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EXTENSION OF ZHU'S SOLUTION TO LOTTO'S CONJECTURE ON THE WEIGHTED BERGMAN SPACES

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We reformulate Lotto's conjecture on the weighted Bergman space A^2_α setting and extend Zhu's solution (on the Hardy space H^2) to the space A^2_α.

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1. Background and terminology. Let H denote the space of analytic maps on the unit disk D and let A^2_α, the weighted Bergman space, be defined (for $\alpha > -1$) as

$$A^2_\alpha = \left\{ f \in H : \iint_D |f(z)|^2 (1-|z|^2)^\alpha dx\,dy < \infty \right\}. \tag{1.1}$$

Given $\phi \in H$ with $\text{Range}(\phi) \subset D$, the composition operator C_ϕ on A^2_α is defined by

$$C_\phi(f)(z) = f(\phi(z)), \quad z \in D. \tag{1.2}$$

The following facts are well known:

(i) A^2_α is a Hilbert space (with the norm $\|f\| = (\iint_D |f(z)|^2(1-|z|^2)^\alpha dx\,dy)^{1/2}$);

(ii) C_ϕ is a bounded linear operator on A^2_α and the compactness of C_ϕ is characterized in [3] as the following theorem illustrates.

Theorem 1.1. *Suppose $0 < p < \infty$ and $\alpha > -1$ are given, then C_ϕ is compact on A^p_α if and only if ϕ has no angular derivative at any point of ∂D.*

The Schatten p-class $\mathcal{S}_p(A^2_\alpha)$ is defined as

$$\mathcal{S}_p(A^2_\alpha) = \left\{ T \in \mathcal{L}(A^2_\alpha) : \sum_{n=0}^{\infty} s_n(T)^p < \infty \right\}, \tag{1.3}$$

where $s_n(T)$ are the singular numbers for T, given by

$$s_n(T) = \inf\{\|T-K\| : K \text{ has rank} \le n\} \tag{1.4}$$

and $\mathcal{L}(A^2_\alpha)$ denotes the space of bounded linear operators on A^2_α. The classes $\mathcal{S}_1(A^2_\alpha)$ (the trace class) and $\mathcal{S}_2(A^2_\alpha)$ (the Hilbert-Schmidt class) are best known.

It is known that $\mathcal{S}_2(A^2_\alpha)$ is a two-sided ideal in $\mathcal{L}(A^2_\alpha)$ [2] and, as a consequence of this, some important comparison properties [4], which are used for the construction of compact but non-Schatten ideals on A^2_α, hold.

Lotto [1] began the investigation of the connection between the geometry of $\phi(D)$ and the membership of C_ϕ in $\mathcal{S}_p(H^2)$. He considered the Riemann map ϕ from D onto the semidisk

$$\left\{z : \operatorname{Im}(z) > 0 \text{ and } \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\} \tag{1.5}$$

which fixes 1 (see [4, Figure 1.1]), and computed an explicit formula for ϕ given by

$$\phi(z) = \frac{1}{1 - ig(z)}, \quad g(z) = \sqrt{i\frac{1-z}{1+z}}. \tag{1.6}$$

Lotto [1] proved that C_ϕ is a compact composition operator on H^2 but not Hilbert-Schmidt (i.e., $C_\phi \notin \mathcal{S}_p(A^2_\alpha)$) and came up with the following conjectures.

Conjecture 1.2. *The composition operator C_ϕ belongs to the Schatten-p ideal $\mathcal{S}_p(H^2)$ if $p > 2$.*

Conjecture 1.3. *Given p, $0 < p < \infty$, there exists a simple example of a domain G_p with $G_p \subseteq D$, or there are easily verifiable geometric conditions on G_p, such that the Riemann map from D onto G_p induces a compact operator that is not in $\mathcal{S}_p(H^2)$.*

Zhu [4] proved both Lotto's conjectures and constructed a Riemann map that induces a compact composition operator which is not in any of the Schatten ideals on H^2.

The goal of this paper is to extend Zhu's solution of Lotto's conjectures on the weighted Bergman space $\mathcal{S}_p(A^2_\alpha)$.

In the $\mathcal{S}_p(A^2_\alpha)$ setting, Lotto's question can be summarized as follows: consider the Riemann map ϕ described above.

(1) Find p, $0 < p < \infty$, such that $C_\phi \notin \mathcal{S}_p(A^2_\alpha)$.

(2) Given p, $0 < p < \infty$, look for analogous geometric conditions on $G_p \subseteq D$ such that the Riemann map $\phi_p : D \to G_p$ induces a compact composition operator that is not in $\mathcal{S}_p(A^2_\alpha)$, and use this fact to construct C_ϕ which is compact but not in any $\mathcal{S}_p(A^2_\alpha)$ for all $0 < p < \infty$.

The compactness criterion (Theorem 1.1) assures us that C_ϕ is compact on A^2_α. And note here that the compactness of C_ϕ is independent of α.

In the next section, we address both of these questions. For $\alpha = 0$, we extend Zhu's solution [4] to prove that $C_\phi \in \mathcal{S}_p(A^2_0) \leftrightarrow p > 1$, showing that the trace class $\mathcal{S}_1(A^2_0)$ "draws" the "borderline" of membership of the C_ϕ's in the Schatten ideals on $\mathcal{S}_p(A^2_0)$. Likewise, we extend Zhu's results on Conjecture 1.3 firstly in $\mathcal{S}_p(A^2_0)$ and then for the general $\mathcal{S}_p(A^2_\alpha)$, $\alpha > -1$.

2. Extension of Zhu's solution to weighted Bergman spaces A^2_α. To answer the first question, we first need Luecking-Zhu theorem [2] to characterize membership in $\mathcal{S}_p(A^2_\alpha)$ which reads

$$C_\phi \in \mathcal{S}_p(A^2_\alpha) \Longleftrightarrow N_{\phi,\alpha+2}(z)\left(\log\left(\frac{1}{|z|}\right)\right)^{-\alpha-2} \in \mathcal{L}^{p/2}(d\lambda), \tag{2.1}$$

where

$$N_{\phi,\beta}(z) = \sum_{\omega \in \phi^{-1}(z)} \log\left(\frac{1}{|\omega|}\right)^{\beta}, \tag{2.2}$$

the generalized Nevanlinna counting function, and $d\lambda(z) = (1-|z|^2)^{-2}dx\,dy$, the Möbius invariant measure on D.

For ϕ a univalent self-map of D into itself,

$$N_{\phi,\beta}(z) = \left(\log\left(\frac{1}{|\phi^{-1}(z)|}\right)\right)^{\beta} \approx (1-|\phi^{-1}(z)|)^{\beta}, \quad \text{for } |\phi^{-1}(z)| \longrightarrow 1. \tag{2.3}$$

Thus, we have the following lemma.

Lemma 2.1. *For ϕ univalent with $\phi(1) = 1$,*

$$C_{\phi} \in \mathcal{S}_p(A^2_{\alpha}) \Longleftrightarrow \chi_{\phi(D)} \cdot \left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{\alpha+2} \in \mathcal{L}^{p/2}(d\lambda). \tag{2.4}$$

We use Lemma 2.1 to update [4, Theorem 3.1] on $\mathcal{S}_p(A^2_{\alpha})$ setting. To emphasize the case $\alpha = 0$, we differentiate two cases.

(1) $\alpha = 0$: for the case $\alpha = 0$, the analogue of [4, Theorem 3.1] reads as follows.

Theorem 2.2. *Let ϕ be a Riemann map from D onto the semidisk*

$$G = \left\{z : \mathrm{Im}(z) > 0 \text{ and } \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\} \tag{2.5}$$

such that $\phi(1) = 1$. Then the composition operator C_{ϕ} belongs to the Schatten ideals $\mathcal{S}_p(A^2_0)$ if and only if $p > 1$.

Remark 2.3. It is interesting to compare Theorem 2.2 with the corresponding result in the H^2 case (see [4, Theorem 3.1]) which holds for $p > 2$ showing here that the trace class $\mathcal{S}_1(A^2_0)$ is the "borderline" case for membership of the C_{ϕ}'s in the Schatten-p ideals. For the proof, see the general case next.

(2) $-1 < \alpha$ arbitrary: we start with Lemma 2.1. That is, check if (or when) the integral

$$\iint_G \left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{((\alpha+2)/2)p} \frac{dA(z)}{(1-|z|^2)^2} < \infty. \tag{2.6}$$

Since $\partial G \cap \partial D = \{1\}$, (2.6) is equivalent to

$$\iint_{G\cap\Delta(\epsilon)} \left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{((\alpha+2)/2)p} \frac{dA(z)}{(1-|z|^2)^2} < \infty, \tag{2.7}$$

where $\Delta(\epsilon) = \{z; |z-1| < \epsilon\}$ (for $\epsilon > 0$ small) as in the proof of [4, Theorem 3.1], and ϕ is the Riemann map from $D \to G$. For $\alpha = 0$, the left-hand side of (2.7) reduces to

$$\iint_{G\cap\Delta(\epsilon)} \left(\frac{1-|\phi^{-1}(z)|}{1-|z|}\right)^{p} \frac{dA(z)}{(1-|z|^2)^2} \tag{2.8}$$

which converges if and only if $p > 1$ (see equations (3.2), (3.7), and (3.8) in the proof of [4, Theorem 3.1] replacing the parameter p with $p/2$), which proves Theorem 2.2.

Once more, replacing $p/2$ by $((\alpha+2)/2)p$ in equations (3.2) and (3.7) in the proof of [4, Theorem 3.1] reveals that (2.7) is finite if and only if

$$\iint_G \left(\frac{r^2\sin(2\theta)}{r\cos\theta}\right)^{((\alpha+2)/2)p} \frac{r\,dr\,d\theta}{(r\cos\theta)^2} < \infty, \tag{2.9}$$

where r is such that $z = 1 - re(i\theta) \in G$ as in the proof of [4, Theorem 3.1]. Again, replacing $p/2$ by $((\alpha+2)/2)p$ in [4, equations (3.7) and (3.8)],

$$\iint_G \left(\frac{r^2\sin(2\theta)}{r\cos\theta}\right)^{((\alpha+2)/2)p} \frac{r\,dr\,d\theta}{(r\cos\theta)^2} \approx \int_0^{\pi/2} \frac{d\theta}{(\cos\theta)^{(2-((\alpha+2)/2)p)}}. \tag{2.10}$$

But then the right-hand side converges if and only if $p > 2/(\alpha+2)$, which certainly agrees with case (1), when $\alpha = 0$. Thus, we proved the following theorem.

Theorem 2.4. *For $-1 < \alpha$, under the assumptions of Theorem 1.1, $C_\phi \in \mathcal{S}_p(A^2_\alpha)$ if and only if $p > 2/(\alpha+2)$.*

In the following, we address the second question.

For $0 < \beta < 1$, let G_β be the crescent-shaped region bounded by

$$G = \left\{z : \operatorname{Im}(z) > 0 \text{ and } \left|z - \frac{1}{2}\right| = \frac{1}{2}\right\} \tag{2.11}$$

and a circular arc in the upper half of D joining 0 to 1, with the two arcs forming an angle of $\beta\pi$ at 0 and 1 (see [4, Figure 1.2]). Let ϕ_β be the Riemann map of D onto G_β with $\phi_\beta(1) = 1$. To see if (when) $C_{\phi_\beta} \in \mathcal{S}_p(A^2_\alpha)$, we only need to look at equation (4.9) and the last line(s) (in all the three cases) of the proof of [4, Theorem 4.1] (and note here that we replace α by β and $p/2$ by $2/(\alpha+2)$), which means

$$C_{\phi_\beta} \in \mathcal{S}_1(A^2_\alpha) \Longleftrightarrow 2 - \left(\frac{1}{\beta} - 1\right)\left(\frac{\alpha+2}{2}p\right) < 1, \tag{2.12}$$

which converges if and only if $p > 2\beta/(1-\beta)(\alpha+2)$ and this conforms to Theorems 2.2 and 2.4 when $\beta = 1/2$. Thus, we proved the following theorem.

Theorem 2.5. (1) *$C_{\phi_\beta} \notin \mathcal{S}_{2\beta/(1-\beta)(\alpha+2)}(A^2_\alpha)$;*
(2) *$C_{\phi_\beta} \in \mathcal{S}_p(A^2_\alpha)$ for all $p > 2\beta/(1-\beta)(\alpha+2)$.*

Remark 2.6. (1) Note that here β characterizes the geometry of $\phi_\beta(D)$.

(2) The same argument as in Zhu's construction of a compact composition operator that is not in any of the Schatten-p ideals (see [4, Section 5]) can be transferred to the Bergman space case with a slight modification. (Here, of course, we use the corresponding facts on A^2_α mentioned in Section 1.)

The modification is as follows.

Rewriting the basic steps of the construction, let $\theta_n = \pi/(n+1)$, $z_n = e^{i\theta_n}$, $r_n = (1/2)\sin\theta_n$, and $c_n = (1-r_n)z_n$, where $n = 1, 2, \ldots$.

Define Ω_n to be the region bounded by the semicircle

$$\{z : \mathrm{Im}(z) \geq 0 \text{ and } |z - |c_n\|| = r_n\} \tag{2.13}$$

and a circular arc that is inside D joining $1 - 2r_n$ to 1 forming an angle of $((n+1)/(n+2))\pi$ (for the $\alpha = 0$ case) and $(n+1)(\alpha+2)/(2+(n+1)(\alpha+2))$ (for the $\alpha > -1$ case). (This modification is made so as to apply Theorem 2.5.)

Let

$$\Omega'_n = \{ze^{i\theta_n} : z \in \Omega_n\}, \tag{2.14}$$

$$\Omega = \bigcup_{n=1'}^{\infty} \Omega'_n. \tag{2.15}$$

The same argument (in the A^2_α setting) as in the proof of [4, Theorem 5] yields the following theorem.

Theorem 2.7. *Suppose Ω is defined as in (2.15), then*

(1) *Ω is a simply connected domain contained in the upper half of D;*

(2) *any Riemann map ϕ that maps D onto Ω induces a compact composition operator C_ϕ that does not belong to any of the Schatten-p ideals $\mathcal{S}_p(A^2_\alpha)$, $p > 0$.*

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