\left(\sum_{k=1}^{m} z_k\right)^{t-1}, \tag{2.7}$$

where $t \ge 1$ is constant and $z_m \ge 0$ $(m \in \mathbb{N})$, it is easy to observe that

$$\left(\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\right)^{1/p} \le \frac{1}{p\Lambda_n^{1/p}}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p} \tag{2.8}$$

for $\Lambda_n \ge 1$ and $0 < p \le 1$. Then, by (2.6) and (2.8), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}\\
&\le\frac{1}{p}\sum_{n=1}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n^{1/p}\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p}\\
&=\frac{1}{p}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\sum_{n=m}^{\infty}\left(\frac{\lambda_{n+1}}{\Lambda_n^{1/p}\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}}\right)\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1/p)/p}.
\end{aligned} \tag{2.9}$$

Choosing $c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n} = (\Lambda_n^{(1-1/p)}\Lambda_{n+1}\nu_{n+1}^{-1})^{\Lambda_n}$ $(n\in\mathbb{N})$ and setting $\Lambda_0 = 1$, from (2.1), it follows that

$$\begin{aligned} c_n &= \left(\frac{\left(\Lambda_n^{(1-1/p)}\Lambda_{n+1}\nu_{n+1}^{-1}\right)^{\Lambda_n}}{\left(\Lambda_{n-1}^{(1-1/p)}\Lambda_n\nu_n^{-1}\right)^{\Lambda_{n-1}}}\right)^{1/\lambda_n\nu_n} \\ &= \left(\frac{\Lambda_{n+1}}{\Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)}\Lambda_n^{1/p}\nu_n^{-(\Lambda_{n-1}/\Lambda_n)}\cdot\nu_{n+1}}\right)^{\Lambda_n/\lambda_n\nu_n}\Lambda_n \\ &\le \left(1+\frac{\lambda_n\nu_n}{\Lambda_n}\right)^{\Lambda_n/\lambda_n\nu_n}\Lambda_n. \end{aligned} \tag{2.10}$$

This implies that

$$\begin{aligned} &\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n} \\ &\quad\le \frac{1}{p}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\sum_{n=m}^{\infty}\frac{\lambda_{n+1}\nu_{n+1}}{\Lambda_n\Lambda_{n+1}}\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p} \\ &\quad= \frac{1}{p}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\sum_{n=m}^{\infty}\left(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\right)\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p} \\ &\quad= \frac{1}{p}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\frac{1}{\Lambda_n}\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p} \\ &\quad\le \frac{1}{p}\sum_{m=1}^{n}\left(1+\frac{\lambda_m\nu_m}{\Lambda_m}\right)^{p\Lambda_m/\lambda_m\nu_m}\lambda_m\nu_m a_m^p\Lambda_m^{p-1}\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p}. \end{aligned} \tag{2.11}$$

Hence, using the inequality [19, Lemma 3.1]

$$\left(1+\frac{1}{x}\right)^x \le e\frac{(1-\beta/x)}{(1+1/x)^\alpha} \tag{2.12}$$

for $x > 1$, and α, β satisfying $0\le\alpha\le 1/\ln 2-1$, $0\le\beta\le 1-2/e$, and $e\beta+2^{1+\alpha}=e$, we have (2.2). Thus Theorem 2.1 is proved. □

Taking $\nu_n = 1$ $(n\in\mathbb{N})$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let* $\lambda_n > 0$, $a_n \ge 0$, $\Lambda_n = \sum_{m=1}^{n}\lambda_m$ $(n\in\mathbb{N})$, $0 < p \le 1$, *and* $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$. *If*

$$\frac{\Lambda_{n+1}\Lambda_n}{\Lambda_n+\lambda_n} \le \Lambda_{n-1}^{(1-1/p)(\Lambda_{n-1}/\Lambda_n)}\Lambda_n^{1/p}, \tag{2.13}$$

then

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n}\le\frac{e^p}{p}\sum_{n=1}^{\infty}\left(\frac{\Lambda_n^{\alpha}(\Lambda_n-\lambda_n\beta)}{\Lambda_n(\Lambda_n+\lambda_n)^{\alpha}}\right)^p\lambda_n a_n^p. \tag{2.14}$$

Remark 2.3. Corollary 2.2 is a result corresponding to Theorem 1.1 with a weaker condition for λ_n $(n\in\mathbb{N})$. Further, setting $p=1$ in Corollary 2.2, we obtain an inequality corresponding to inequality (1.5) with a weaker condition for Λ_n $(n\in\mathbb{N})$.

Setting $p=1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let $\lambda_n>0$, $\nu_n>0$, $a_n\ge 0$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m\nu_m$ $(n\in\mathbb{N})$, and $0<\sum_{n=1}^{\infty}\lambda_n\nu_n a_n<\infty$. If*

$$\frac{\Lambda_{n+1}}{\Lambda_n+\lambda_n\nu_n}\le\nu_n^{-(\Lambda_{n-1}/\Lambda_n)}\nu_{n+1}, \tag{2.15}$$

then

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}\le e\sum_{n=1}^{\infty}\left(\frac{\Lambda_n^{\alpha}(\Lambda_n-\lambda_n\nu_n\beta)}{\Lambda_n(\Lambda_n+\lambda_n\nu_n)^{\alpha}}\right)\lambda_n\nu_n a_n. \tag{2.16}$$

Remark 2.5. Inequality (2.16) is a new refinement and generalization of inequality (1.3).

Corollary 2.6. *Under the assumptions of Theorem 2.1, if $\alpha=0$, $\beta=1-2/e$, then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}\\&\quad\le\frac{e^p}{p}\sum_{n=1}^{\infty}\left(1-\frac{\lambda_n\nu_n(1-2/e)}{\Lambda_n}\right)^p\lambda_n\nu_n a_n^p\Lambda_n^{p-1}\left(\sum_{k=1}^{n}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p},\end{aligned} \tag{2.17}$$

where (2.3) holds.

Remark 2.7. Corollary 2.6 is a new refinement and generalization of [10, Corollary 2.3]. Further, setting $\nu_n=1$ $(n\in\mathbb{N})$ in Corollary 2.6, we obtain a result corresponding to [10, Corollary 2.3] with a weaker condition for λ_n $(n\in\mathbb{N})$.

Corollary 2.8. *Under the assumptions of Theorem 2.1, if $\alpha=1/\ln 2-1$, $\beta=0$, then*

$$\begin{aligned}&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{1/\Lambda_n}\\&\quad\le\frac{e^p}{p}\sum_{n=1}^{\infty}\left(\left(1+\frac{\lambda_n\nu_n}{\Lambda_n}\right)^{1-1/\ln 2}\right)^p\lambda_n\nu_n a_n^p\Lambda_n^{p-1}\left(\sum_{k=1}^{n}\lambda_k\nu_k(c_ka_k)^p\right)^{(1-p)/p},\end{aligned} \tag{2.18}$$

where (2.3) holds.

Remark 2.9. Corollary 2.8 is a new refinement and generalization of [10, Corollary 2.4]. Further, setting $\nu_n = 1$ $(n \in \mathbb{N})$ in Corollary 2.8, we obtain a result corresponding to [10, Corollary 2.4] with a weaker condition for λ_n $(n \in \mathbb{N})$.

A new general refined Hardy's inequality is introduced to the following theorem.

Theorem 2.10. *Let* $\lambda_n > 0$, $\nu_n > 0$, $a_n \geq 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m \nu_m$ $(n \in \mathbb{N})$, *and* $0 < \sum_{n=1}^{\infty} \lambda_n \nu_n a_n < \infty$ *for* $0 < p \leq t < \infty$. *If*

$$\frac{\Lambda_{n+1}\Lambda_n}{\Lambda_n + \lambda_n \nu_n} \leq \Lambda_{n-1}^{((p-t)/p)(\Lambda_{n-1}/\Lambda_n)} \Lambda_n^{t/p} \nu_n^{-(\Lambda_{n-1}/\Lambda_n)} \nu_{n+1}, \tag{2.19}$$

then

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n} \right)^{t/\Lambda_n} \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(\frac{\Lambda_n^{\alpha}(\Lambda_n - \lambda_n \nu_n \beta)}{\Lambda_n (\Lambda_n + \lambda_n \nu_n)^{\alpha}} \right)^{p/t} \lambda_n \nu_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p \right)^{(t-p)/p}, \tag{2.20}$$

where

$$c_k = \left[\frac{\left(\Lambda_k^{(1-t/p)} \Lambda_{k+1}\right)^{\Lambda_k} \nu_k^{\Lambda_{k-1}}}{\left(\Lambda_{k-1}^{(1-t/p)} \Lambda_k\right)^{\Lambda_{k-1}} \nu_{k+1}^{\Lambda_k}} \right]^{1/\lambda_k \nu_k t}, \tag{2.21}$$

and α, β *satisfy* $0 \leq \alpha \leq 1/\ln 2 - 1$, $0 \leq \beta \leq 1 - 2/e$, *and* $e\beta + 2^{1+\alpha} = e$.

Proof. The proof is almost the same as in Theorem 2.1. By the power mean inequality [11, page 15], we have (2.4) for $\alpha_m \geq 0$, $p > 0$, and $q_m > 0$ $(m \in \mathbb{N})$ with $\sum_{m=1}^{n} q_m = 1$. By (2.4), we obtain

$$\left(\alpha_1^{q_1} \alpha_2^{q_2} \cdots \alpha_n^{q_n}\right)^t \leq \left(\sum_{m=1}^{n} q_m \alpha_m^p \right)^{t/p} \tag{2.22}$$

for $t > 0$. Taking $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m \nu_m / \Lambda_n$, we have

$$\left((c_1 a_1)^{\lambda_1 \nu_1 / \Lambda_n} (c_2 a_2)^{\lambda_2 \nu_2 / \Lambda_n} \cdots (c_n a_n)^{\lambda_n \nu_n / \Lambda_n} \right)^t \leq \left(\sum_{m=1}^{n} \frac{\lambda_m \nu_m}{\Lambda_n} (c_m a_m)^p \right)^{t/p}. \tag{2.23}$$

Using the above inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n}\\
&\quad=\sum_{n=1}^{\infty}\lambda_{n+1}\frac{\left((c_1a_1)^{\lambda_1\nu_1/\Lambda_n}(c_2a_2)^{\lambda_2\nu_2/\Lambda_n}\cdots(c_na_n)^{\lambda_n\nu_n/\Lambda_n}\right)^t}{\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n}}\\
&\quad\le\sum_{n=1}^{\infty}\left(\frac{\lambda_{n+1}}{\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n}}\right)\left(\sum_{m=1}^{n}\frac{\lambda_m\nu_m}{\Lambda_n}(c_ma_m)^p\right)^{t/p}.
\end{aligned}\tag{2.24}$$

By using the inequality (see [4, 12])

$$\left(\sum_{m=1}^{n}z_m\right)^t\le t\sum_{n=1}^{\infty}z_m\left(\sum_{k=1}^{m}z_k\right)^{t-1},\tag{2.25}$$

where $t\ge 1$ is constant and $z_m>0$ $(n\in\mathbb{N})$, it is easy to observe that

$$\frac{1}{\Lambda_n^{t/p}}\left(\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\right)^{t/p}\le\frac{t}{p\Lambda_n^{t/p}}\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(t-p)/p}\tag{2.26}$$

for $0<p\le t$. Then, by (2.24) and (2.26), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1\nu_1}a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n}\\
&\le\frac{t}{p}\sum_{n=1}^{\infty}\left(\frac{\lambda_{n+1}}{\Lambda_n^{t/p}\left(c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n}}\right)\sum_{m=1}^{n}\lambda_m\nu_m(c_ma_m)^p\left(\sum_{k=1}^{m}\lambda_k\nu_k(c_ka_k)^p\right)^{(t-p)/p}.
\end{aligned}\tag{2.27}$$

Choosing $c_1^{\lambda_1\nu_1}c_2^{\lambda_2\nu_2}\cdots c_n^{\lambda_n\nu_n}=(\Lambda_n^{(1-t/p)}\Lambda_{n+1}\nu_{n+1}^{-1})^{\Lambda_n/t}$ $(n\in\mathbb{N})$ and setting $\Lambda_0=1$, from (2.19), it follows that

$$\begin{aligned}
c_n&=\left(\frac{\left(\Lambda_n^{(1-t/p)}\Lambda_{n+1}\nu_{n+1}^{-1}\right)^{\Lambda_n/t}}{\left(\Lambda_{n-1}^{(1-t/p)}\Lambda_n\nu_n^{-1}\right)^{\Lambda_{n-1}/t}}\right)^{1/\lambda_n\nu_n}\\
&=\left(\frac{\Lambda_{n+1}\nu_n^{(\Lambda_{n-1}/\Lambda_n)}}{\Lambda_{n-1}^{(1-t/p)(\Lambda_{n-1}/\Lambda_n)}\Lambda_n^{t/p}\nu_{n+1}}\right)^{\Lambda_n/\lambda_n\nu_n t}\Lambda_n^{1/t}\\
&\le\left(1+\frac{\lambda_n\nu_n}{\Lambda_n}\right)^{\Lambda_n/\lambda_n\nu_n t}\Lambda_n^{1/t}.
\end{aligned}\tag{2.28}$$

This implies that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1}\left(a_1^{\lambda_1\nu_1} a_2^{\lambda_2\nu_2}\cdots a_n^{\lambda_n\nu_n}\right)^{t/\Lambda_n} \\
&\quad\le \frac{t}{p}\sum_{m=1}^{n}\lambda_m\nu_m\left(c_m a_m\right)^p \sum_{n=m}^{\infty}\frac{\lambda_{n+1}\nu_{n+1}}{\Lambda_n\Lambda_{n+1}}\left(\sum_{k=1}^{m}\lambda_k\nu_k\left(c_k a_k\right)^p\right)^{(t-p)/p} \\
&\quad\le \frac{t}{p}\sum_{m=1}^{n}\left(1+\frac{\lambda_m\nu_m}{\Lambda_m}\right)^{p\Lambda_m/\lambda_m\nu_m t}\lambda_m\nu_m a_m^p \Lambda_n^{p/t} \\
&\qquad\times \sum_{n=m}^{\infty}\left(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\right)\left(\sum_{k=1}^{m}\lambda_k\nu_k\left(c_k a_k\right)^p\right)^{(t-p)/p} \\
&\quad\le \frac{t}{p}\sum_{m=1}^{\infty}\left(1+\frac{\lambda_m\nu_m}{\Lambda_m}\right)^{p\Lambda_m/\lambda_m\nu_m t}\lambda_m\nu_m a_m^p \Lambda_m^{(p-t)/t}\left(\sum_{k=1}^{m}\lambda_k\nu_k\left(c_k a_k\right)^p\right)^{(t-p)/p}.
\end{aligned}
\tag{2.29}
$$

Hence, using inequality (2.12) [19, Lemma 3.1] for $x > 1$, and α, β satisfy $0 \le \alpha \le 1/\ln 2 - 1$, $0 \le \beta \le 1 - 2/e$, and $e\beta + 2^{1+\alpha} = e$, we have (2.20). Thus Theorem 2.10 is proved. □

Remark 2.11. Theorem 2.10 reduces to Theorem 2.1 when $t = 1$. Hence, inequality (2.20) is a new generalization of Hardy's inequality. Taking $\nu_n = 1$ $(n \in \mathbb{N})$ in Theorem 2.10, we obtain a result corresponding to [10, Theorem 2.6] with a weak condition for λ_n $(n \in \mathbb{N})$. Also assuming that $\lambda_n = 1$ in Theorem 2.10, we have an extension of [10, Corollary 2.7] as in the following corollary.

Corollary 2.12. *Let $\nu_n > 0$, $a_n \ge 0$, $\Lambda_n = \sum_{m=1}^{n}\nu_m$ $(n \in \mathbb{N})$, and $0 < \sum_{n=1}^{\infty}\nu_n a_n < \infty$ for $0 < p \le t < \infty$. If*

$$
\frac{\Lambda_{n+1}\Lambda_n}{\Lambda_n+\nu_n} \le \Lambda_{n-1}^{((p-t)/p)(\Lambda_n/\Lambda_n)}\Lambda_n^{t/p}\nu_n^{-(\Lambda_{n-1}/\Lambda_n)}\nu_{n+1},
\tag{2.30}
$$

then

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(a_1^{\nu_1}a_2^{\nu_2}\cdots a_n^{\nu_n}\right)^{t/\Lambda_n} \\
&\quad\le \frac{te^{p/t}}{p}\sum_{n=1}^{\infty}\left(\frac{\Lambda_n^{\alpha}\left(\Lambda_n-\nu_n\beta\right)}{\Lambda_n\left(\Lambda_n+\nu_n\right)^{\alpha}}\right)^{p/t}\nu_n a_n^p\Lambda_n^{(p-t)/t}\left(\sum_{k=1}^{n}\nu_k\left(c_k a_k\right)^p\right)^{(t-p)/p},
\end{aligned}
\tag{2.31}
$$

where

$$
c_k = \left(\frac{\left(\Lambda_k^{(1-t/p)}\Lambda_{k+1}\right)^{\Lambda_k}\nu_k^{\Lambda_{k-1}}}{\left(\Lambda_k^{(1-t/p)}\Lambda_k\right)^{\Lambda_{k-1}}\nu_{k+1}^{\Lambda_k}}\right)^{1/\nu_k t}.
\tag{2.32}
$$

Corollary 2.13. *Under the assumptions of Theorem 2.10, if $\alpha = 0$, $\beta = 1 - 2/e$, then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n}\right)^{t/\Lambda_n} \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n \nu_n (1 - 2/e)}{\Lambda_n}\right)^{p/t} \lambda_n \nu_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p\right)^{(t-p)/p}, \tag{2.33}$$

where

$$c_k^{\lambda_k \nu_k t} = \frac{\left(\Lambda_k^{(1-t/p)} \Lambda_{k+1}\right)^{\Lambda_k} \nu_k^{\Lambda_{k-1}}}{\left(\Lambda_{k-1}^{(1-t/p)} \Lambda_k\right)^{\Lambda_{k-1}} \nu_{k+1}^{\Lambda_k}}. \tag{2.34}$$

Corollary 2.14. *Under the assumptions of Theorem 2.1, if $\alpha = 1/\ln 2 - 1$, $\beta = 0$, then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1 \nu_1} a_2^{\lambda_2 \nu_2} \cdots a_n^{\lambda_n \nu_n}\right)^{t/\Lambda_n} \leq \frac{t e^{p/t}}{p} \sum_{n=1}^{\infty} \left(\left(1 + \frac{\lambda_n \nu_n}{\Lambda_n}\right)^{1-1/\ln 2}\right)^{p/t} \lambda_n \nu_n a_n^p \Lambda_n^{(p-t)/t} \left(\sum_{k=1}^{n} \lambda_k \nu_k (c_k a_k)^p\right)^{(t-p)/p}, \tag{2.35}$$

where (2.34) holds.

Remark 2.15. Corollaries 2.13 and 2.14 are new refinements and generalizations of [10, Corollaries 2.8 and 2.9], respectively. Further, taking $\nu_n = 1$ $(n \in \mathbb{N})$ in Corollaries 2.13 and 2.14, we obtain results corresponding to [10, Corollaries 2.8 and 2.9] with a weaker condition for λ_n $(n \in \mathbb{N})$.

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