IJMMS 2004:42, 2231–2264
PII. S0161171204308203
http://ijmms.hindawi.com

MAKING NONTRIVIALLY ASSOCIATED MODULAR CATEGORIES FROM FINITE GROUPS

M. M. AL-SHOMRANI and E. J. BEGGS

Received 15 August 2003

We show that the double $\mathcal{D}$ of the nontrivially associated tensor category constructed from left coset representatives of a subgroup of a finite group X is a modular category. Also we give a definition of the character of an object in this category as an element of a braided Hopf algebra in the category. This definition is shown to be adjoint invariant and multiplicative on tensor products. A detailed example is given. Finally, we show an equivalence of categories between the nontrivially associated double $\mathcal{D}$ and the trivially associated category of representations of the Drinfeld double of the group $D(X)$.

2000 Mathematics Subject Classification: 18D10, 16W30.

1. Introduction. This paper will make continual use of formulae and ideas from [2], and these definitions and formulae will not be repeated, as they would add very considerably to the length of the paper. The paper [2] is itself based on the papers [3, 4], but is mostly self-contained in terms of notation and definitions. The book [6] has been used as a standard reference for Hopf algebras, and [1, 8] as references for modular categories.

In [2], there is a construction of a nontrivially associated tensor category $\mathcal{C}$ from data which is a choice of left coset representatives M for a subgroup G of a finite group X. This introduces a binary operation "$\cdot$" and a G-valued "cocycle" τ on M. There is also a double construction where X is viewed as a subgroup of a larger group. This gives rise to a braided category $\mathcal{D}$, which is the category of reps of an algebra D, which is itself in the category, and it is the category that we concentrate on in this paper.

It is our aim to show that the nontrivially associated algebra D has reps which have characters in the same way that the reps of a finite group have characters, and also that the category of its representations has a modular structure in the same way that the category of reps of the double of a group has a modular structure.

We begin by describing the indecomposable objects in $\mathcal{C}$, in a manner similar to that used in [4]. A detailed example is given using the group D_6. Then we show how to find the dual objects in the category, and again illustrate this with an example.

Next, we show that the rigid braided category $\mathcal{D}$ is a ribbon category. The ribbon maps are calculated for the indecomposable objects in our example category.

In the next section, we explicitly evaluate in $\mathcal{D}$ the standard diagram for trace in a ribbon category [6]. Then we define the character of an object in $\mathcal{D}$ as an element of the dual of the braided Hopf algebra D. This element is shown to be right adjoint invariant. Also we show that the character is multiplicative for the tensor product of objects. A formula is found for the character in $\mathcal{D}$ in terms of characters of group representations.

The last ingredient needed for a modular category is the trace of the double braiding, and this is calculated in $\mathcal{D}$ in terms of group characters. Then the matrices S, T, and C, implementing the modular representation, are calculated explicitly in our example.

Finally, we show an equivalence of categories between the nontrivially associated double $\mathcal{D}$ and the category of representations of the Drinfeld double of the group $D(X)$.

Throughout the paper, we assume that all groups mentioned are finite, and that all vector spaces are finite-dimensional. We take the base field to be the complex numbers $\mathbb{C}$.

2. Indecomposable objects in $\mathcal{C}$.

The objects of $\mathcal{C}$ are the right representations of the algebra A described in [2]. We now look at the indecomposable objects in $\mathcal{C}$, or the irreducible representations of A, in a manner similar to that used in [4].

Theorem 2.1. *The indecomposable objects in $\mathcal{C}$ are of the form*

$$V = \bigoplus_{s \in \mathbb{O}} V_s, \tag{2.1}$$

where $\mathbb{O}$ is an orbit in M under the G action $\triangleleft$, and each V_s is an irreducible right representation of the stabilizer of s, $\operatorname{stab}(s)$. *Every object T in $\mathcal{C}$ can be written as a direct sum of indecomposable objects in $\mathcal{C}$.*

Proof. For an object T in $\mathcal{C}$, we can use the M-grading to write

$$T = \bigoplus_{s \in M} T_s, \tag{2.2}$$

but as M is a disjoint union of orbits $\mathbb{O}_s = \{s \triangleleft u : u \in G\}$ for $s \in M$, T can be rewritten as a disjoint sum over orbits:

$$T = \bigoplus_{\mathbb{O}} T_{\mathbb{O}}, \tag{2.3}$$

where

$$T_{\mathbb{O}} = \bigoplus_{s \in \mathbb{O}} T_s. \tag{2.4}$$

Now we will define the stabilizer of $s \in \mathbb{O}$, which is a subgroup of G, as

$$\operatorname{stab}(s) = \{u \in G : s \triangleleft u = s\}. \tag{2.5}$$

As $\langle \eta \bar{\triangleleft} u \rangle = \langle \eta \rangle \triangleleft u$ for all $\eta \in T$, T_s is a representation of the group $\operatorname{stab}(s)$. Now fix a base point $t \in \mathbb{O}$. Because $\operatorname{stab}(t)$ is a finite group, T_t is a direct sum of irreducible group representations W_i for $i = 1, \dots, m$, that is,

$$T_t = \bigoplus_{i=1}^{m} W_i. \tag{2.6}$$

Suppose that $\mathbb{O} = \{t_1, t_2, \dots, t_n\}$, where $t_1 = t$, and take $u_i \in G$ so that $t_i = t \triangleleft u_i$. Define

$$U_i = \bigoplus_{j=1}^{n} W_i \bar{\triangleleft} u_j \subset \bigoplus_{s \in \mathbb{O}} T_s. \tag{2.7}$$

We claim that each U_i is an indecomposable object in $\mathscr{C}$. For any $v \in G$ and $\xi \bar{\triangleleft} u_k \in W_i \bar{\triangleleft} u_k$,

$$(\xi \bar{\triangleleft} u_k) \bar{\triangleleft} v = (\xi \bar{\triangleleft} (u_k v u_j{}^{-1})) \bar{\triangleleft} u_j, \tag{2.8}$$

where $u_k v u_j{}^{-1} \in \mathrm{stab}(t)$ for some $u_j \in G$. This shows that U_i is a representation of G. By the definition of U_i, any subrepresentation of U_i which contains W_i must be all of U_i. Thus U_i is an indecomposable object in $\mathscr{C}$ and

$$T_{\mathbb{O}} = \bigoplus_{i=1}^{m} U_i. \tag{2.9}$$

□

Theorem 2.2 (Schur's lemma). *Let V and W be two indecomposable objects in $\mathscr{C}$ and let $\alpha : V \to W$ be a morphism. Then α is zero or a scalar multiple of the identity.*

Proof. V and W are associated to orbits $\mathbb{O}$ and $\mathbb{O}'$ so that $V = \bigoplus_{s\in\mathbb{O}} V_s$ and $W = \bigoplus_{s\in\mathbb{O}'} W_s$. As morphisms preserve grade, if $\alpha \neq 0$, then $\mathbb{O} = \mathbb{O}'$. Now, if we take $s \in \mathbb{O}$, we will find that $\alpha : V_s \to W_s$ is a map of irreps of $\mathrm{stab}(s)$, so by Schur's lemma for groups, any nonzero map is a scalar multiple of the identity, and we have $V_s = W_s$ as representations of $\mathrm{stab}(s)$. Now we need to check that the multiple of the identity is the same for each $s \in \mathbb{O}$. Suppose that α is a multiplication by λ on V_s. Given $t \in \mathbb{O}$, there is a $u \in G$ so that $t \triangleleft u = s$. Then, for $\eta \in V_t$,

$$\alpha(\eta) = \alpha(\eta \bar{\triangleleft} u) \bar{\triangleleft} u^{-1} = \lambda(\eta \bar{\triangleleft} u) \bar{\triangleleft} u^{-1} = \lambda \eta. \tag{2.10}$$

□

Lemma 2.3. *Let V be an indecomposable object in $\mathscr{C}$ associated to the orbit $\mathbb{O}$. Choose $s, t \in \mathbb{O}$ and $u \in G$ so that $s \triangleleft u = t$. Then V_s and V_t are irreps of* $\mathrm{stab}(s)$ *and* $\mathrm{stab}(t)$, *respectively, and the group characters obey $\chi_{V_t}(v) = \chi_{V_s}(u v u^{-1})$.*

Proof. Note that $\bar{\triangleleft} u$ is an invertible map from V_s to V_t. Then we have the commuting diagram

$$\begin{array}{ccc} V_s & \xrightarrow{\bar{\triangleleft} u v u^{-1}} & V_s \\ \downarrow{\scriptstyle \bar{\triangleleft} u} & & \downarrow{\scriptstyle \bar{\triangleleft} u} \\ V_t & \xrightarrow{\bar{\triangleleft} v} & V_t \end{array} \tag{2.11}$$

which implies that $\mathrm{trace}(\bar{\triangleleft} u v u^{-1} : V_s \to V_s) = \mathrm{trace}(\bar{\triangleleft} v : V_t \to V_t)$. □

3. An example of indecomposable objects. We give an example of indecomposable objects in the categories discussed in the last section. As we will later want to have a

TABLE 3.1

Irreps		$\{e\}$	$\{a^3\}$	$\{b,ba^2,ba^4\}$	$\{ba,ba^3,ba^5\}$	$\{a^2,a^4\}$	$\{a,a^5\}$
1_1	2_1	1	1	1	1	1	1
1_2	2_2	1	−1	−1	1	1	−1
1_3	2_3	1	−1	1	−1	1	−1
1_4	2_4	1	1	−1	−1	1	1
1_5	2_5	2	−2	0	0	−1	1
1_6	2_6	2	2	0	0	−1	−1

TABLE 3.2

Irreps		e	a	a^2	a^3	a^4	a^5
3_0	4_0	1	1	1	1	1	1
3_1	4_1	1	ω^1	ω^2	ω^3	ω^4	ω^5
3_2	4_2	1	ω^2	ω^4	1	ω^2	ω^4
3_3	4_3	1	ω^3	1	ω^3	1	ω^3
3_4	4_4	1	ω^4	ω^2	1	ω^4	ω^2
3_5	4_5	1	ω^5	ω^4	ω^3	ω^2	ω^1

category with braiding, we use the double construction in [2]. We also use Lemma 2.3 to list the group characters [5] for every point in the orbit in terms of the given base points.

Take X to be the dihedral group $D_6 = \langle a,b : a^6 = b^2 = e,\ ab = ba^5\rangle$, whose elements we list as $\{e,a,a^2,a^3,a^4,a^5,b,ba,ba^2,ba^3,ba^4,ba^5\}$, and G to be the nonabelian normal subgroup of order 6 generated by a^2 and b, that is, $G = \{e,a^2,a^4,b,ba^2,ba^4\}$. We choose $M = \{e,a\}$. The center of D_6 is the subgroup $\{e,a^3\}$, and it has the following conjugacy classes: $\{e\}$, $\{a^3\}$, $\{a^2,a^4\}$, $\{a,a^5\}$, $\{b,ba^2,ba^4\}$, and $\{ba,ba^3,ba^5\}$.

The category $\mathcal{D}$ consists of right representations of the group $X = D_6$ which are graded by $Y = D_6$ (as a set), using the actions $\tilde{\triangleleft} : Y \times X \to Y$ and $\tilde{\triangleright} : Y \times X \to X$ which are defined as follows:

$$y \tilde{\triangleleft} x = x^{-1} y x, \qquad vt \tilde{\triangleright} x = v^{-1} x v' = t x t'^{-1}, \tag{3.1}$$

for $x \in X$, $y \in Y$, $v,v' \in G$, and $t,t' \in M$, where $vt \tilde{\triangleleft} x = v't'$.

Now let V be an indecomposable object in $\mathcal{D}$. We get the following cases.

Case (1). Take the orbit $\{e\}$ with base point e, whose stabilizer is the whole of D_6. There are six possible irreducible group representations of the stabilizer, with their characters given by Table 3.1 [7].

Case (2). Take the orbit $\{a^3\}$ with base point a^3, whose stabilizer is the whole of D_6. There are six possible irreps $\{2_1,2_2,2_3,2_4,2_5,2_6\}$, with characters given by Table 3.1.

Case (3). Take the orbit $\{a^2,a^4\}$ with base point a^2, whose stabilizer is $\{e,a,a^2,a^3,a^4,a^5\}$. There are six irreps $\{3_0,3_1,3_2,3_3,3_4,3_5\}$, with characters given by Table 3.2, where $\omega = e^{i\pi/3}$. Applying Lemma 2.3 gives $\chi_{V_{a^4}}(v) = \chi_{V_{a^2}}(bvb)$.

TABLE 3.3

Irreps	e	a^3	b	ba^3
5_{++}	1	1	1	1
5_{+-}	1	1	−1	−1
5_{-+}	1	−1	1	−1
5_{--}	1	−1	−1	1

TABLE 3.4

Irreps	e	a^3	ba	ba^4
6_{++}	1	1	1	1
6_{-+}	1	−1	1	−1
6_{+-}	1	1	−1	−1
6_{--}	1	−1	−1	1

Case (4). Take the orbit $\{a,a^5\}$ with base point a, whose stabilizer is $\{e,a,a^2,a^3,a^4,a^5\}$. There are six irreps $\{4_0,4_1,4_2,4_3,4_4,4_5\}$ with characters given in Table 3.2. Applying Lemma 2.3 gives $\chi_{V_{a^5}}(v)=\chi_{V_a}(ba^2vba^2)$.

Case (5). Take the orbit $\{b,ba^2,ba^4\}$ with base point b, whose stabilizer is $\{e,a^3,b,ba^3\}$. There are four irreps with characters given by Table 3.3. Applying Lemma 2.3 gives $\chi_{V_{ba^2}}(v)=\chi_{V_b}(a^4va^2)$ and $\chi_{V_{ba^4}}(v)=\chi_{V_b}(a^2va^4)$.

Case (6). Take the orbit $\{ba,ba^3,ba^5\}$ with base point ba, whose stabilizer is $\{e,a^3,ba,ba^4\}$. There are four irreps with characters given by Table 3.4. Applying Lemma 2.3 gives $\chi_{V_{ba^3}}(v)=\chi_{V_{ba}}(a^4va^2)$ and $\chi_{V_{ba^5}}(v)=\chi_{V_{ba}}(a^2va^4)$.

4. Duals of indecomposable objects in $\mathscr{C}$. Given an irreducible object V with associated orbit $\mathbb{O}$ in $\mathscr{C}$, how do we find its dual V^*? The dual would be described, as in Section 2, by an orbit, a base point in the orbit, and a right group representation of the stabilizer of the base point. Using the formula $(s^L\cdot s)\triangleleft u=(s^L\triangleleft(s\triangleright u))\cdot(s\triangleleft u)=e$, we see that the left inverse of a point in the orbit containing s is in the orbit containing s^L. By using the evaluation map from $V^*\otimes V$ to the field, we can take $(V^*)_{s^L}=(V_s)^*$ as vector spaces. We use $\check{\triangleleft}$ as the action of $\mathrm{stab}(s)$ on $(V_s)^*$, that is, $(\alpha\check{\triangleleft}z)(\xi\bar{\triangleleft}z)=\alpha(\xi)$ for $\alpha\in(V_s)^*$ and $\xi\in V_s$. The action $\bar{\triangleleft}$ of $\mathrm{stab}(s^L)$ on $(V^*)_{s^L}$ is given by $\alpha\bar{\triangleleft}(s\triangleright z)=\alpha\check{\triangleleft}z$ for $z\in\mathrm{stab}(s)$. In terms of group characters, this gives

$$\chi_{(V^*)_{s^L}}(s\triangleright z)=\chi_{(V_s)^*}(z),\quad z\in\mathrm{stab}(s).\tag{4.1}$$

If we take $\mathbb{O}^L=\{s^L:s\in\mathbb{O}\}$ to have base point p, and choose $u\in G$ so that $p\triangleleft u=s^L$, then using Lemma 2.3 gives

$$\chi_{(V^*)_{s^L}}(s\triangleright z)=\chi_{(V_s)^*}(z)=\chi_{(V^*)_p}(u(s\triangleright z)u^{-1}),\quad z\in\mathrm{stab}(s).\tag{4.2}$$

This formula allows us to find the character of V^* at its base point p as a representation of $\mathrm{stab}(p)$ in terms of the character of the dual of V_s as a representation of $\mathrm{stab}(s)$.

Lemma 4.1. *In $\mathscr{C}$, $(V\otimes W)^*$ can be regarded as $W^*\otimes V^*$ with the evaluation*

$$(\alpha\otimes\beta)(\xi\otimes\eta)=(\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi\rangle\cdot\langle\eta\rangle))(\eta)(\beta\bar{\triangleleft}\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1})(\xi). \tag{4.3}$$

Given a basis $\{\xi\}$ of V and a basis $\{\eta\}$ of W, the dual basis $\{\widehat{\xi\otimes\eta}\}$ of $W^*\otimes V^*$ can be written in terms of the dual basis of V^* and W^* as

$$\widehat{\xi\otimes\eta}=\hat{\eta}\bar{\triangleleft}\tau(\langle\xi\rangle^L\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle),\langle\xi\rangle\cdot\langle\eta\rangle)^{-1}\otimes\hat{\xi}\bar{\triangleleft}\tau(\langle\xi\rangle,\langle\eta\rangle). \tag{4.4}$$

Proof. Applying the associator to $(\alpha\otimes\beta)\otimes(\xi\otimes\eta)$ gives

$$\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi\rangle\cdot\langle\eta\rangle)\otimes(\beta\otimes(\xi\otimes\eta)), \tag{4.5}$$

and then applying the inverse associator gives

$$\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi\rangle\cdot\langle\eta\rangle)\otimes((\beta\bar{\triangleleft}\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}\otimes\xi)\otimes\eta). \tag{4.6}$$

Applying the evaluation map first to $\beta\bar{\triangleleft}\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}\otimes\xi$ then to $\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi\rangle\cdot\langle\eta\rangle)\otimes\eta$ gives the first equation. For the evaluation to be nonzero, we need $(\langle\beta\rangle\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1})\cdot\langle\xi\rangle=e$ which implies $\langle\beta\rangle\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)^{-1}=\langle\xi\rangle^L$ or, equivalently, $\langle\beta\rangle=\langle\xi\rangle^L\triangleleft\tau(\langle\xi\rangle,\langle\eta\rangle)$. This gives the second equation. □

Example 4.2. Using (4.2), we calculate the duals of the objects given in the last section.

Case (1). The orbit $\{e\}$ has left inverse $\{e\}$, so $\chi_{(V^*)_e}=\chi_{(V_e)^*}$. By a calculation with group characters, all the listed irreps of $\mathrm{stab}(e)$ are self-dual, so $1_r^*=1_r$ for $r\in\{1,\dots,6\}$.

Case (2). The orbit $\{a^3\}$ has left inverse $\{a^3\}$, so $\chi_{(V^*)_{a^3}}=\chi_{(V_{a^3})^*}$. As in the last case, the group representations are self-dual, so $2_r^*=2_r$ for $r\in\{1,\dots,6\}$.

Case (3). The left inverse of the base point a^2 is a^4, which is still in the orbit. As *group* representations, the dual of 3_r is $3_{6-r}\,(\mathrm{mod}\,6)$. Applying Lemma 2.3 to move the base point, we see that the dual of 3_r *in the category* is 3_r.

Case (4). The left inverse of the base point a is a^5, which is still in the orbit. As in the last case, the dual of 4_r *in the category* is 4_r.

Case (5). The left inverse of the base point is itself, and as group representations, all Case (5) irreps are self-dual. We deduce that in the category the objects are self-dual.

Case (6). Self-dual as in Case (5).

5. The ribbon map on the category $\mathscr{D}$

Theorem 5.1. *The ribbon transformation $\theta_V:V\to V$ for any object V in $\mathscr{D}$ can be defined by $\theta_V(\xi)=\xi\hat{\triangleleft}\|\xi\|$.*

Proof. In the following lemmas, we show that the required properties hold. □

Lemma 5.2. *θ_V is a morphism in the category.*

Proof. Begin by checking the X-grade: for $\xi \in V$,

$$||\theta_V(\xi)|| = ||\xi \hat{\triangleleft} \|\xi\| || = \|\xi\| \tilde{\triangleleft} \|\xi\| = \|\xi\|. \tag{5.1}$$

Now we check the X-action, that is, that $\theta_V(\xi \hat{\triangleleft} x) = \theta_V(\xi) \hat{\triangleleft} x$:

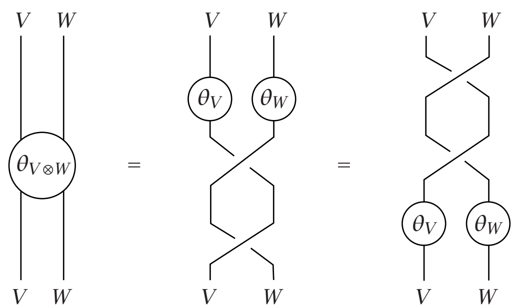
$$\begin{aligned} \theta_V(\xi \hat{\triangleleft} x) &= (\xi \hat{\triangleleft} x) \hat{\triangleleft} ||\xi \hat{\triangleleft} x|| = (\xi \hat{\triangleleft} x) \hat{\triangleleft} (\|\xi\| \tilde{\triangleleft} x) \\ &= \xi \hat{\triangleleft} x x^{-1} \|\xi\| x = (\xi \hat{\triangleleft} \|\xi\|) \hat{\triangleleft} x = \theta_V(\xi) \hat{\triangleleft} x. \end{aligned} \tag{5.2}$$

□

Lemma 5.3. *For any two objects V and W in $\mathcal{D}$,*

$$\theta_{V\otimes W} = \Psi^{-1}_{V\otimes W} \circ \Psi^{-1}_{W\otimes V} \circ (\theta_V \otimes \theta_W) = (\theta_V \otimes \theta_W) \circ \Psi^{-1}_{V\otimes W} \circ \Psi^{-1}_{W\otimes V}. \tag{5.3}$$

This can also be described by the following:



(5.4)

Proof. First calculate $\Psi(\Psi(\xi \otimes \eta))$ for $\xi \in V$ and $\eta \in W$, beginning with

$$\Psi(\Psi(\xi \otimes \eta)) = \Psi(\eta \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1} \otimes \xi \hat{\triangleleft} |\eta|). \tag{5.5}$$

To simplify what follows, we will use the substitutions

$$\eta' = \xi \hat{\triangleleft} |\eta|, \qquad \xi' = \eta \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1}, \tag{5.6}$$

so (5.5) can be rewritten as

$$\Psi(\Psi(\xi \otimes \eta)) = \Psi(\xi' \otimes \eta') = \eta' \hat{\triangleleft} (\langle \xi' \rangle \triangleleft |\eta'|)^{-1} \otimes \xi' \hat{\triangleleft} |\eta'|. \tag{5.7}$$

As $\eta' = \xi \hat{\triangleleft} |\eta| = \xi \bar{\triangleleft} |\eta|$, then $|\eta'| = |\xi \bar{\triangleleft} |\eta|| = (\langle \xi \rangle \triangleright |\eta|)^{-1} |\xi||\eta|$, so

$$\begin{aligned} \xi' \hat{\triangleleft} |\eta'| &= \eta \hat{\triangleleft} (\langle \xi \rangle \triangleleft |\eta|)^{-1} (\langle \xi \rangle \triangleright |\eta|)^{-1} |\xi||\eta| \\ &= \eta \hat{\triangleleft} ((\langle \xi \rangle \triangleright |\eta|)(\langle \xi \rangle \triangleleft |\eta|))^{-1} |\xi||\eta| \\ &= \eta \hat{\triangleleft} |\eta|^{-1} \langle \xi \rangle^{-1} |\xi||\eta|. \end{aligned} \tag{5.8}$$

Hence, if we put $y = \|\xi \otimes \eta\| = \|\xi\| \circ \|\eta\| = |\eta|^{-1} |\xi|^{-1} \langle \xi \rangle \langle \eta \rangle$,

$$\Psi(\Psi(\xi \otimes \eta)) \hat{\triangleleft} \|\xi \otimes \eta\| = \xi \hat{\triangleleft} |\eta| (\langle \xi' \rangle \triangleleft |\eta'|)^{-1} (p \tilde{\triangleright} \|\xi \otimes \eta\|) \otimes \eta \hat{\triangleleft} |\eta|^{-1} \langle \eta \rangle, \tag{5.9}$$

where, using (5.8),

$$
\begin{aligned}
p = \|\xi'\hat{\triangleleft}|\eta'|\| = |\xi'\bar{\triangleleft}|\eta'||^{-1}\langle \xi'\bar{\triangleleft}|\eta'|\rangle = \|\eta\|\tilde{\triangleleft}\|\eta\|y^{-1} = \|\eta\|\tilde{\triangleleft}y^{-1},\\
p\tilde{\triangleright}\|\xi\otimes\eta\| = (\|\eta\|\tilde{\triangleleft}y^{-1})\tilde{\triangleright}y = (\|\eta\|\tilde{\triangleright}y^{-1})^{-1}.
\end{aligned}
\tag{5.10}
$$

As $\|\xi'\bar{\triangleleft}|\eta'|\| = v't' = \|\eta\|\tilde{\triangleleft}y^{-1}$, by unique factorization, $t' = \langle\xi'\rangle\triangleleft|\eta'|$. Then $\|\eta\|\tilde{\triangleright}y^{-1} = \langle\eta\rangle y^{-1}t'^{-1}$, which implies that

$$
|\eta|(\langle\xi'\rangle\triangleleft|\eta'|)^{-1}(\|\eta\|\tilde{\triangleright}y^{-1})^{-1} = |\eta|t'^{-1}t'y\langle\eta\rangle^{-1} = \|\xi\|. \tag{5.11}
$$

Substituting this into (5.9) gives

$$
\Psi(\Psi(\xi\otimes\eta))\hat{\triangleleft}\|\xi\otimes\eta\| = \xi\hat{\triangleleft}\|\xi\|\otimes\eta\hat{\triangleleft}\|\eta\|. \tag{5.12}
$$

□

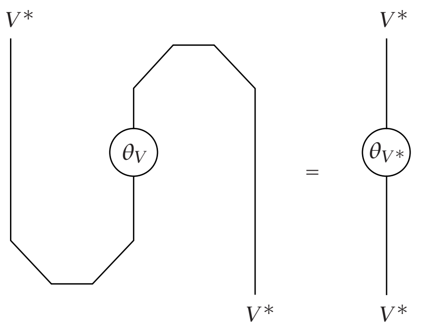
Lemma 5.4. *For the unit object $\underline{1} = \mathbb{C}$ in $\mathcal{D}$, $\theta_{\underline{1}}$ is the identity.*

Proof. For any object V in $\mathcal{D}$, $\theta_V : V \to V$ is defined by

$$
\theta_V(\xi) = \xi\hat{\triangleleft}\|\xi\| \quad \text{for } \xi\in V. \tag{5.13}
$$

If we choose $V = \underline{1} = \mathbb{C}$, then $\theta_{\underline{1}}(\xi) = \xi\hat{\triangleleft}e = \xi$ as $\|\xi\| = e$. □

Lemma 5.5. *For any object V in $\mathcal{D}$, $(\theta_V)^* = \theta_{V^*}$:*



(5.14)

Proof. Begin with

$$
\begin{aligned}
\mathrm{coev}_V(1) &= \sum_{\xi\in \text{ basis of } V} \xi\hat{\triangleleft}\tilde{\tau}(\|\xi\|^L,\|\xi\|)^{-1}\otimes\hat{\xi}\\
&= \sum_{\xi\in \text{ basis of } V} \xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi}.
\end{aligned}
\tag{5.15}
$$

For $\alpha\in V^*$, we follow (5.14) and calculate

$$
(\theta_V)^*(\alpha) = (\mathrm{eval}_V\otimes\mathrm{id})\sum_{\xi\in \text{ basis of } V}\Phi^{-1}(\alpha\otimes(\theta_V(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1})\otimes\hat{\xi})). \tag{5.16}
$$

Now, as $\tau(\langle\xi\rangle^L,\langle\xi\rangle)=\langle\xi\rangle^L\langle\xi\rangle$,

$$\begin{aligned}||\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}|| &= \|\xi\|\tilde{\triangleleft}(\langle\xi\rangle^L\langle\xi\rangle)^{-1}\\ &= \langle\xi\rangle^L\langle\xi\rangle|\xi|^{-1}\langle\xi\rangle\langle\xi\rangle^{-1}\langle\xi\rangle^{L-1}\\ &= \langle\xi\rangle^L\langle\xi\rangle|\xi|^{-1}\langle\xi\rangle^{L-1},\\ \theta_V(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}) &= (\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1})\hat{\triangleleft}||\xi\hat{\triangleleft}\tilde{\tau}(\|\xi\|^L,\|\xi\|)^{-1}||\\ &= \xi\hat{\triangleleft}\langle\xi\rangle^{-1}\langle\xi\rangle^{L-1}\langle\xi\rangle^L\langle\xi\rangle|\xi|^{-1}\langle\xi\rangle^{L-1}\\ &= \xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}.\end{aligned}\tag{5.17}$$

The next step is to find

$$\begin{aligned}&\Phi^{-1}(\alpha\otimes((\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1})\otimes\hat{\xi}))\\ &\quad=(\alpha\hat{\triangleleft}\tilde{\tau}(||\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}||,||\hat{\xi}||)^{-1}\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}))\otimes\hat{\xi}.\end{aligned}\tag{5.18}$$

As

$$\begin{aligned}&||\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}||\\ &\quad=\|\xi\|\tilde{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}\\ &\quad=\langle\xi\rangle^L|\xi||\xi|^{-1}\langle\xi\rangle|\xi|^{-1}\langle\xi\rangle^{L-1}\\ &\quad=\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}\langle\xi\rangle^{L-1}\\ &\quad=\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}\langle\xi\rangle\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\\ &\quad=\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}(\langle\xi\rangle\triangleright\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1})(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}),\end{aligned}\tag{5.19}$$

then, as $\|\hat{\xi}\|=\|\xi\|^L=|\xi|\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\langle\xi\rangle^L$,

$$\begin{aligned}&\Phi^{-1}(\alpha\otimes((\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1})\otimes\hat{\xi}))\\ &\quad=(\alpha\hat{\triangleleft}\tau(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1},\langle\xi\rangle^L)^{-1}\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}))\otimes\hat{\xi}.\end{aligned}\tag{5.20}$$

Put $v=\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}=\langle\xi\rangle^{-1}\langle\xi\rangle^{L-1}$ and $w=\tau(\langle\xi\rangle\triangleleft v,\langle\xi\rangle^L)^{-1}=((\langle\xi\rangle\triangleleft v)\langle\xi\rangle^L)^{-1}$; then substituting in (5.16) gives

$$(\theta_V)^*(\alpha)=(\mathrm{eval}_V\otimes\mathrm{id})\sum_{\xi\in\text{ basis of }V}((\alpha\hat{\triangleleft}w)\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}))\otimes\hat{\xi}.\tag{5.21}$$

For a given term in the sum to be nonzero, we require that

$$\|\alpha\|=||\hat{\xi}||=\|\xi\|^L=|\xi|\langle\xi\rangle^{-1},\tag{5.22}$$

and we proceed under this assumption. Now calculate

$$\mathrm{eval}_V((\alpha\hat{\triangleleft}w)\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle^{L-1}))=(\beta\hat{\triangleleft}(\|\xi\|\tilde{\triangleright}p))(\xi\tilde{\triangleleft}p)=\beta(\xi),\tag{5.23}$$

where $p = |\xi|^{-1}\langle\xi\rangle^{L-1}$ and $\beta = \alpha\hat{\triangleleft}w(\|\xi\|\tilde{\triangleright}p)^{-1}$. Next, we want to find $\|\xi\|\tilde{\triangleright}p$. To do this, we first find

$$\begin{aligned}\|\xi\|\tilde{\triangleleft}p &= \langle\xi\rangle^{L}|\xi||\xi|^{-1}\langle\xi\rangle|\xi|^{-1}\langle\xi\rangle^{L-1}\\ &= v^{-1}|\xi|^{-1}\langle\xi\rangle v = v^{-1}|\xi|^{-1}(\langle\xi\rangle\triangleright v)(\langle\xi\rangle\triangleleft v),\end{aligned} \tag{5.24}$$

and hence

$$\begin{aligned}\|\xi\|\tilde{\triangleright}p &= \langle\xi\rangle p(\langle\xi\rangle\triangleleft v)^{-1}\\ &= \langle\xi\rangle|\xi|^{-1}\langle\xi\rangle v(\langle\xi\rangle\triangleleft v)^{-1}\\ &= \langle\xi\rangle|\xi|^{-1}(\langle\xi\rangle\triangleright v).\end{aligned} \tag{5.25}$$

Thus

$$\begin{aligned}\beta &= \alpha\hat{\triangleleft}w(\langle\xi\rangle\triangleright v)^{-1}|\xi|\langle\xi\rangle^{-1}\\ &= \alpha\hat{\triangleleft}\langle\xi\rangle^{L-1}(\langle\xi\rangle\triangleleft v)^{-1}(\langle\xi\rangle\triangleright v)^{-1}|\xi|\langle\xi\rangle^{-1}\\ &= \alpha\hat{\triangleleft}\langle\xi\rangle v(\langle\xi\rangle v)^{-1}|\xi|\langle\xi\rangle^{-1} = \alpha\hat{\triangleleft}|\xi|\langle\xi\rangle^{-1}.\end{aligned} \tag{5.26}$$

Now, substituting these last equations in (5.21) gives

$$(\theta_V)^{*}(\alpha) = \sum_{\xi\in\text{ basis of }V,\ |\xi|\langle\xi\rangle^{-1}=\|\alpha\|}(\alpha\hat{\triangleleft}\|\alpha\|)(\xi)\cdot\hat{\xi}. \tag{5.27}$$

Take a basis $\xi_1,\xi_2,\dots,\xi_n$ with $(\alpha\hat{\triangleleft}\|\alpha\|)(\xi_i)$ being 1 if $i=1$, and 0 otherwise. Then

$$(\theta_V)^{*}(\alpha) = \hat{\xi}_1 + 0 = \alpha\hat{\triangleleft}\|\alpha\| = \theta_{V^*}(\alpha), \tag{5.28}$$

where $\hat{\xi}_1,\hat{\xi}_2,\dots,\hat{\xi}_n$ is the dual basis of V^* defined by $\hat{\xi}_i(\xi_j) = \delta_{i,j}$. □

Example 5.6. We return to the example of Section 3. First, we calculate the value of the ribbon map on the indecomposable objects. For an irreducible representation V, we have $\theta_V : V \to V$ defined by $\theta_V(\xi) = \xi\hat{\triangleleft}\|\xi\|$ for $\xi\in V$. At the base point $s\in\mathbb{O}$, we have $\theta_V(\xi) = \xi\bar{\triangleleft}s$ for $\xi\in V$ and $\theta : V_s \to V_s$ is a multiple Θ_V, say, of the identity or, more explicitly, $\operatorname{trace}(\theta : V_s \to V_s) = \Theta_V \dim_{\mathbb{C}}(V_s)$, that is,

$$\Theta_V = \frac{\text{group character }(s)}{\dim_{\mathbb{C}}(V_s)}. \tag{5.29}$$

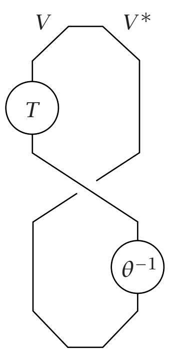
And then, for the different cases we will get Table 5.1.

TABLE 5.1

Irreps	Θ_V	Irreps	Θ_V
1_1	1	3_4	ω^2
1_2	1	3_5	ω^4
1_3	1	4_0	1
1_4	1	4_1	ω^1
1_5	1	4_2	ω^2
1_6	1	4_3	-1
2_1	1	4_4	ω^4
2_2	-1	4_5	ω^5
2_3	-1	5_{++}	1
2_4	1	5_{+-}	-1
2_5	-1	5_{-+}	1
2_6	1	5_{--}	-1
3_0	1	6_{++}	1
3_1	ω^2	6_{-+}	1
3_2	ω^4	6_{+-}	-1
3_3	1	6_{--}	-1

6. Traces in the category $\mathcal{D}$

DEFINITION 6.1 [8]**.** The trace of a morphism $T : V \to V$ for any object V in $\mathcal{D}$ is defined by the following diagram:



(6.1)

THEOREM 6.2. *If the diagram of Definition 6.1 is evaluated in $\mathcal{D}$, the following is found:*

$$\operatorname{trace}(T) = \sum_{\xi \in \text{ basis of } V} \hat{\xi}(T(\xi)). \tag{6.2}$$

PROOF. Begin with

$$\begin{aligned}\operatorname{coev}_V(1) &= \sum_{\xi \in \text{ basis of } V} \xi \hat{\triangleleft} \tilde{\tau}\left(\|\xi\|^L, \|\xi\|\right)^{-1} \otimes \hat{\xi} \\ &= \sum_{\xi \in \text{ basis of } V} \xi \hat{\triangleleft} \tau\left(\langle \xi \rangle^L, \langle \xi \rangle\right)^{-1} \otimes \hat{\xi},\end{aligned} \tag{6.3}$$

and applying $T\otimes\mathrm{id}$ to this gives

$$\sum_{\xi\in\text{ basis of }V} T\big(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\big)\otimes\hat{\xi}=\sum_{\xi\in\text{ basis of }V} T(\xi)\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi}. \tag{6.4}$$

Next, apply the braiding map to the last equation to get

$$\sum_{\xi\in\text{ basis of }V}\Psi\big(T(\xi)\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi}\big)=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}\big(\langle\xi'\rangle\triangleleft|\hat{\xi}|\big)^{-1}\otimes\xi'\hat{\triangleleft}|\hat{\xi}|, \tag{6.5}$$

where $\xi'=T(\xi)\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}$, so

$$\begin{aligned}\langle\xi'\rangle&=\big\langle T(\xi)\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\big\rangle=\big\langle T(\xi)\bar{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\big\rangle\\&=\big\langle T(\xi)\big\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}=\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}.\end{aligned} \tag{6.6}$$

To calculate $|\hat{\xi}|$, we start with

$$||\hat{\xi}||=\|\xi\|^L=\big(|\xi|^{-1}\langle\xi\rangle\big)^L=|\xi|\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\langle\xi\rangle^L, \tag{6.7}$$

which implies that $|\hat{\xi}|=\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}$. Then

$$\begin{aligned}\hat{\xi}\hat{\triangleleft}\big(\langle\xi'\rangle\triangleleft|\hat{\xi}|\big)^{-1}&=\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}\big)^{-1}\\&=\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1},\\ \xi'\triangleleft|\hat{\xi}|=\big(T(\xi)\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\big)&\hat{\triangleleft}\big(\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}\big)=T(\xi)\hat{\triangleleft}|\xi|^{-1},\end{aligned} \tag{6.8}$$

which gives

$$\begin{aligned}&\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}\big(\langle\xi'\rangle\triangleleft|\hat{\xi}|\big)^{-1}\otimes\xi'\hat{\triangleleft}|\hat{\xi}|\\&\quad=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}\otimes T(\xi)\hat{\triangleleft}|\xi|^{-1}.\end{aligned} \tag{6.9}$$

Next,

$$\begin{aligned}\theta^{-1}\big(T(\xi)\hat{\triangleleft}|\xi|^{-1}\big)&=\big(T(\xi)\hat{\triangleleft}|\xi|^{-1}\big)\hat{\triangleleft}||T(\xi)\hat{\triangleleft}|\xi|^{-1}||^{-1}\\&=\big(T(\xi)\hat{\triangleleft}|\xi|^{-1}\big)\hat{\triangleleft}\big(||T(\xi)||\tilde{\triangleleft}|\xi|^{-1}\big)^{-1}\\&=T(\xi)\hat{\triangleleft}|\xi|^{-1}\big(\|\xi\|\tilde{\triangleleft}|\xi|^{-1}\big)^{-1}\\&=T(\xi)\hat{\triangleleft}|\xi|^{-1}\big(|\xi||\xi|^{-1}\langle\xi\rangle|\xi|^{-1}\big)^{-1}\\&=T(\xi)\hat{\triangleleft}|\xi|^{-1}|\xi|\langle\xi\rangle^{-1}=T(\xi)\hat{\triangleleft}\langle\xi\rangle^{-1},\end{aligned} \tag{6.10}$$

and finally we need to calculate

$$\mathrm{eval}\big(\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}\otimes T(\xi)\hat{\triangleleft}\langle\xi\rangle^{-1}\big)=\big(\hat{\xi}\hat{\triangleleft}\big(\langle\xi\rangle\triangleleft|\xi|^{-1}\big)^{-1}\big)\big(T(\xi)\hat{\triangleleft}\langle\xi\rangle^{-1}\big). \tag{6.11}$$

We know from the definition of the action on V^* that

$$\big(\hat{\xi}\hat{\triangleleft}(||T(\xi)||\tilde{\triangleright}x)\big)\big(T(\xi)\hat{\triangleleft}x\big)=\hat{\xi}\big(T(\xi)\big). \tag{6.12}$$

If we put $x = \langle\xi\rangle^{-1}$, we want to show that $\|T(\xi)\|\tilde{\triangleright} x = (\langle\xi\rangle \triangleleft |\xi|^{-1})^{-1}$, so

$$\|\xi\|\tilde{\triangleleft} x = |\xi|^{-1}\langle\xi\rangle\tilde{\triangleleft}\langle\xi\rangle^{-1} = \langle\xi\rangle|\xi|^{-1} = (\langle\xi\rangle\triangleright|\xi|^{-1})(\langle\xi\rangle\triangleleft|\xi|^{-1}) = v't', \tag{6.13}$$

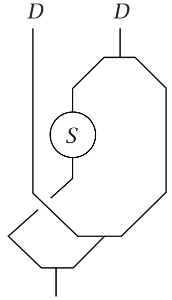
which implies that $t' = \langle\xi\rangle \triangleleft |\xi|^{-1}$, and hence

$$\begin{aligned} \|T(\xi)\|\tilde{\triangleright} x &= \|\xi\|\tilde{\triangleright} x = |\xi|^{-1}\langle\xi\rangle\tilde{\triangleright}\langle\xi\rangle^{-1} = t\langle\xi\rangle^{-1}t'^{-1} \\ &= \langle\xi\rangle\langle\xi\rangle^{-1}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1} = (\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}. \end{aligned} \tag{6.14}$$

□

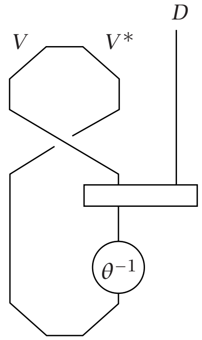
7. Characters in the category $\mathfrak{D}$

Definition 7.1 [6]**.** The right adjoint action in $\mathfrak{D}$ of the algebra D on itself is defined by the following diagram:



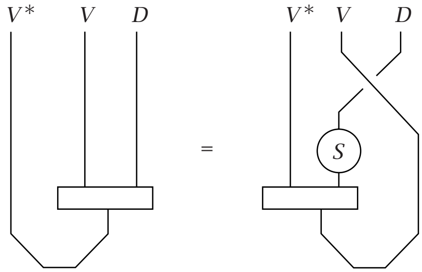
(7.1)

Definition 7.2. The character χ_V of an object V in $\mathfrak{D}$ is defined by the following diagram:



(7.2)

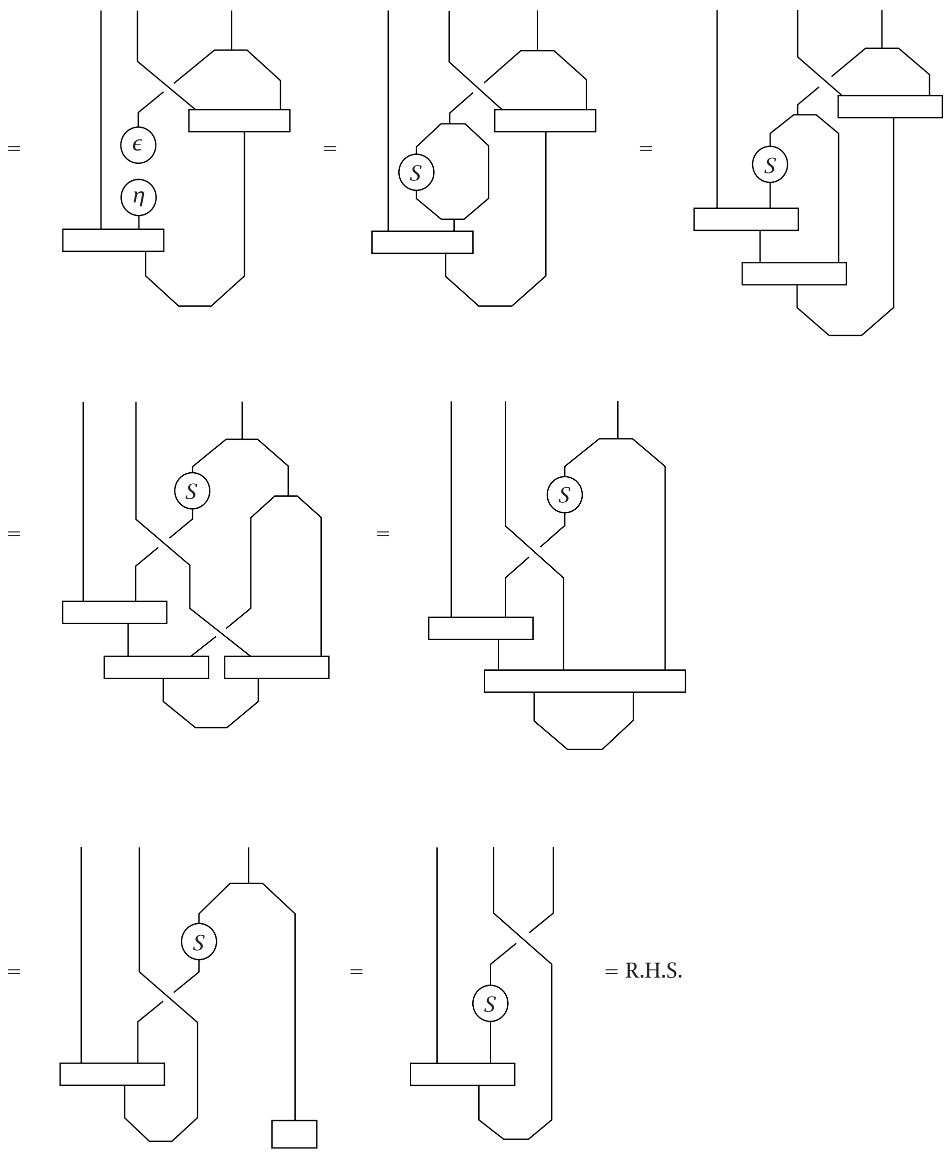
Lemma 7.3. *For an object V in $\mathfrak{D}$, the following holds:*



(7.3)

Proof.

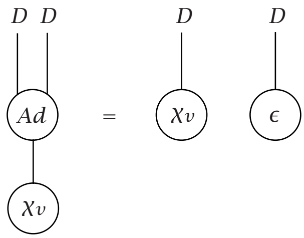
L.H.S.



(7.4)

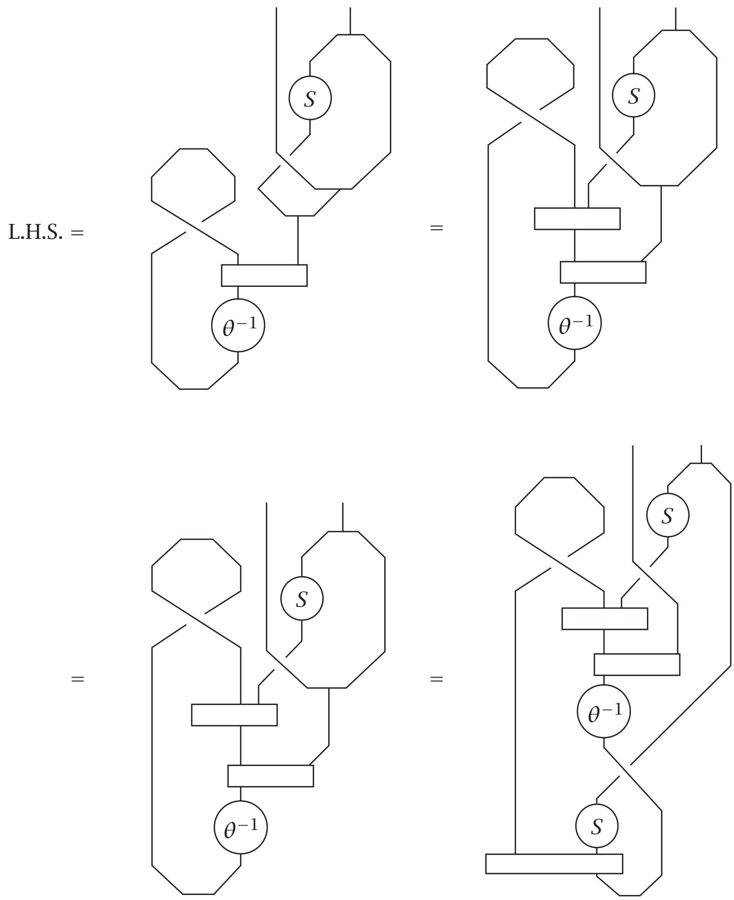
□

Proposition 7.4. *The character is right adjoint invariant, that is, for an object V in $\mathfrak{D}$, the following holds:*

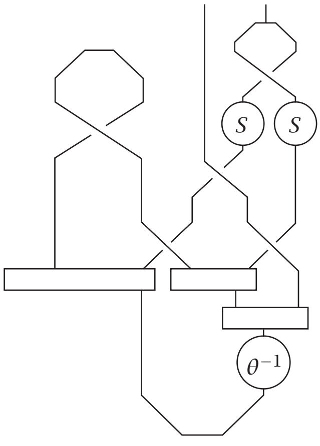


(7.5)

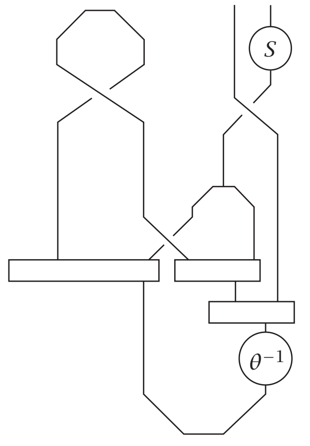
Proof.



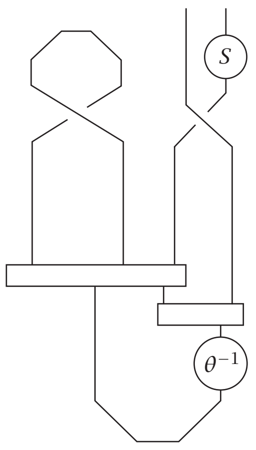
=



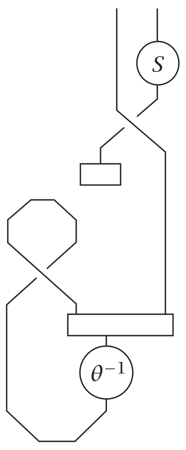
=



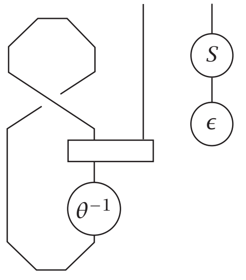
=



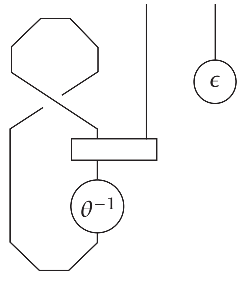
=



=



=

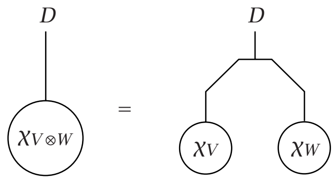


= R.H.S.

(7.6)

□

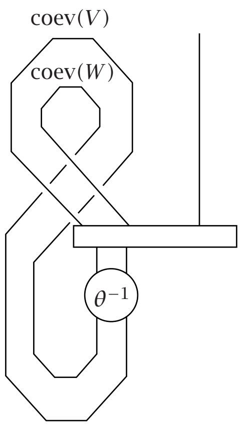
Proposition 7.5. *The character of a tensor product of representations is the product of the characters, that is, for two objects V and W in $\mathcal{D}$, the following holds:*



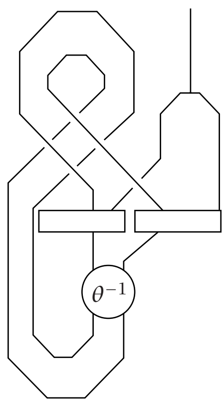
(7.7)

Proof.

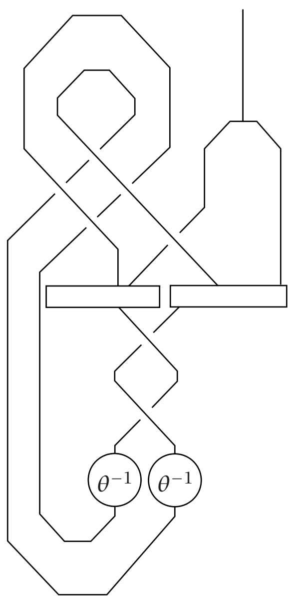
L.H.S. =



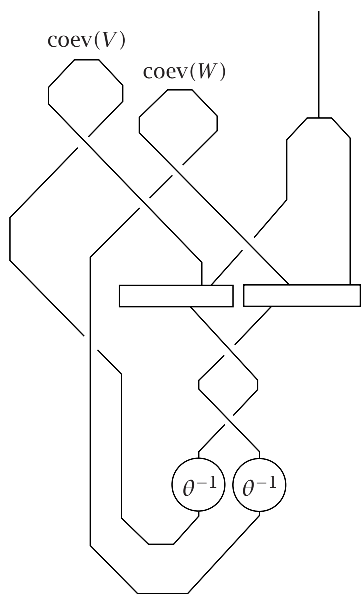
=

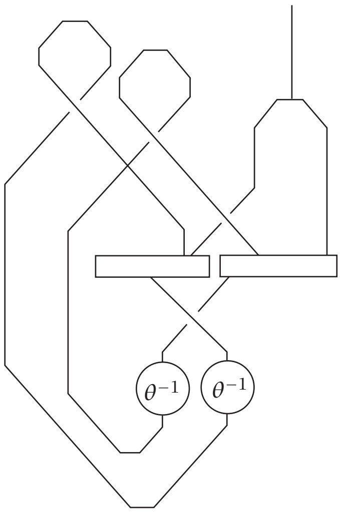


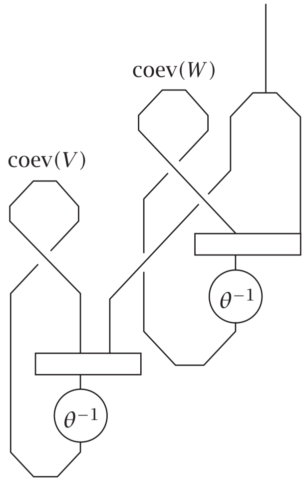
=

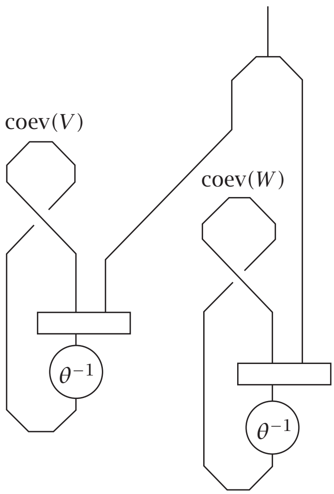


=



= 

= 

= 

= R.H.S.

(7.8)

□

Theorem 7.6. *The following formula holds for the character:*

$$\chi_V(\delta_y \otimes x) = \sum_{\xi \in \text{ basis of } V,\ y = \langle \xi \rangle |\xi|^{-1}} \hat{\xi}(\xi \hat{\triangleleft} \langle \xi \rangle^{-1} x \langle \xi \rangle), \tag{7.9}$$

for $xy = yx$, otherwise $\chi_V(\delta_y \otimes x) = 0$.

Proof. Set $a=\delta_y\otimes x$. To have $\chi_V(a)\neq 0$, we must have $\|a\|=e$, that is, $y=y\tilde{\triangleleft}x$, which implies that x and y commute. Assuming this, we continue with the diagrammatic definition of the character, starting with

$$\left(\sum_{\xi\in\text{ basis of }V}\xi\hat{\triangleleft}\tilde{\tau}(\|\xi\|^L,\|\xi\|)^{-1}\otimes\hat{\xi}\right)\otimes a=\sum_{\xi\in\text{ basis of }V}(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi})\otimes a. \tag{7.10}$$

Next, we calculate

$$\Psi(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi})=\hat{\xi}\hat{\triangleleft}(\langle\xi'\rangle\triangleleft|\hat{\xi}|)^{-1}\otimes\xi'\hat{\triangleleft}|\hat{\xi}|, \tag{7.11}$$

where $\xi'=\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}$, so

$$\langle\xi'\rangle=\langle\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\rangle=\langle\xi\bar{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\rangle=\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}. \tag{7.12}$$

From a previous calculation, we know that $|\hat{\xi}|=\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1}$, so

$$\begin{aligned}\hat{\xi}\hat{\triangleleft}(\langle\xi'\rangle\triangleleft|\hat{\xi}|)^{-1}&=\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1})^{-1}\\&=\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1},\\ \xi'\triangleleft|\hat{\xi}|&=(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1})\hat{\triangleleft}(\tau(\langle\xi\rangle^L,\langle\xi\rangle)|\xi|^{-1})=\xi\hat{\triangleleft}|\xi|^{-1},\end{aligned} \tag{7.13}$$

which gives the next stage in the evaluation of the diagram:

$$\begin{aligned}&\sum_{\xi\in\text{ basis of }V}\Psi(\xi\hat{\triangleleft}\tau(\langle\xi\rangle^L,\langle\xi\rangle)^{-1}\otimes\hat{\xi})\otimes a\\&\quad=\sum_{\xi\in\text{ basis of }V}(\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes\xi\hat{\triangleleft}|\xi|^{-1})\otimes a.\end{aligned} \tag{7.14}$$

Now we apply the associator to the last equation to get

$$\begin{aligned}&\sum_{\xi\in\text{ basis of }V}\Phi((\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes\xi\hat{\triangleleft}|\xi|^{-1})\otimes a)\\&\quad=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\tilde{\tau}(\|\xi\hat{\triangleleft}|\xi|^{-1}\|^L,\|a\|)\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\otimes a)\\&\quad=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\tau(\langle\xi\hat{\triangleleft}|\xi|^{-1}\rangle,e)\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\otimes a)\\&\quad=\sum_{\xi\in\text{ basis of }V}\hat{\xi}\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes(\xi\hat{\triangleleft}|\xi|^{-1}\otimes(\delta_y\otimes x))\end{aligned} \tag{7.15}$$

as $\tau(\langle\xi\hat{\triangleleft}|\xi|^{-1}\rangle,e)=e$. Now apply the action $\hat{\triangleleft}$ to $\xi\hat{\triangleleft}|\xi|^{-1}\otimes(\delta_y\otimes x)$ to get

$$(\xi\hat{\triangleleft}|\xi|^{-1})\hat{\triangleleft}(\delta_y\otimes x)=\delta_{y,\|\xi\hat{\triangleleft}|\xi|^{-1}\|}(\xi\hat{\triangleleft}|\xi|^{-1})\hat{\triangleleft}x=\delta_{y,\|\xi\|\tilde{\triangleleft}|\xi|^{-1}}\xi\hat{\triangleleft}|\xi|^{-1}x, \tag{7.16}$$

and to get a nonzero answer, we must have

$$y=\|\xi\|\tilde{\triangleleft}|\xi|^{-1}=|\xi|^{-1}\langle\xi\rangle\tilde{\triangleleft}|\xi|^{-1}=|\xi||\xi|^{-1}\langle\xi\rangle|\xi|^{-1}=\langle\xi\rangle|\xi|^{-1}. \tag{7.17}$$

Thus the character of V is given by

$$\chi_V(\delta_y\otimes x)=\sum_{\xi\in\text{ basis of }V,\ y=\langle\xi\rangle|\xi|^{-1}}\text{eval}\left(\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\otimes\theta^{-1}\left(\xi\,\hat{\triangleleft}|\xi|^{-1}x\right)\right).\qquad(7.18)$$

Next,

$$\begin{aligned}\theta^{-1}\left(\xi\,\hat{\triangleleft}|\xi|^{-1}x\right)&=\left(\xi\,\hat{\triangleleft}|\xi|^{-1}x\right)\hat{\triangleleft}\left\|\xi\,\hat{\triangleleft}|\xi|^{-1}x\right\|^{-1}\\&=\left(\xi\,\hat{\triangleleft}|\xi|^{-1}x\right)\hat{\triangleleft}\left(\|\xi\|\,\tilde{\triangleleft}|\xi|^{-1}x\right)^{-1}\\&=\left(\xi\,\hat{\triangleleft}|\xi|^{-1}x\right)\hat{\triangleleft}\left(x^{-1}|\xi||\xi|^{-1}\langle\xi\rangle|\xi|^{-1}x\right)^{-1}\\&=\xi\,\hat{\triangleleft}|\xi|^{-1}xx^{-1}|\xi|\langle\xi\rangle^{-1}x=\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x.\end{aligned}\qquad(7.19)$$

Now we need to calculate $\text{eval}(\hat{\xi}\,\hat{\triangleleft}(\langle\xi\rangle\triangleleft|\xi|^{-1})^{-1}\otimes\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x)$. Start with $\|\xi\|\,\tilde{\triangleleft}\langle\xi\rangle^{-1}x=\langle\xi\rangle|\xi|^{-1}\,\tilde{\triangleleft}x=\langle\xi\rangle|\xi|^{-1}$, as we only have nonzero summands for $y=\langle\xi\rangle|\xi|^{-1}$. Then

$$\begin{aligned}&\text{eval}\left(\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\otimes\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x\right)\\&\quad=\text{eval}\left(\left(\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\otimes\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x\right)\hat{\triangleleft}\langle\xi\rangle\right)\\&\quad=\text{eval}\left(\hat{\xi}\,\hat{\triangleleft}\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)^{-1}\left(\langle\xi\rangle|\xi|^{-1}\tilde{\triangleright}\langle\xi\rangle\right)\otimes\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x\langle\xi\rangle\right).\end{aligned}\qquad(7.20)$$

To find $\langle\xi\rangle|\xi|^{-1}\tilde{\triangleright}\langle\xi\rangle$, first find $\langle\xi\rangle|\xi|^{-1}\,\tilde{\triangleleft}\langle\xi\rangle=|\xi|^{-1}\langle\xi\rangle$, so

$$\begin{aligned}\langle\xi\rangle|\xi|^{-1}\tilde{\triangleright}\langle\xi\rangle&=\left(\langle\xi\rangle\triangleright|\xi|^{-1}\right)\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)\tilde{\triangleright}\langle\xi\rangle\\&=\left(\langle\xi\rangle\triangleleft|\xi|^{-1}\right)\langle\xi\rangle\langle\xi\rangle^{-1}=\langle\xi\rangle\triangleleft|\xi|^{-1}.\end{aligned}\qquad(7.21)$$

□

Lemma 7.7. *Let V be an object in $\mathfrak{D}$. For $\delta_y\otimes x\in D$, the character of V is given by the following formula, where $y=su^{-1}$ with $s\in M$ and $u\in G$:*

$$\chi_V(\delta_y\otimes x)=\sum_{\xi\in\text{ basis of }V_{u^{-1}s}}\hat{\xi}\left(\xi\,\hat{\triangleleft}s^{-1}xs\right)=\chi_{V_{u^{-1}s}}\left(s^{-1}xs\right),\qquad(7.22)$$

where $xy=yx$, otherwise $\chi_V(\delta_y\otimes x)=0$. Here, $\chi_{V_{u^{-1}s}}$ is the group representation character of the representation $V_{u^{-1}s}$ of the group $\text{stab}(u^{-1}s)$.

Proof. From Theorem 7.6, we know that

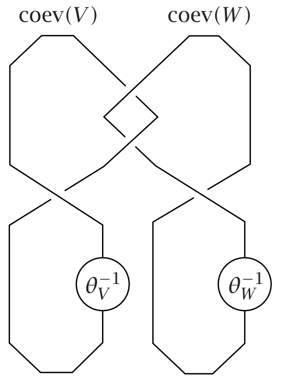
$$\chi_V(\delta_y\otimes x)=\sum_{\xi\in\text{ basis of }V,\ y=\langle\xi\rangle|\xi|^{-1}}\hat{\xi}\left(\xi\,\hat{\triangleleft}\langle\xi\rangle^{-1}x\langle\xi\rangle\right),\qquad(7.23)$$

for $xy=yx$. Set $s=\langle\xi\rangle$ and $u=|\xi|$, so $y=su^{-1}$. We note that $s^{-1}xs$ is in $\text{stab}(u^{-1}s)$, because

$$u^{-1}s\,\tilde{\triangleleft}s^{-1}xs=s^{-1}x^{-1}su^{-1}ss^{-1}xs=s^{-1}x^{-1}xsu^{-1}s=u^{-1}s.\qquad(7.24)$$

It just remains to note that $\|\xi\|=|\xi|^{-1}\langle\xi\rangle=u^{-1}s$. □

8. Modular categories. Let $\mathcal{M}$ be a semisimple ribbon category. For objects V and W in $\mathcal{M}$, define $\tilde{S}_{VW} \in \underline{\mathbf{1}}$ as follows:

$\tilde{S}_{VW} =$  (8.1)

There are standard results [1, 8]:

$$\tilde{S}_{VW} = \tilde{S}_{WV} = \tilde{S}_{V*W*} = \tilde{S}_{W*V*}, \qquad \tilde{S}_{V\underline{1}} = \dim(V). \tag{8.2}$$

Here, $\dim(V)$ is the trace in $\mathcal{M}$ of the identity map on V.

Definition 8.1. Call an object U in an abelian category $\mathcal{M}$ simple if, for any V in $\mathcal{M}$, any injection $V \hookrightarrow U$ is either 0 or an isomorphism [1]. A semisimple category is an abelian category whose objects split as direct sums of simple objects [8].

Definition 8.2 [1]**.** A modular category is a semisimple ribbon category $\mathcal{M}$ satisfying the following properties:

(1) there are only a finite number of isomorphism classes of simple objects in $\mathcal{M}$,
(2) Schur's lemma holds, that is, the morphisms between simple objects are zero unless they are isomorphic, in which case the morphisms are a multiple of the identity,
(3) the matrix $\tilde{S}_{VW}$ with indices in isomorphism classes of simple objects is invertible.

Definition 8.3 [1]**.** For a simple object V, the ribbon map on V is a multiple of the identity, and Θ_V is used for the scalar multiple. The numbers $P^{\pm}$ are defined as the following sums over simple isomorphism classes:

$$P^{\pm} = \sum_V \Theta_V{}^{\pm 1}\left(\dim(V)\right)^2, \tag{8.3}$$

and the matrices T and C are defined using the Kronecker delta function by

$$T_{VW} = \delta_{VW}\Theta_V, \qquad C_{VW} = \delta_{VW*}. \tag{8.4}$$

Theorem 8.4 [1]. *In a modular category, if the matrix S is defined by*

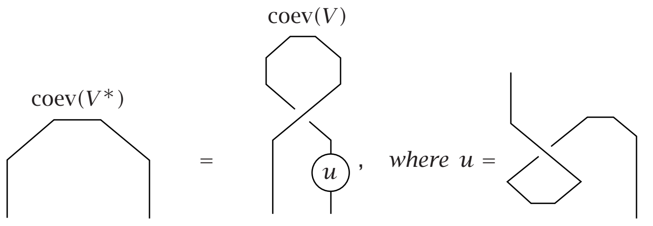
$$S = \frac{\tilde{S}}{\sqrt{P^+P^-}}, \tag{8.5}$$

then the following matrix equations hold:

$$(ST)^3 = \sqrt{\frac{P^+}{P^-}}S^2, \qquad S^2 = C, \qquad CT = TC, \qquad C^2 = 1. \tag{8.6}$$

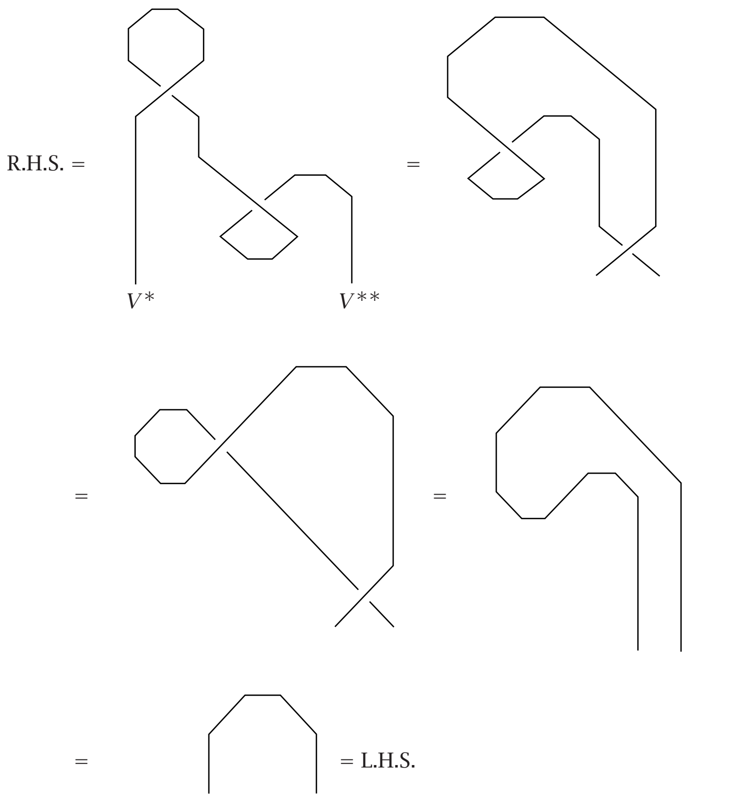
We now give some results which allow us to calculate the matrix $\tilde{S}$ in $\mathcal{D}$.

Lemma 8.5.



(8.7)

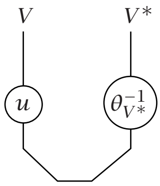
Proof.



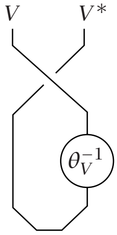
(8.8)

□

Lemma 8.6.

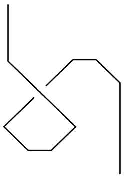


=

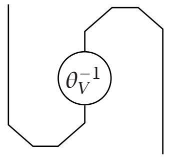


(8.9)

where $u =$

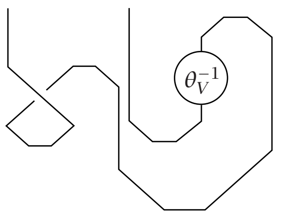


and $\theta_{V^*}^{-1} =$

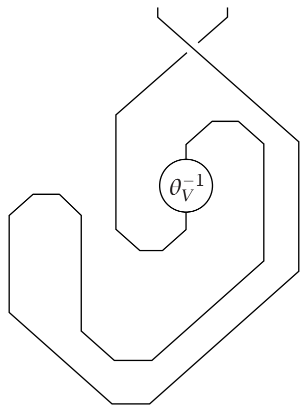


Proof.

L.H.S. =

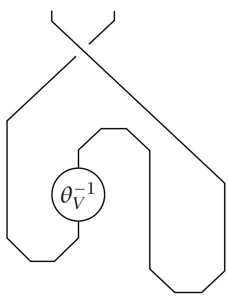


=

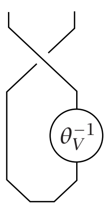


(8.10)

=



=

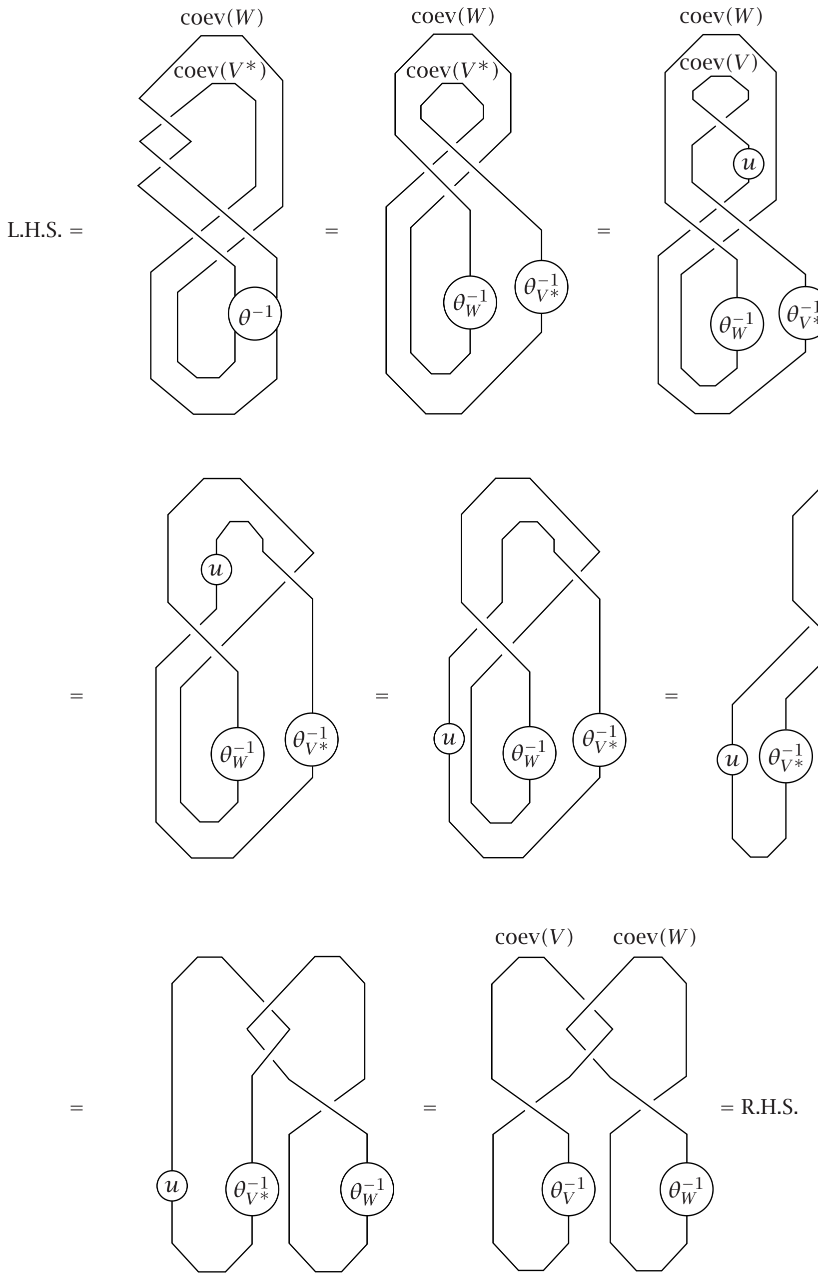


= R.H.S.

□

Lemma 8.7. *For V, W indecomposable objects in $\mathcal{D}$,* $\operatorname{trace}(\Psi_{V^*W} \circ \Psi_{WV^*}) = \tilde{S}_{VW}$.

Proof.


L.H.S. =
coev(W)
coev(V*)
θ^{-1}
=
coev(W)
coev(V*)
θ_W^{-1}
θ_{V*}^{-1}
=
coev(W)
coev(V)
u
θ_W^{-1}
θ_{V*}^{-1}
=
u
θ_W^{-1}
θ_{V*}^{-1}
=
u
θ_W^{-1}
θ_{V*}^{-1}
=
u
θ_{V*}^{-1}
θ_W^{-1}
=
u
θ_{V*}^{-1}
θ_W^{-1}
=
coev(V)
coev(W)
θ_V^{-1}
θ_W^{-1}
= R.H.S.

(8.11)

□

Lemma 8.8. *For two objects V and W in $\mathcal{D}$,*

$$\operatorname{trace}(\Psi_{W\otimes V}\circ\Psi_{V\otimes W}) = \sum_{\substack{\xi\otimes\eta\,\in\,\text{basis of } V\otimes W\\ |\xi|^{-1}\langle\xi\rangle \text{ commutes with } |\eta|\langle\eta\rangle^{-1}}} \hat{\eta}(\eta\hat{\triangleleft}|\eta|^{-1}\langle\xi\rangle^{-1}|\xi||\eta|)\hat{\xi}(\xi\hat{\triangleleft}|\eta|\langle\eta\rangle^{-1}). \tag{8.12}$$

Proof. From Theorem 6.2, we know that

$$\operatorname{trace}(\Psi_{W\otimes V}\circ\Psi_{V\otimes W}) = \sum_{(\xi\otimes\eta)\,\in\,\text{basis of } V\otimes W} (\widehat{\xi\otimes\eta})(\Psi^2(\xi\otimes\eta)). \tag{8.13}$$

From the definition of the ribbon map, we know that $\Psi(\Psi(\xi\otimes\eta))\hat{\triangleleft}\|\xi\otimes\eta\| = \xi\hat{\triangleleft}\|\xi\|\otimes\eta\hat{\triangleleft}\|\eta\|$, so

$$\begin{aligned}\Psi(\Psi(\xi\otimes\eta)) &= (\xi\hat{\triangleleft}\|\xi\|\otimes\eta\hat{\triangleleft}\|\eta\|)\hat{\triangleleft}\|\xi\otimes\eta\|^{-1}\\ &= (\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\eta\rangle)\hat{\triangleleft}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\ &= (\xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle)\hat{\triangleleft}(||\eta\hat{\triangleleft}\|\eta\|||\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|)\\ &\quad\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\ &= \xi\hat{\triangleleft}|\xi|^{-1}\langle\xi\rangle(\|\eta\|\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|)\otimes\eta\hat{\triangleleft}|\eta|^{-1}\langle\xi\rangle^{-1}|\xi||\eta|.\end{aligned} \tag{8.14}$$

Put $\Psi(\Psi(\xi\otimes\eta)) = \xi'\otimes\eta'$ and $\widehat{\xi\otimes\eta} = \alpha\otimes\beta$, and then from Lemma 4.1 we get

$$(\widehat{\xi\otimes\eta})(\xi'\otimes\eta') = (\alpha\bar{\triangleleft}\tau(\langle\beta\rangle,\langle\xi'\rangle\cdot\langle\eta'\rangle))(\eta')(\beta\bar{\triangleleft}\tau(\langle\xi'\rangle,\langle\eta'\rangle)^{-1})(\xi'). \tag{8.15}$$

As $\widehat{\xi\otimes\eta}$ is part of a dual basis, the last expression can only be nonzero if $\|\xi'\| = \|\xi\|$ and $\|\eta'\| = \|\eta\|$. A simple calculation shows that $\|\eta'\| = \|\eta\|$ if and only if $|\xi|^{-1}\langle\xi\rangle$ commutes with $|\eta|\langle\eta\rangle^{-1}$. We use this to find

$$\begin{aligned}\|\eta\|\tilde{\triangleleft}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta| &= |\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\eta\rangle|\eta|^{-1}\langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\\ &= |\eta|^{-1}\langle\eta\rangle|\eta|^{-1}|\xi|^{-1}\langle\xi\rangle\langle\xi\rangle^{-1}|\xi||\eta| = |\eta|^{-1}\langle\eta\rangle,\end{aligned} \tag{8.16}$$

and then

$$\|\eta\|\tilde{\triangleright}\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta| = \langle\eta\rangle\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}|\xi||\eta|\langle\eta\rangle^{-1} = \langle\xi\rangle^{-1}|\xi||\eta|\langle\eta\rangle^{-1}. \tag{8.17}$$

Now, using the formula for $\widehat{\xi\otimes\eta} = \alpha\otimes\beta$ from Lemma 4.1 gives the result. □

Lemma 8.9. *Let V and W be objects in $\mathcal{D}$. Then in terms of group characters,*

$$\operatorname{trace}\left(\Psi^2_{V\otimes W}\right) = \sum_{\substack{u,v\in G,\, s,t\in M \\ su \text{ commutes with } vt}} \chi_{W_{us}}\left(s^{-1}t^{-1}v^{-1}s\right)\chi_{V_{vt}}\left(u^{-1}s^{-1}\right). \tag{8.18}$$

Proof. This is more or less immediate from Lemma 8.8. Put $\|\eta\| = u^{-1}s$ and $\|\xi\| = v^{-1}t$ and sum over basis elements of constant degree first. □

9. An example of a modular category. Using the order of the indecomposable objects in Table 5.1, we get T to be a diagonal 32×32 matrix whose diagonal entries are taken from the table. As every indecomposable object in our example is self-dual, the matrix C is the 32×32 identity matrix.

To find S, we calculate the trace of the double braiding trace$(\Psi_{VW}\circ\Psi_{WV})$. We do this using the result from Lemma 8.8, split into different cases for the objects V and W, and move the points the characters are evaluated at to the base points for each orbit using Lemma 2.3. The following examples are given.

(I) Case (1) ⊗ Case (1) (i.e., the orbit of W is $\{e\}$ and the orbit of V is $\{e\}$):

$$\operatorname{trace}\left(\Psi^2\right) = \chi_{W_e}(e)\chi_{V_e}(e). \tag{9.1}$$

(II) Case (2) ⊗ Case (5) (i.e., the orbit of W is $\{a^3\}$ and the orbit of V is $\{b, ba^2, ba^4\}$):

$$\operatorname{trace}\left(\Psi^2\right) = \left(\chi_{W_{a^3}}\left(ba^2\right) + \chi_{W_{a^3}}\left(ba^4\right) + \chi_{W_{a^3}}(b)\right)\chi_{V_b}\left(a^3\right). \tag{9.2}$$

(III) Case (5) ⊗ Case (3) (i.e., the orbit of W is $\{b, ba^2, ba^4\}$ and the orbit of V is $\{a^2, a^4\}$):

$$\operatorname{trace}\left(\Psi^2\right) = 0. \tag{9.3}$$

(IV) Case (6) ⊗ Case (5) (i.e., the orbit of W is $\{ba, ba^3, ba^5\}$ and the orbit of V is $\{b, ba^2, ba^4\}$):

$$\operatorname{trace}\left(\Psi^2\right) = 3\left(\chi_{W_{ba}}\left(ba^4\right)\chi_{V_b}\left(ba^3\right)\right). \tag{9.4}$$

Noting that the dimension in D of each V is the same as its usual dimension, we get $P^+ = P^- = 12$.

From these cases, we get S to be one twelfth of the following 32×32 symmetric matrix:

$$\tilde{S}=\begin{pmatrix}
1&1&1&1&2&2&1&1&1&1&2&2&2&2&2&2&2&2&2&2&2&2&2&2&3&3&3&3&3&3&3&3\\
1&1&1&1&2&2&-1&-1&-1&-1&-2&-2&2&2&2&2&2&2&-2&-2&-2&-2&-2&-2&-3&-3&-3&-3&3&3&3&3\\
1&1&1&1&2&2&-1&-1&-1&-1&-2&-2&2&2&2&2&2&2&-2&-2&-2&-2&-2&-2&3&3&3&3&-3&-3&-3&-3\\
1&1&1&1&2&2&1&1&1&1&2&2&2&2&2&2&2&2&2&2&2&2&2&2&-3&-3&-3&-3&-3&-3&-3&-3\\
2&2&2&2&4&4&-2&-2&-2&-2&-4&-4&-2&-2&-2&-2&-2&-2&2&2&2&2&2&2&0&0&0&0&0&0&0&0\\
2&2&2&2&4&4&2&2&2&2&4&4&-2&-2&-2&-2&-2&-2&-2&-2&-2&-2&-2&-2&0&0&0&0&0&0&0&0\\
1&-1&-1&1&-2&2&1&-1&-1&1&-2&2&2&-2&2&-2&2&-2&2&-2&2&-2&2&-2&3&3&-3&-3&3&-3&3&-3\\
1&-1&-1&1&-2&2&-1&1&1&-1&2&-2&2&-2&2&-2&2&-2&-2&2&-2&2&-2&2&-3&-3&3&3&3&-3&3&-3\\
1&-1&-1&1&-2&2&-1&1&1&-1&2&-2&2&-2&2&-2&2&-2&-2&2&-2&2&-2&2&3&3&-3&-3&-3&3&-3&3\\
1&-1&-1&1&-2&2&1&-1&-1&1&-2&2&2&-2&2&-2&2&-2&2&-2&2&-2&2&-2&-3&-3&3&3&-3&3&-3&3\\
2&-2&-2&2&-4&4&-2&2&2&-2&4&-4&-2&2&-2&2&-2&2&2&-2&2&-2&2&-2&0&0&0&0&0&0&0&0\\
2&-2&-2&2&-4&4&2&-2&-2&2&-4&4&-2&2&-2&2&-2&2&-2&2&-2&2&-2&2&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&2&2&2&2&-2&-2&4&-2&-2&4&-2&-2&4&-2&-2&4&-2&-2&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&-2&-2&-2&-2&2&2&-2&-2&4&-2&-2&4&2&-4&2&2&-4&2&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&2&2&2&2&-2&-2&-2&4&-2&-2&4&-2&-2&-2&4&-2&-2&4&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&-2&-2&-2&-2&2&2&4&-2&-2&4&-2&-2&-4&2&2&-4&2&2&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&2&2&2&2&-2&-2&-2&-2&4&-2&-2&4&-2&4&-2&-2&4&-2&0&0&0&0&0&0&0&0\\
2&2&2&2&-2&-2&-2&-2&-2&-2&2&2&-2&4&-2&-2&4&-2&2&2&-4&2&2&-4&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&2&-2&-2&2&2&-2&4&2&-2&-4&-2&2&4&2&-2&-4&-2&2&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&-2&2&2&-2&-2&2&-2&-4&-2&2&4&2&2&-2&-4&-2&2&4&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&2&-2&-2&2&2&-2&-2&2&4&2&-2&-4&-2&-4&-2&2&4&2&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&-2&2&2&-2&-2&2&4&2&-2&-4&-2&2&-4&-2&2&4&2&-2&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&2&-2&-2&2&2&-2&-2&-4&-2&2&4&2&-2&2&4&2&-2&-4&0&0&0&0&0&0&0&0\\
2&-2&-2&2&2&-2&-2&2&2&-2&-2&2&-2&2&4&2&-2&-4&2&4&2&-2&-4&-2&0&0&0&0&0&0&0&0\\
3&-3&3&-3&0&0&3&-3&3&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&-3&3&-3&3&-3&-3&3\\
3&-3&3&-3&0&0&3&-3&3&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-3&3&-3&3&-3&3&3&-3\\
3&-3&3&-3&0&0&-3&3&-3&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&-3&3&-3&-3&3&3&-3\\
3&-3&3&-3&0&0&-3&3&-3&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-3&3&-3&3&3&-3&-3&3\\
3&3&-3&-3&0&0&3&3&-3&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&-3&-3&3&3&3&-3&-3\\
3&3&-3&-3&0&0&-3&-3&3&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-3&3&3&-3&3&3&-3&-3\\
3&3&-3&-3&0&0&3&3&-3&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-3&3&3&-3&-3&-3&3&3\\
3&3&-3&-3&0&0&-3&-3&3&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&-3&-3&3&-3&-3&3&3
\end{pmatrix}. \tag{9.5}$$

Now it is possible to check that the matrices S, T, and C satisfy the following relations:

$$S^2=(ST)^3,\qquad CS=SC,\qquad CT=TC. \tag{9.6}$$

10. An equivalence of tensor categories. In this section, we will generalize some results of [3] which considered group double cross products, that is, a group X factoring into two subgroups G and M.

Definition 10.1 [3]**.** For the double cross product group $X = GM$, there is a quantum double $D(X) = k(X) \rtimes kX$ which has the following operations:

$$(\delta_y \otimes x)(\delta_{y'} \otimes x') = \delta_{x^{-1}yx,y'}(\delta_y \otimes xx'), \qquad \Delta(\delta_y \otimes x) = \sum_{ab=y} \delta_a \otimes x \otimes \delta_b \otimes x,$$
$$1 = \sum_y \delta_y \otimes e, \qquad \epsilon(\delta_y \otimes x) = \delta_{y,e}, \qquad S(\delta_y \otimes x) = \delta_{x^{-1}y^{-1}x} \otimes x^{-1}, \tag{10.1}$$
$$(\delta_y \otimes x)^* = \delta_{x^{-1}yx} \otimes x^{-1}, \qquad R = \sum_{x,z} \delta_x \otimes e \otimes \delta_z \otimes x.$$

The representations of $D(X)$ are given by X-graded left kX-modules. The kX-action will be denoted by $\dot{\triangleright}$ and the grading by $|||\cdot|||$. The grading and X-action are related by

$$|||x \dot{\triangleright} \xi||| = x|||\xi|||x^{-1}, \quad x \in X,\ \xi \in V, \tag{10.2}$$

and the action of $(\delta_y \otimes x) \in D(X)$ is given by

$$(\delta_y \otimes x)\dot{\triangleright}\xi = \delta_{y,|||x\dot{\triangleright}\xi|||}\, x \dot{\triangleright} \xi. \tag{10.3}$$

Proposition 10.2. *There is a functor χ from $\mathcal{D}$ to the category of representations of $D(X)$ given by the following: as vector spaces, $\chi(V)$ is the same as V, and χ is the identity map. The X-grading $|||\cdot|||$ on $\chi(V)$ and the action of $us \in kX$ are defined by*

$$\begin{gathered} |||\chi(\eta)||| = \langle \eta \rangle^{-1}|\eta| \quad \text{for } \eta \in V, \\ us \dot{\triangleright} \chi(\eta) = \chi\big(((s \triangleleft |\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft} u^{-1}\big), \quad s \in M,\ u \in G. \end{gathered} \tag{10.4}$$

A morphism $\phi : V \to W$ in $\mathcal{D}$ is sent to the morphism $\chi(\phi) : \chi(V) \to \chi(W)$ defined by $\chi(\phi)(\chi(\xi)) = \chi(\phi(\xi))$.

Proof. First, we show that $\dot{\triangleright}$ is an action, that is, $vt\dot{\triangleright}(us\dot{\triangleright}\chi(\eta)) = vtus\dot{\triangleright}\chi(\eta)$ for all $s,t \in M$ and $u,v \in G$. Note that

$$\begin{aligned} vt\dot{\triangleright}(us\dot{\triangleright}\chi(\eta)) &= vt\dot{\triangleright}\chi\big(((s \triangleleft |\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft} u^{-1}\big) \\ &= \chi\big(((t \triangleleft |\bar{\eta}|^{-1})\bar{\triangleright}\bar{\eta})\bar{\triangleleft} v^{-1}\big), \end{aligned} \tag{10.5}$$

where $\bar{\eta} = ((s \triangleleft |\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft} u^{-1}$. On the other hand, we have

$$vtus = v(t \triangleright u)\tau(t \triangleleft u, s)\big((t \triangleleft u)\cdot s\big), \tag{10.6}$$

where $v(t \triangleright u)\tau(t \triangleleft u, s) \in G$ and $(t \triangleleft u)\cdot s \in M$, so

$$vtus\dot{\triangleright}\chi(\eta) = \chi\big(((((t \triangleleft u)\cdot s) \triangleleft |\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}\tau(t \triangleleft u, s)^{-1}(t \triangleright u)^{-1}v^{-1}\big). \tag{10.7}$$

We need to show that

$$\begin{aligned} (t \triangleleft |\bar{\eta}|^{-1})\bar{\triangleright}\bar{\eta} &= \big((((t \triangleleft u)\cdot s) \triangleleft |\eta|^{-1})\bar{\triangleright}\eta\big)\bar{\triangleleft}\tau(t \triangleleft u, s)^{-1}(t \triangleright u)^{-1} \\ &= \big(((t \triangleleft u(s \triangleright |\eta|^{-1}))\cdot(s \triangleleft |\eta|^{-1}))\bar{\triangleright}\eta\big)\bar{\triangleleft}\tau(t \triangleleft u, s)^{-1}(t \triangleright u)^{-1}. \end{aligned} \tag{10.8}$$

Put $\bar{s} = s \triangleleft |\eta|^{-1}$ and $\eta' = \bar{s} \bar{\triangleright} \eta$ which give $\bar{\eta} = \eta' \bar{\triangleleft} u^{-1}$. Then, using the connections between the gradings and actions,

$$|\bar{\eta}| = |\eta' \bar{\triangleleft} u^{-1}| = (\langle \eta' \rangle \triangleright u^{-1})^{-1} |\eta'| u^{-1}. \tag{10.9}$$

Putting $\bar{t} = t \triangleleft u |\eta'|^{-1}$, the left-hand side of (10.8) will become

$$\begin{aligned}(t \triangleleft |\bar{\eta}|^{-1}) \bar{\triangleright} \bar{\eta} &= (t \triangleleft u |\eta'|^{-1} (\langle \eta' \rangle \triangleright u^{-1})) \bar{\triangleright} (\eta' \bar{\triangleleft} u^{-1}) \\ &= (\bar{t} \triangleleft (\langle \eta' \rangle \triangleright u^{-1})) \bar{\triangleright} (\eta' \bar{\triangleleft} u^{-1}) \\ &= (\bar{t} \bar{\triangleright} \eta') \bar{\triangleleft} ((\bar{t} \triangleleft |\eta'|) \triangleright u^{-1}).\end{aligned} \tag{10.10}$$

Now, from (10.8) and the fact that $(t \triangleright u)^{-1} = (\bar{t} \triangleleft |\eta'|) \triangleright u^{-1}$, we only need to show that

$$\bar{t} \bar{\triangleright} \eta' = (((t \triangleleft u(s \triangleright |\eta|^{-1})) \cdot (s \triangleleft |\eta|^{-1})) \bar{\triangleright} \eta) \bar{\triangleleft} \tau(t \triangleleft u, s)^{-1}. \tag{10.11}$$

From the formula for the composition of the M "action," the right-hand side of (10.11) becomes $\bar{p} \bar{\triangleright} (\bar{s} \bar{\triangleright} \eta) = \bar{p} \bar{\triangleright} \eta'$, where $\bar{p}' = t \triangleleft u(s \triangleright |\eta|^{-1})$ and $\bar{p} = \bar{p}' \triangleleft \tau(\bar{s}, \langle \eta \rangle) \tau(\langle \bar{s} \bar{\triangleright} \eta \rangle, \bar{s} \triangleleft |\eta|)^{-1}$. We have used the fact that $\tau(t \triangleleft u, s) = \tau(\bar{p}' \triangleleft (\bar{s} \triangleright |\eta|), \bar{s} \triangleleft |\eta|)$. Now we just have to prove that $\bar{p} = \bar{t}$. Because $\tau(\bar{s}, \langle \eta \rangle)^{-1} (\bar{s} \triangleright |\eta|) = \tau(\langle \bar{s} \bar{\triangleright} \eta \rangle, \bar{s} \triangleleft |\eta|)^{-1} |\bar{s} \bar{\triangleright} \eta|$ and knowing that $(\bar{s} \triangleright |\eta|) = (s \triangleright |\eta|^{-1})^{-1}$, we can write $\bar{p}$ as follows:

$$\begin{aligned}\bar{p} &= \bar{p}' \triangleleft (\bar{s} \triangleright |\eta|) \, |\bar{s} \triangleright \eta|^{-1} \\ &= t \triangleleft u(s \triangleright |\eta|^{-1}) (s \triangleright |\eta|^{-1})^{-1} |\eta'|^{-1} \\ &= t \triangleleft u |\eta'|^{-1} = \bar{t}.\end{aligned} \tag{10.12}$$

Next, we show that $|||us \dot{\triangleright} \chi(\eta)||| = us|||\chi(\eta)|||(us)^{-1}$, where $u \in G$ and $s \in M$:

$$\begin{aligned}|||x \dot{\triangleright} \chi(\eta)||| &= |||\chi(((s \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta) \bar{\triangleleft} u^{-1})||| \\ &= \langle \eta' \bar{\triangleleft} u^{-1} \rangle^{-1} |\eta' \bar{\triangleleft} u^{-1}| \\ &= u \langle \eta' \rangle^{-1} |\eta'| u^{-1} \\ &= u \langle \bar{s} \bar{\triangleright} \eta \rangle^{-1} |\bar{s} \bar{\triangleright} \eta| u^{-1} \\ &= u (\bar{s} \triangleleft |\eta|) \langle \eta \rangle^{-1} |\eta| (\bar{s} \triangleleft |\eta|)^{-1} u^{-1} \\ &= us \langle \eta \rangle^{-1} |\eta| s^{-1} u^{-1}.\end{aligned} \tag{10.13}$$

□

Theorem 10.3. *The functor χ is invertible.*

Proof. We have already proved in Proposition 10.2 that the X-grading $|||\cdot|||$ and the action $\dot{\triangleright}$ give a representation of $D(X)$, so we only need to show that χ is a one-to-one correspondence, which we do by giving its inverse χ^{-1} as follows: let W be a representation of $D(X)$, with kX-action $\dot{\triangleright}$ and X-grading $|||\cdot|||$. Define a D representation as follows: $\chi^{-1}(W)$ will be the same as W as a vector space. There will be G- and M-gradings given by the factorization

$$|||\xi|||^{-1} = |\chi^{-1}(\xi)|^{-1} \langle \chi^{-1}(\xi) \rangle, \quad \xi \in W, \ \langle \chi^{-1}(\xi) \rangle \in M, \ |\chi^{-1}(\xi)| \in G. \tag{10.14}$$

The actions of $s \in M$ and $u \in G$ are given by

$$s \bar{\triangleright} \chi^{-1}(\xi) = \chi^{-1}((s \triangleleft |\chi^{-1}(\xi)|) \dot{\triangleright} \xi), \qquad \chi^{-1}(\xi) \bar{\triangleleft} u = \chi^{-1}(u^{-1} \hat{\triangleright} \xi). \tag{10.15}$$

Checking the rest is left to the reader. □

Proposition 10.4. *For $\delta_y \otimes x \in \mathcal{D}$, $\chi(\xi \hat{\triangleleft} (\delta_y \otimes x)) = \delta_{y,\|\xi\|} x^{-1} \dot{\triangleright} \chi(\xi)$.*

Proof. Starting with the left-hand side,

$$\chi(\xi \hat{\triangleleft} (\delta_y \otimes x)) = \chi(\delta_{y,\|\xi\|} \xi \hat{\triangleleft} x) = \delta_{y,\|\xi\|} \chi(\xi \hat{\triangleleft} x). \tag{10.16}$$

Putting $x = us$ for $u \in G$ and $s \in M$,

$$\xi \hat{\triangleleft} x = \xi \hat{\triangleleft} us = (\xi \bar{\triangleleft} u) \hat{\triangleleft} s = ((s^L \triangleleft u^{-1} |\xi|^{-1}) \bar{\triangleright} \xi) \bar{\triangleleft} (s^L \triangleright u^{-1})^{-1} \tau(s^L, s). \tag{10.17}$$

Now put $\bar{u} = \tau(s^L, s)^{-1}(s^L \triangleright u^{-1})$ and $\bar{s} = s^L \triangleleft u^{-1}$. Then

$$\begin{aligned} \chi(\xi \hat{\triangleleft} (\delta_y \otimes x)) &= \delta_{y,\|\xi\|} \chi(((\bar{s} \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi) \bar{\triangleleft} \bar{u}^{-1}) = \delta_{y,\|\xi\|} \bar{u} \bar{s} \dot{\triangleright} \chi(\xi) \\ &= \delta_{y,\|\xi\|} \tau(s^L, s)^{-1} (s^L \triangleright u^{-1})(s^L \triangleleft u^{-1}) \dot{\triangleright} \chi(\xi) \\ &= \delta_{y,\|\xi\|} s^{-1} s^{L^{-1}} s^L u^{-1} \dot{\triangleright} \chi(\xi) \\ &= \delta_{y,\|\xi\|} (us)^{-1} \dot{\triangleright} \chi(\xi) \\ &= \delta_{y,\|\xi\|} x^{-1} \dot{\triangleright} \chi(\xi). \end{aligned} \tag{10.18}$$

□

Proposition 10.5. *Define a map $\psi : D \to D(X)$ by $\psi(\delta_y \otimes x) = \delta_{x^{-1}yx} \otimes x^{-1}$. Then ψ satisfies the equation $\chi(\xi \hat{\triangleleft} (\delta_y \otimes x)) = \psi(\delta_y \otimes x) \dot{\triangleright} \chi(\xi)$.*

Proof. Use the previous proposition. □

The reader will recall that D is in general a nontrivially associated algebra (i.e., it is only associative in the category $\mathcal{D}$ with its nontrivial associator). Thus, in general, it cannot be isomorphic to $D(X)$, which is really associative. In general, ψ cannot be an algebra map.

Proposition 10.6. *For a and b elements of the algebra D in the category $\mathcal{D}$,*

$$\psi(b)\psi(a) = \psi(ab) \left(\sum_{y \in Y} \delta_y \otimes \tau(\langle a \rangle, \langle b \rangle)^{-1} \right). \tag{10.19}$$

Proof. By Proposition 10.5, we have

$$\begin{aligned} \chi((\xi \hat{\triangleleft} a) \hat{\triangleleft} b) &= \psi(b) \dot{\triangleright} \chi(\xi \hat{\triangleleft} a) = \psi(b) \dot{\triangleright} (\psi(a) \dot{\triangleright} \chi(\xi)) \\ &= \psi(b)\psi(a) \dot{\triangleright} \chi(\xi). \end{aligned} \tag{10.20}$$

But also, where $f = \sum_y \delta_y \otimes \tau(\langle a \rangle, \langle b \rangle)$,

$$\begin{aligned} \chi((\xi \hat{\triangleleft} a) \hat{\triangleleft} b) &= \chi((\xi \hat{\triangleleft} \tilde{\tau}(\|a\|, \|b\|)) \hat{\triangleleft} ab) = \psi(ab) \dot{\triangleright} \chi(\xi \hat{\triangleleft} \tilde{\tau}(\|a\|, \|b\|)) \\ &= \psi(ab) \dot{\triangleright} \chi(\xi \hat{\triangleleft} \tilde{\tau}(\langle a \rangle, \langle b \rangle)) = \psi(ab)\psi(f) \dot{\triangleright} \chi(\xi). \end{aligned} \tag{10.21}$$

□

Definition 10.7. Let V and W be objects of $\mathcal{D}$. The map $c:\chi(V)\otimes\chi(W)\to\chi(V\otimes W)$ is defined by

$$c(\chi(\eta)\otimes\chi(\xi))=\chi(((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\otimes\xi). \tag{10.22}$$

Proposition 10.8. *The map c, defined above, is a $D(X)$ module map, that is,*

$$\begin{aligned}|||c(\chi(\eta)\otimes\chi(\xi))|||&=|||\chi(\eta)\otimes\chi(\xi)|||,\\ x\dot{\triangleright}c(\chi(\eta)\otimes\chi(\xi))&=c(x\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi)))\quad\forall x\in X.\end{aligned} \tag{10.23}$$

Proof. We will begin with the grading first. It is known that

$$|||\chi(\eta)\otimes\chi(\xi)|||=|||\chi(\eta)|||\,|||\chi(\xi)|||=\langle\eta\rangle^{-1}|\eta|\langle\xi\rangle^{-1}|\xi|. \tag{10.24}$$

But, on the other hand, we know from the definition of c that

$$\begin{aligned}|||c(\chi(\eta)\otimes\chi(\xi))|||&=|||\chi(((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\otimes\xi)|||\\ &=\langle((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\otimes\xi\rangle^{-1}|((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\otimes\xi|\\ &=\langle\xi\rangle^{-1}\langle\bar{\eta}\rangle^{-1}|\bar{\eta}||\xi|\\ &=\langle\xi\rangle^{-1}\langle\bar{s}\bar{\triangleright}\eta\rangle^{-1}|\bar{s}\bar{\triangleright}\eta||\xi|\\ &=\langle\xi\rangle^{-1}(\bar{s}\triangleleft|\eta|)\langle\eta\rangle^{-1}|\eta|(\bar{s}\triangleleft|\eta|)^{-1}|\xi|\\ &=\langle\eta\rangle^{-1}|\eta|\langle\xi\rangle^{-1}|\xi|,\end{aligned} \tag{10.25}$$

where $\bar{s}=\langle\xi\rangle\triangleleft|\eta|^{-1}$ and $\bar{\eta}=(\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta=\bar{s}\bar{\triangleright}\eta$, which gives the result.

For the G-action, we know from the definitions that

$$\begin{aligned}u\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi))&=\chi(\eta\bar{\triangleleft}u^{-1})\otimes\chi(\xi\bar{\triangleleft}u^{-1}),\\ c(u\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi)))&=\chi((((\langle\xi\bar{\triangleleft}u^{-1}\rangle\triangleleft|\eta\bar{\triangleleft}u^{-1}|^{-1})\bar{\triangleright}(\eta\bar{\triangleleft}u^{-1}))\otimes(\xi\bar{\triangleleft}u^{-1})).\end{aligned} \tag{10.26}$$

By using the properties of the G- and M-gradings,

$$\begin{aligned}\langle\xi\bar{\triangleleft}u^{-1}\rangle\triangleleft|\eta\bar{\triangleleft}u^{-1}|^{-1}&=(\langle\xi\rangle\triangleleft u^{-1})\triangleleft u|\eta|^{-1}(\langle\eta\rangle\triangleright u^{-1})\\ &=\langle\xi\rangle\triangleleft|\eta|^{-1}(\langle\eta\rangle\triangleright u^{-1}),\\ ((\langle\xi\bar{\triangleleft}u^{-1}\rangle\triangleleft|\eta\bar{\triangleleft}u^{-1}|^{-1})\bar{\triangleright}(\eta\bar{\triangleleft}u^{-1})&=(((\langle\xi\rangle\triangleleft|\eta|^{-1})\triangleleft(\langle\eta\rangle\triangleright u^{-1}))\bar{\triangleright}(\eta\bar{\triangleleft}u^{-1})\\ &=(((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}((((\langle\xi\rangle\triangleleft|\eta|^{-1})\triangleleft|\eta|)\triangleright u^{-1})\\ &=(((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}(\langle\xi\rangle\triangleright u^{-1}).\end{aligned} \tag{10.27}$$

Now we can write

$$c(u\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi)))=\chi((((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}(\langle\xi\rangle\triangleright u^{-1})\otimes(\xi\bar{\triangleleft}u^{-1})). \tag{10.28}$$

On the other hand,

$$u\dot{\triangleright}c(\chi(\eta)\otimes\chi(\xi)) = u\dot{\triangleright}\chi((((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\otimes\xi) \\ = \chi(((((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\otimes\xi)\bar{\triangleleft}u^{-1}), \tag{10.29}$$

which gives the same as (10.28).

Now we show that c preserves the M-action. For $s\in M$,

$$s\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi)) = \chi((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\otimes\chi((s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi),$$

$$c(s\dot{\triangleright}(\chi(\eta)\otimes\chi(\xi))) = \chi((\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft\big|(s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big|^{-1})\bar{\triangleright}((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta) \\ \otimes((s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi)). \tag{10.30}$$

Using the "action" property for $\bar{\triangleright}$, we get

$$(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft\big|(s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big|^{-1})\bar{\triangleright}((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta) \\ = ((p'\cdot\bar{t})\bar{\triangleright}\eta)\bar{\triangleleft}\tau(p'\triangleleft(\bar{t}\triangleright|\eta|),\bar{t}\triangleleft|\eta|)^{-1}, \tag{10.31}$$

where $\bar{t}=s\triangleleft|\eta|^{-1}$ and

$$p' = \langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft\big|\bar{t}\bar{\triangleright}\eta\big|^{-1}\tau(\langle\bar{t}\bar{\triangleright}\eta\rangle,\bar{t}\triangleleft|\eta|)\tau(\bar{t},\langle\eta\rangle)^{-1}. \tag{10.32}$$

But, using the connections between the gradings and the actions, we know that $|\bar{t}\bar{\triangleright}\eta|^{-1}=(\bar{t}\triangleright|\eta|)^{-1}\tau(\bar{t},\langle\eta\rangle)\tau(\langle\bar{t}\bar{\triangleright}\eta\rangle,\bar{t}\triangleleft|\eta|)^{-1}$, so

$$p' = \langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft(\bar{t}\triangleright|\eta|)^{-1} \\ = \langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft((s\triangleleft|\eta|^{-1})\triangleright|\eta|)^{-1} \\ = \langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft(s\triangleright|\eta|^{-1}). \tag{10.33}$$

Substituting in the equation above gives

$$(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft\big|(s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta\big|^{-1})\bar{\triangleright}((s\triangleleft|\eta|^{-1})\bar{\triangleright}\eta) \\ = ((((\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\triangleleft(s\triangleright|\eta|^{-1}))\cdot(s\triangleleft|\eta|^{-1}))\bar{\triangleright}\eta)\bar{\triangleleft}\tau(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s)^{-1} \\ = ((((\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle\cdot s)\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}\tau(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s)^{-1} \\ = ((((s\triangleleft|\xi|^{-1})\cdot\langle\xi\rangle)\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\bar{\triangleleft}\tau(\langle(s\triangleleft|\xi|^{-1})\bar{\triangleright}\xi\rangle,s)^{-1}. \tag{10.34}$$

On the other hand, we know that

$$s\dot{\triangleright}c(\chi(\eta)\otimes\chi(\xi)) = s\dot{\triangleright}\chi((((\langle\xi\rangle\triangleleft|\eta|^{-1})\bar{\triangleright}\eta)\otimes\xi) \\ = s\dot{\triangleright}\chi(\bar{\eta}\otimes\xi) = \chi((s\triangleleft|\bar{\eta}\otimes\xi|^{-1})\bar{\triangleright}(\bar{\eta}\otimes\xi)), \tag{10.35}$$

where $\bar{\eta} = (\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta$. Next, we calculate

$$
\begin{aligned}
|\bar{\eta} \otimes \xi| &= \tau(\langle \bar{\eta} \rangle, \langle \xi \rangle)^{-1} |\bar{\eta}||\xi|, \\
s \triangleleft |\bar{\eta} \otimes \xi|^{-1} &= s \triangleleft |\xi|^{-1} |\bar{\eta}|^{-1} \tau(\langle \bar{\eta} \rangle, \langle \xi \rangle).
\end{aligned} \tag{10.36}
$$

If we put $\bar{s} = s \triangleleft |\xi|^{-1} |\bar{\eta}|^{-1}$, then

$$
\begin{aligned}
&(s \triangleleft |\bar{\eta} \otimes \xi|^{-1}) \bar{\triangleright} (\bar{\eta} \otimes \xi) \\
&\quad = (\bar{s} \triangleleft \tau(\langle \bar{\eta} \rangle, \langle \xi \rangle)) \bar{\triangleright} (\bar{\eta} \otimes \xi) \\
&\quad = (\bar{s} \bar{\triangleright} \bar{\eta}) \bar{\triangleleft} \tau(\bar{s} \triangleleft |\bar{\eta}|, \langle \xi \rangle) \tau(\langle (\bar{s} \triangleleft |\bar{\eta}|) \bar{\triangleright} \xi \rangle, \bar{s} \triangleleft |\bar{\eta}||\xi|)^{-1} \otimes (\bar{s} \triangleleft |\bar{\eta}|) \bar{\triangleright} \xi \\
&\quad = (\bar{s} \bar{\triangleright} \bar{\eta}) \bar{\triangleleft} \tau(s \triangleleft |\xi|^{-1}, \langle \xi \rangle) \tau(\langle (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \rangle, s)^{-1} \otimes (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi.
\end{aligned} \tag{10.37}
$$

Using the "action" property again,

$$
\begin{aligned}
\bar{s} \bar{\triangleright} \bar{\eta} &= (s \triangleleft |\xi|^{-1} |\bar{\eta}|^{-1}) \bar{\triangleright} ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta) \\
&= ((q' \cdot (\langle \xi \rangle \triangleleft |\eta|^{-1})) \bar{\triangleright} \eta) \bar{\triangleleft} \tau(q' \triangleleft ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleright |\eta|), \langle \xi \rangle)^{-1} \\
&= ((q' \cdot (\langle \xi \rangle \triangleleft |\eta|^{-1})) \bar{\triangleright} \eta) \bar{\triangleleft} \tau(q' \triangleleft (\langle \xi \rangle \triangleright |\eta|^{-1})^{-1}, \langle \xi \rangle)^{-1},
\end{aligned} \tag{10.38}
$$

where

$$
\begin{aligned}
q' &= (s \triangleleft |\xi|^{-1} |\bar{\eta}|^{-1}) \triangleleft \tau(\langle ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleright \eta \rangle, \langle \xi \rangle) \tau(\langle \xi \rangle \triangleleft |\eta|^{-1}, \langle \eta \rangle)^{-1} \\
&= (s \triangleleft |\xi|^{-1}) \triangleleft (\langle \xi \rangle \triangleright |\eta|^{-1}),
\end{aligned} \tag{10.39}
$$

as

$$
|\bar{\eta}|^{-1} = ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleright |\eta|)^{-1} \tau(\langle \xi \rangle \triangleleft |\eta|^{-1}, \langle \eta \rangle) \tau(\langle ((\langle \xi \rangle \triangleleft |\eta|^{-1}) \triangleright \eta \rangle, \langle \xi \rangle)^{-1}. \tag{10.40}
$$

Hence, substituting with the value of q', we get

$$
\begin{aligned}
\bar{s} \bar{\triangleright} \bar{\eta} &= ((((s \triangleleft |\xi|^{-1}) \triangleleft (\langle \xi \rangle \triangleright |\eta|^{-1})) \cdot (\langle \xi \rangle \triangleleft |\eta|^{-1})) \bar{\triangleright} \eta) \bar{\triangleleft} \tau((s \triangleleft |\xi|^{-1}), \langle \xi \rangle)^{-1} \\
&= ((((s \triangleleft |\xi|^{-1}) \cdot \langle \xi \rangle) \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta) \bar{\triangleleft} \tau(s \triangleleft |\xi|^{-1}, \langle \xi \rangle)^{-1},
\end{aligned} \tag{10.41}
$$

giving the required result

$$
\begin{aligned}
&(\bar{s} \bar{\triangleright} \bar{\eta}) \bar{\triangleleft} \tau(s \triangleleft |\xi|^{-1}, \langle \xi \rangle) \tau(\langle (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \rangle, s)^{-1} \\
&\quad = ((((s \triangleleft |\xi|^{-1}) \cdot \langle \xi \rangle) \triangleleft |\eta|^{-1}) \bar{\triangleright} \eta) \bar{\triangleleft} \tau(\langle (s \triangleleft |\xi|^{-1}) \bar{\triangleright} \xi \rangle, s)^{-1}.
\end{aligned} \tag{10.42}
$$

□

References

[1] B. Bakalov and A. Kirillov Jr., *Lectures on Tensor Categories and Modular Functors*, University Lecture Series, vol. 21, American Mathematical Society, Rhode Island, 2001.

[2] E. J. Beggs, *Making non-trivially associated tensor categories from left coset representatives*, J. Pure Appl. Algebra **177** (2003), no. 1, 5–41.

[3] E. J. Beggs, J. D. Gould, and S. Majid, *Finite group factorizations and braiding*, J. Algebra **181** (1996), no. 1, 112–151.

[4] E. J. Beggs and S. Majid, *Quasitriangular and differential structures on bicrossproduct Hopf algebras*, J. Algebra **219** (1999), no. 2, 682–727.
[5] L. C. Grove, *Groups and Characters*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1997.
[6] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.
[7] A. D. Thomas and G. V. Wood, *Group Tables*, Shiva Mathematics Series, vol. 2, Shiva Publishing, Nantwich, 1980.
[8] V. Turaev and H. Wenzl, *Semisimple and modular categories from link invariants*, Math. Ann. **309** (1997), no. 3, 411–461.

M. M. Al-Shomrani: Department of Mathematics, University of Wales, Swansea, Singleton Park, SA2 8PP, UK
E-mail address: malshomrani@hotmail.com

E. J. Beggs: Department of Mathematics, University of Wales, Swansea, Singleton Park, SA2 8PP, UK
E-mail address: e.j.beggs@swansea.ac.uk