IJMMS 2004:42, 2219–2230
PII. S0161171204309208
http://ijmms.hindawi.com

ON DIFFERENTIAL SUBORDINATIONS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

V. RAVICHANDRAN, HERB SILVERMAN, S. SIVAPRASAD KUMAR, and K. G. SUBRAMANIAN

Received 25 September 2003

We obtain several results concerning the differential subordination between analytic functions and a linear operator defined for a certain family of analytic functions which are introduced here by means of these linear operators. Also, some special cases are considered.

2000 Mathematics Subject Classification: 30C45, 30C80.

1. Introduction. Let $\mathcal{A}_0$ be the class of normalized analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ which are defined in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of all analytic functions $p(z)$ with $p(0) = 1$ which are defined on Δ. The class $\mathcal{P}$ of *Carathéodory* functions consists of functions $p(z) \in \mathcal{A}$ having positive real part. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

their Hadamard product (or convolution) is defined, as usual, by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.2}$$

Define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0, -1, -2, \ldots;\ z \in \Delta), \tag{1.3}$$

where $(x)_n$ is the Pochhammer symbol or the shifted factorial defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n \in \mathbb{N} := \{1,2,3,\ldots\}. \end{cases} \tag{1.4}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] introduced a linear operator $L(a,c)$ on $\mathcal{A}_0$ by the following convolution:

$$L(a,c)f(z) := \phi(a,c;z) * f(z), \tag{1.5}$$

or, equivalently, by

$$L(a,c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \Delta). \tag{1.6}$$

It follows from (1.6) that

$$z\big(L(a,c)f(z)\big)' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z). \tag{1.7}$$

For two functions f and g analytic in Δ, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta), \tag{1.8}$$

if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \Delta), \tag{1.9}$$

such that

$$f(z) = g\big(w(z)\big) \quad (z \in \Delta). \tag{1.10}$$

In particular, if the function g is *univalent* in Δ, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\Delta) \subset g(\Delta). \tag{1.11}$$

Over the past few decades, several authors have obtained criteria for univalence and starlikeness depending on bounds of the functionals $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$. See [4, 5, 7] and the references in [7]. In [2, 6], certain results involving linear operators were considered. In this paper, we obtain sufficient conditions involving

$$\frac{L(a+1,c)f(z)}{L(a,c)f(z)}, \qquad \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \tag{1.12}$$

for functions to satisfy the subordination

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z), \quad \left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\beta} \prec q(z) \quad (q(z) \in \mathcal{A}). \tag{1.13}$$

Also, we obtain sufficient conditions involving

$$\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}, \qquad \frac{L(a+1,c)f(z)}{z} \tag{1.14}$$

for functions to satisfy the subordination

$$\left(\frac{L(a,c)f(z)}{z}\right)^{\beta} \prec q(z), \quad \frac{z}{L(a+1,c)f(z)} \prec q(z) \quad (q(z) \in \mathcal{A}). \tag{1.15}$$

Since $L(n+1,1)f(z) = D^n f(z)$, where $D^n f(z)$ is the Ruscheweyh derivative of $f(z)$, our results can be specialized to the Ruscheweyh derivative and we omit these details. Note that the Ruscheweyh derivative of order δ is defined by

$$D^{\delta} f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \quad (f \in \mathcal{A}_0;\ \delta > -1) \tag{1.16}$$

or, equivalently, by

$$D^{\delta} f(z) := z + \sum_{k=2}^{\infty} \binom{\delta+k-1}{k+1} a_k z^k \quad (f \in \mathcal{A}_0;\ \delta > -1). \tag{1.17}$$

In our present investigation, we need the following result of Miller and Mocanu [3] to prove our main results.

Theorem 1.1 (cf. [3, Theorem 3.4h, page 132])**.** *Let $q(z)$ be univalent in the unit disk $\triangle$ and let ϑ and φ be analytic in a domain $\mathbb{D} \supset q(\triangle)$ with $\varphi(w) \neq 0$, when $w \in q(\triangle)$. Set*

$$Q(z) := zq'(z)\varphi(q(z)), \qquad h(z) := \vartheta(q(z)) + Q(z). \tag{1.18}$$

Suppose that

(1) *Q is starlike univalent in Δ;*

(2) *$\Re(zh'(z)/Q(z)) > 0$ for $z \in \triangle$.*

If $p(z)$ is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subset \mathbb{D}$, and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{1.19}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2. Main results.

We begin with the following.

Theorem 2.1. *Let α, β, and γ be real numbers, $\beta \neq 0$, and $(1+a)\beta\gamma < 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\triangle$ and let it satisfy the following condition for $z \in \triangle$:*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{\beta + (1+a)\gamma}{\beta} & \text{if } \dfrac{\beta + \gamma(a+1)}{\beta} \geq 0, \\ 0 & \text{if } \dfrac{\beta + \gamma(a+1)}{\beta} \leq 0. \end{cases} \tag{2.1}$$

If $f(z) \in \mathcal{A}_0$ and

$$\begin{aligned} &\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left\{ \alpha \frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \beta \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \gamma \right\} \\ &\quad \prec \frac{1}{a+1} \{ \alpha(a+1) + a\beta + [\beta + \gamma(a+1)]q(z) - \beta z q'(z) \}, \end{aligned} \tag{2.2}$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z) \tag{2.3}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{L(a,c)f(z)}{L(a+1,c)f(z)}. \tag{2.4}$$

Then, clearly, $p(z)$ is analytic in Δ. Also, by a simple computation, we find from (2.4) that

$$\frac{zp'(z)}{p(z)} = \frac{z\big(L(a,c)f(z)\big)'}{L(a,c)f(z)} - \frac{z\big(L(a+1,c)f(z)\big)'}{L(a+1,c)f(z)}. \tag{2.5}$$

By making use of the familiar identity (1.7) in (2.5), we get

$$\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{1}{a+1}\left(1+\frac{a}{p(z)}-\frac{zp'(z)}{p(z)}\right). \tag{2.6}$$

By using (2.4) and (2.6), we obtain

$$\begin{aligned}
&\left[\alpha\frac{L(a+1,c)f(z)}{L(a,c)f(z)}+\beta\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+\gamma\right]\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\\
&\quad=\left[\frac{\alpha}{p(z)}+\frac{\beta}{a+1}\left(1+\frac{a}{p(z)}-\frac{zp'(z)}{p(z)}\right)+\gamma\right]p(z)\\
&\quad=\frac{1}{a+1}\{(a+1)\alpha+a\beta+[\beta+\gamma(a+1)]p(z)-\beta zp'(z)\}.
\end{aligned} \tag{2.7}$$

In view of (2.7), the subordination (2.2) becomes

$$[\beta+\gamma(a+1)]p(z)-\beta zp'(z) \prec [\beta+\gamma(a+1)]q(z)-\beta zq'(z) \tag{2.8}$$

and this can be written as (1.19), where

$$\vartheta(w) := [\beta+\gamma(a+1)]w, \qquad \varphi(w) := -\beta. \tag{2.9}$$

Note that $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C}$. Since $\beta \neq 0$, we have $\varphi(w) \neq 0$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned}
Q(z) &:= zq'(z)\varphi(q(z)) = -\beta zq'(z),\\
h(z) &:= \vartheta(q(z))+Q(z) = [\beta+(a+1)\gamma]q(z)-\beta zq'(z).
\end{aligned} \tag{2.10}$$

In light of hypothesis (2.1) stated in Theorem 2.1, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\gamma(a+1)+\beta}{-\beta}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right\} > 0. \tag{2.11}$$

The result of Theorem 2.1 now follows by an application of Theorem 1.1. □

Note that

$$L(1,1)f(z) = f(z),$$
$$L(2,1)f(z) = zf'(z), \tag{2.12}$$
$$L(3,1)f(z) = zf'(z) + \frac{z^2 f''(z)}{2}.$$

By taking $a = c = 1$ in Theorem 2.1 and after a change in the parameters, we have the following.

Corollary 2.2. *Let α be a real number, $1+\alpha > 0$, and let $q(z)$ be univalent in Δ, and let it satisfy*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \begin{cases} -\alpha & \text{if } \alpha \le 0, \\ 0 & \text{if } \alpha \ge 0. \end{cases} \tag{2.13}$$

If $f \in A_0$ and

$$\frac{f(z)}{zf'(z)}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} - \left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha\right\} \prec zq'(z) + \alpha q(z) - \alpha, \tag{2.14}$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z) \tag{2.15}$$

and $q(z)$ is the best dominant.

If we take

$$q(z) = 1 + \frac{\lambda}{1+\alpha} z \tag{2.16}$$

in Corollary 2.2, we obtain a recent result of Singh [7, Theorem 1(i), page 571].

By using Theorem 1.1, we can show the following.

Lemma 2.3. *Let γ, β be real numbers, $\beta \ne 0$, and $1 > \gamma/\beta$. Let $q(z) \in \mathcal{A}$ be univalent in Δ and let it satisfy*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{\gamma}{\beta} & \text{if } \dfrac{\gamma}{\beta} \ge 0, \\ 0 & \text{if } \dfrac{\gamma}{\beta} \le 0. \end{cases} \tag{2.17}$$

If $p(z) \in \mathcal{A}$ satisfies

$$\gamma p(z) - \beta z p'(z) \prec \gamma q(z) - \beta z q'(z), \tag{2.18}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

By using Lemma 2.3, or from Theorem 2.1, we have the following.

Corollary 2.4. *Let α, β, γ be real numbers, $\beta \neq 0$, and $1 > \gamma/\beta$. Let $q(z) \in \mathcal{A}$ be univalent in Δ and let it satisfy (2.17). If $f(z) \in \mathcal{A}_0$ satisfies*

$$\frac{f(z)}{zf'(z)}\left\{\alpha\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)+\gamma\right\} \prec \alpha+\beta-\beta zq'(z)+\gamma q(z), \tag{2.19}$$

then (2.15) holds and $q(z)$ is the best dominant.

By using Theorem 1.1, we obtain the following.

Theorem 2.5. *Let $a \neq -1$. Let α, β, γ, and δ be real numbers, $\alpha \neq 0$, and $1+\delta(a+1)(\alpha+\gamma)/\alpha > 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\triangle$ and let it satisfy the following condition for $z \in \triangle$:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \begin{cases} -\dfrac{\delta(a+1)(\alpha+\gamma)}{\alpha} & \text{if } \dfrac{\delta(a+1)(\alpha+\gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \dfrac{\delta(a+1)(\alpha+\gamma)}{\alpha} \geq 0. \end{cases} \tag{2.20}$$

If $f(z) \in \mathcal{A}_0$ and

$$\begin{aligned}&\left\{\alpha\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+\beta\left(\frac{z}{L(a+1,c)f(z)}\right)^{\delta}+\gamma\right\}\left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta} \\ &\quad \prec \frac{\alpha}{\delta(a+1)}zq'(z)+(\alpha+\gamma)q(z)+\beta,\end{aligned} \tag{2.21}$$

then

$$\left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta} \prec q(z) \tag{2.22}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta}. \tag{2.23}$$

Then, clearly, $p(z)$ is analytic in Δ. Also, by a simple computation, we find from (2.23) that

$$\frac{zp'(z)}{p(z)} = \frac{\delta z\big(L(a+1,c)f(z)\big)'}{L(a+1,c)f(z)} - \delta. \tag{2.24}$$

By making use of the familiar identity (1.7) in (2.24), we get

$$\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{1}{\delta(a+1)}\frac{zp'(z)}{p(z)}+1. \tag{2.25}$$

By using (2.23) and (2.25), we obtain

$$\begin{aligned}&\left\{\alpha\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+\beta\left(\frac{z}{L(a+1,c)f(z)}\right)^{\delta}+\gamma\right\}\left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta}\\&\quad=\left\{\alpha\left(\frac{1}{\delta(a+1)}\frac{zp'(z)}{p(z)}+1\right)+\frac{\beta}{p(z)}+\gamma\right\}p(z)\\&\quad=\frac{\alpha}{\delta(a+1)}zp'(z)+(\alpha+\gamma)p(z)+\beta.\end{aligned}\tag{2.26}$$

In view of (2.26), the subordination (2.21) becomes

$$\delta(a+1)(\alpha+\gamma)p(z)+\alpha zp'(z)\prec\delta(a+1)(\alpha+\gamma)q(z)+\alpha zq'(z)\tag{2.27}$$

and this can be written as (1.19), where

$$\vartheta(w):=\delta(a+1)(\alpha+\gamma)w,\qquad\varphi(w):=\alpha.\tag{2.28}$$

Note that $\varphi(w)\neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C}$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$\begin{aligned}Q(z)&:=zq'(z)\varphi(q(z))=\alpha zq'(z),\\h(z)&:=\vartheta(q(z))+Q(z)=\delta(a+1)(\alpha+\gamma)q(z)+\alpha zq'(z).\end{aligned}\tag{2.29}$$

By hypothesis (2.20) stated in Theorem 2.5, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\}=\Re\left\{\frac{\delta(a+1)(\alpha+\gamma)}{\alpha}+\left(1+\frac{zq''(z)}{q'(z)}\right)\right\}>0.\tag{2.30}$$

Thus, by an application of Theorem 1.1, the proof of Theorem 2.5 is completed. □

By taking $a=c=1$ in Theorem 2.5 and after a suitable change in the parameters, we have the following.

Corollary 2.6. *Let $\alpha,\beta\neq 0$ be real and $1+\alpha>0$. Let $q(z)$ be univalent in Δ and let it satisfy (2.13). If $f\in A_0$ and*

$$\left\{\beta\frac{zf''(z)}{f'(z)}+\alpha\left(1-[f'(z)]^{-\beta}\right)\right\}[f'(z)]^{\beta}\prec zq'(z)+\alpha q(z)-\alpha,\tag{2.31}$$

then

$$[f'(z)]^{\beta}\prec q(z)\tag{2.32}$$

and $q(z)$ is the best dominant.

If we take (2.16) in Corollary 2.6, we obtain a recent result of Singh [7, Theorem 1(ii), page 571].

Theorem 2.7. *Let $a \neq -1$. Let α, β, and γ be real numbers and let $\beta, \gamma \neq 0$ and $1+\alpha/\gamma > 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\triangle$ and let it satisfy the following condition for $z \in \triangle$:*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \begin{cases} -\dfrac{\alpha}{\gamma} & \text{if } \dfrac{\alpha}{\gamma} \leq 0, \\ 0 & \text{if } \dfrac{\alpha}{\gamma} \geq 0. \end{cases} \tag{2.33}$$

If $f(z) \in \mathcal{A}_0$ and

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\beta}\left\{\beta\gamma\left[(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right] + \alpha\right\} \prec \gamma zq'(z) + \alpha q(z), \tag{2.34}$$

then

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\beta} \prec q(z) \tag{2.35}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\beta}. \tag{2.36}$$

Then, clearly, $p(z)$ is analytic in Δ. Also, by a simple computation together with the use of the familiar identity (1.7), we find from (2.36) that

$$\frac{1}{\beta}\frac{zp'(z)}{p(z)} = (a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1. \tag{2.37}$$

Therefore, it follows from (2.36) and (2.37) that

$$\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)^{\beta}\left\{\beta\gamma\left[(a+1)\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right] + \alpha\right\} = \gamma zp'(z) + \alpha p(z). \tag{2.38}$$

In view of (2.38), the subordination (2.34) becomes

$$\gamma zp'(z) + \alpha p(z) \prec \gamma zq'(z) + \alpha q(z) \tag{2.39}$$

and this can be written as (1.19), where

$$\vartheta(w) := \alpha w, \qquad \varphi(w) := \gamma. \tag{2.40}$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $\mathbb{C}$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$Q(z) := zq'(z)\varphi(q(z)) = \gamma zq'(z),$$
$$h(z) := \vartheta(q(z)) + Q(z) = \alpha q(z) + \gamma zq'(z). \tag{2.41}$$

In light of hypothesis (2.33) stated in Theorem 2.7, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\frac{\alpha}{\gamma} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0. \tag{2.42}$$

Since ϑ and φ satisfy the conditions of Theorem 1.1, the result follows by an application of Theorem 1.1. □

By taking $a = c = 1$ in Theorem 2.7 and after a suitable change in the parameters, we have the following.

Corollary 2.8. *Let $\alpha, \beta \neq 0$ and γ be real with $1 + \alpha/\gamma > 0$. Let $q(z)$ be univalent in Δ and let it satisfy (2.33).*

If $f \in A_0$ and

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta}\left\{\beta\gamma\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] + \alpha\right\} \prec \gamma zq'(z) + \alpha q(z), \tag{2.43}$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta} \prec q(z) \tag{2.44}$$

and $q(z)$ is the best dominant.

If we take (2.16) and $\gamma = 1$ in Corollary 2.8, we obtain a recent result of Singh [7, Theorem 1(iii), page 571] and, by setting

$$q(z) = \int_0^1 \frac{1 - \lambda zt^{\alpha}}{1 + \lambda zt^{\alpha}}dt \tag{2.45}$$

and $\alpha = 1$ in Corollary 2.8, we obtain another recent result of Singh [7, Theorem 3, page 573].

Theorem 2.9. *Let $\alpha \neq 0$ and γ be real numbers, $(a+1)\alpha\gamma < 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\triangle$ and let it satisfy the following condition for $z \in \triangle$:*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \dfrac{\alpha + \gamma(a+1)}{\alpha} & \text{if } \dfrac{\alpha + \gamma(a+1)}{\alpha} \geq 0, \\ 0 & \text{if } \dfrac{\alpha + \gamma(a+1)}{\alpha} \leq 0. \end{cases} \tag{2.46}$$

If $f(z) \in \mathcal{A}_0$ *and*

$$\alpha \frac{L(a,c)f(z)}{L(a+1,c)f(z)}\left(\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}\right)+\gamma \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec \frac{a\alpha}{a+1}+\left(\frac{\alpha}{a+1}+\gamma\right)q(z)-\frac{\alpha}{a+1}zq'(z), \tag{2.47}$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z) \tag{2.48}$$

and $q(z)$ *is the best dominant.*

The proof of this theorem is similar to that of Theorem 2.1 and hence it is omitted.

By taking $a=c=1$ in Theorem 2.9 and after a suitable change in the parameters, we have the following.

Corollary 2.10. *Let* $0 \le \alpha \le 1$ *and* $q(z)$ *be univalent in* Δ *and let them satisfy*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \begin{cases} \alpha & \text{if } \alpha \ge 0 \\ 0 & \text{if } \alpha \le 0. \end{cases} \tag{2.49}$$

If $f \in A_0$ *and*

$$\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha\left(\frac{f(z)}{zf'(z)}-1\right) \prec (1+\alpha)-\alpha q(z)-zq'(z), \tag{2.50}$$

then (2.15) holds and $q(z)$ *is the best dominant.*

Let

$$q(z)=1+\frac{\lambda z}{k}\int_0^1 \frac{t^\alpha}{1+(z/k)t}dt. \tag{2.51}$$

After a change of variable in (2.51), we get

$$q(z)=1+\frac{\lambda}{z^\alpha}\int_0^z \frac{\eta^\alpha}{k+\eta}d\eta. \tag{2.52}$$

By differentiating (2.52), we have

$$zq'(z)=\frac{\lambda z}{k+z}-\alpha q(z)+\alpha \tag{2.53}$$

or

$$\alpha-\alpha q(z)-zq'(z)=-\frac{\lambda z}{k+z}. \tag{2.54}$$

Since the bilinear transform

$$w=-\frac{\lambda z}{k+z} \tag{2.55}$$

maps Δ onto the disk

$$\left| w + \frac{\lambda}{1-k^2} \right| \leq \frac{|\lambda| k}{k^2-1}, \tag{2.56}$$

from Corollary 2.10 for the function $q(z)$ given by (2.51), we obtain a recent result of Singh [7, Theorem 2(i), page 572].

Theorem 2.11. *Let $\alpha \neq 0$ and γ be real numbers, $(a+1)\alpha\gamma < 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\triangle$ and let it satisfy (2.46) for $z \in \triangle$.*

If $f(z) \in \mathcal{A}_0$ and

$$\alpha z \frac{L(a+2,c)f(z)}{[L(a+1,c)f(z)]^2} + \gamma \frac{z}{L(a+1,c)f(z)} \prec (\alpha+\gamma)q(z) - \frac{\alpha}{a+1} z q'(z), \tag{2.57}$$

then

$$\frac{z}{L(a+1,c)f(z)} \prec q(z) \tag{2.58}$$

and $q(z)$ is the best dominant.

The proof of this theorem is similar to that of Theorem 2.1 and therefore it is omitted.

By taking $a = c = 1$ in Theorem 2.11 and after a suitable change in the parameters, we have the following.

Corollary 2.12. *Let $0 \leq \alpha \leq 1$ and $q(z)$ be univalent in Δ and let them satisfy (2.49). If $f \in A_0$, $f(z)f'(z)/z \neq 0$, and*

$$\frac{z f''(z)}{f'(z)^2} - \alpha\left(\frac{1}{f'(z)} - 1\right) \prec \alpha - \alpha q(z) - z q'(z), \tag{2.59}$$

then

$$\frac{1}{f'(z)} \prec q(z) \tag{2.60}$$

and $q(z)$ is the best dominant.

On setting (2.51) in Corollary 2.12, we obtain a recent result of Singh [7, Theorem 2(ii), page 572].

Acknowledgment. The authors are thankful to the referees for their comments.

References

[1] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), no. 4, 737–745.

[2] J.-L. Liu and S. Owa, *On a class of multivalent functions involving certain linear operator*, Indian J. Pure Appl. Math. **33** (2002), no. 11, 1713–1722.

[3] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.

[4] M. Obradowič and N. Tuneski, *On the starlike criteria defined by Silverman*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. **181** (2000), no. 24, 59–64.

[5] K. S. Padmanabhan, *On sufficient conditions for starlikeness*, Indian J. Pure Appl. Math. **32** (2001), no. 4, 543–550.
[6] J. Patel and P. Sahoo, *Properties of a class of analytic functions involving a linear operator*, Demonstratio Math. **35** (2002), no. 3, 495–507.
[7] V. Singh, *On some criteria for univalence and starlikeness*, Indian J. Pure Appl. Math. **34** (2003), no. 4, 569–577.

V. Ravichandran: Department of Computer Applications, Sri Venkateswara College of Engineering, Pennalur, Sriperumbudur 602 105, Tamil Nadu, India
E-mail address: vravi@svce.ac.in

Herb Silverman: Department of Mathematics, College of Charleston, Charleston, SC 29424, USA
E-mail address: silvermanH@cofc.edu

S. Sivaprasad Kumar: Department of Mathematics, Sindhi College, 123 P.H. Road, Numbal, Chennai 600 077, Tamil Nadu, India
E-mail address: sivpk71@yahoo.com

K. G. Subramanian: Department of Mathematics, Madras Christian College, Tambaram, Chennai 600 059, Tamil Nadu, India
E-mail address: kgsmani@vsnl.net