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CONSTRUCTION OF NIJENHUIS OPERATORS AND DENDRIFORM TRIALGEBRAS

PHILIPPE LEROUX

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We construct Nijenhuis operators from particular bialgebras called dendriform-Nijenhuis bialgebras. It turns out that such Nijenhuis operators commute with TD-operators, a kind of Baxter-Rota operators, and are therefore closely related to dendriform trialgebras. This allows the construction of associative algebras, called dendriform-Nijenhuis algebras, made out of nine operations and presenting an exotic combinatorial property. We also show that the augmented free dendriform-Nijenhuis algebra and its commutative version have a structure of connected Hopf algebras. Examples are given.

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1. Introduction

Notation. In the sequel, k is a field of characteristic zero. Let $(X,\diamond)$ be a k-algebra and let $(\diamond_i)_{1\le i\le N}: X^{\otimes 2}\to X$ be a family of binary operations on X. The notation $\diamond \to \sum_i \diamond_i$ will mean $x\diamond y=\sum_i x\diamond_i y$, for all $x,y\in X$. We say that the operation $\diamond$ *splits* into the N operations $\diamond_1,\dots,\diamond_N$, or that the operation $\diamond$ is a *cluster of* N *(binary) operations*.

Let $(\mathcal{L},[\cdot,\cdot])$ be a Lie algebra. A *Nijenhuis operator* $N:\mathcal{L}\to\mathcal{L}$ is a linear map verifying

$$[N(x),N(y)]+N^2([x,y])=N([N(x),y]+[x,N(y)]). \tag{1.1}$$

Solutions of this equation can be constructed by producing operators $\beta: A\to A$ verifying the *associative Nijenhuis relation (ANR)* on an associative algebra (A,μ). The *ANR*, that is,

$$(\mathrm{ANR}): \beta(x)\beta(y)+\beta^2(xy)=\beta(\beta(x)y+x\beta(y)), \tag{1.2}$$

for all $x,y\in A$, appears for the first time in [4]; see also [5, 6] and the references therein. Such linear maps β are then Nijenhuis operators since (1.2) implies (1.1) on the Lie algebra $(A,[\cdot,\cdot])$, with $[x,y]:=xy-yx$, for all $x,y\in A$. In the sequel, by Nijenhuis operators we mean a linear map defined on an associative algebra and verifying (1.2).

Section 2 prepares the sequel of this work. Nijenhuis (NS)-algebras are defined and dendriform trialgebras are recalled. We show that Nijenhuis operators on an associative algebra give NS-algebras. The notion of Trigèbre-Dendriforme (TD)-operators is also introduced. Such operators give dendriform trialgebras [6, 9, 14]. In Section 3, the notion of dendriform-Nijenhuis bialgebras (A,μ,Δ) is introduced. This notion is the cornerstone of the paper. Indeed, any dendriform-Nijenhuis bialgebra gives two operators $\beta,\gamma : \mathrm{End}(A) \to \mathrm{End}(A)$, where $\mathrm{End}(A)$ is the k-algebra of linear maps from A to A, which commute with one another, that is, $\beta\gamma = \gamma\beta$. The first one turns out to be a Nijenhuis operator and the last one, a TD-operator. Section 4 gives examples. Section 5 introduces dendriform-Nijenhuis algebras, which are associative algebras whose associative product splits into nine binary operations linked together by 28 constraints. When the nine operations are gathered in a particular way, dendriform-Nijenhuis algebras become NS-algebras and when gathered in another way, dendriform-Nijenhuis algebras become dendriform trialgebras. Otherwise stated, this means the existence of a commutative diagram between the involved categories,

$$\begin{tikzcd} \text{Dend Nijenhuis} \arrow[r] \arrow[d] \arrow[dr] & \text{NS} \arrow[d] \\ \text{TriDend} \arrow[r] & \text{As} \end{tikzcd} \tag{1.3}$$

which will be explained in Section 5. Such associative algebras can be easily constructed from dendriform-Nijenhuis bialgebras. Section 6 shows that the augmented free dendriform-Nijenhuis algebra and the augmented free NS-algebra on a k-vector space V, as well as their commutative versions, have a structure of connected Hopf algebras.

2. NS-algebras and dendriform trialgebras.

We present some results relating NS-algebras and dendriform trialgebras to Nijenhuis operators and TD-operators.

2.1. NS-algebras

Definition 2.1 (NS-algebra)**.** An NS-*algebra* $(A,\prec,\succ,\bullet)$ is a k-vector space equipped with three binary operations $\prec,\succ,\bullet : A^{\otimes 2} \to A$ verifying

$$\begin{aligned} (x \prec y) \prec z &= x \prec (y \star z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x \star y) \succ z &= x \succ y \succ z, \\ (x \star y) \bullet z + (x \bullet y) \prec z &= x \succ (y \bullet z) + x \bullet (y \star z), \end{aligned} \tag{2.1}$$

where $\star \to \prec + \succ + \bullet$. The k-vector space $(A,\star)$ is an associative algebra, that is, there exists a functor $F_1 : \mathrm{NS} \to \mathrm{As}$ where NS is the category of NS-algebras and As the category of associative algebras.

Remark 2.2. Let $(A,\prec,\succ,\bullet)$ be an NS-algebra. Define three operations $\prec^{\mathrm{op}},\succ^{\mathrm{op}},\bullet^{\mathrm{op}} : A^{\otimes 2} \to A$ as follows:

$$x \prec^{\mathrm{op}} y = y \succ x, \quad x \succ^{\mathrm{op}} y = y \prec x, \quad x \bullet^{\mathrm{op}} y = y \bullet x, \quad \forall x,y \in A. \tag{2.2}$$

Then, $(A,\prec^{\mathrm{op}},\succ^{\mathrm{op}},\bullet^{\mathrm{op}})$ is an NS-algebra called the *opposite* of $(A,\prec,\succ,\bullet)$. NS-algebras are said to be *commutative* if they coincide with their opposites, that is, if $x \prec y = y \succ x$ and $x \bullet y = y \bullet x$.

Proposition 2.3. *Let (A,μ) be an associative algebra equipped with a Nijenhuis operator $\beta : A \to A$. Define three binary operations $\prec_\beta, \succ_\beta, \bullet_\beta : A^{\otimes 2} \to A$ as follows:*

$$x \prec_\beta y = x\beta(y), \qquad x \succ_\beta y = \beta(x)y, \qquad x \bullet_\beta y = -\beta(xy), \tag{2.3}$$

for all $x,y \in A$. Then, $A^\beta := (A, \prec_\beta, \succ_\beta, \bullet_\beta)$ is an NS*-algebra.*

Proof. Straightforward. $\square$

Proposition 2.4. *Let (A,μ) be a unital associative algebra with unit i. Suppose $\beta : A \to A$ is a Nijenhuis operator. Define three binary operations $\tilde{\prec}_\beta, \tilde{\succ}_\beta, \tilde{\bullet}_\beta : (A^\beta)^{\otimes 2} \to A^\beta$ as follows:*

$$\begin{aligned} x \tilde{\prec}_\beta y &= x\beta(i) \prec_\beta y \big(= x \prec_\beta \beta(i) y\big), \\ x \tilde{\succ}_\beta y &= x \succ_\beta \beta(i) y \big(= x\beta(i) \succ_\beta y\big), \\ x \tilde{\bullet}_\beta y &= x\beta(i) \bullet_\beta y \big(= x \bullet_\beta \beta(i) y\big), \end{aligned} \tag{2.4}$$

for all $x,y \in A$. Then, $(A, \tilde{\prec}_\beta, \tilde{\succ}_\beta, \tilde{\bullet}_\beta)$ is an NS*-algebra.*

Proof. Observe that for all $x \in A$, $\beta(x)\beta(i) = \beta(x\beta(i))$ and $\beta(i)\beta(x) = \beta(\beta(i)x)$. Therefore, for all $x,y \in A$, $x \tilde{\prec}_\beta y := x\beta(i)\beta(y) = x\beta(\beta(i)y) = x \prec_\beta \beta(i)y$ and similarly for the two other operations. Fix $x,y,z \in A$. We check that

$$(x \tilde{\prec}_\beta y) \tilde{\prec} z = x \tilde{\prec}_\beta (y \tilde{\star}_\beta z), \tag{2.5}$$

where $\tilde{\star}_\beta \to \tilde{\prec}_\beta + \tilde{\succ}_\beta + \tilde{\bullet}_\beta$. Indeed,

$$\begin{aligned} (x \tilde{\prec}_\beta y) \tilde{\prec}_\beta z &:= x \prec_\beta \big(\beta(i) y \star_\beta \beta(i) z\big) \\ &= x \prec_\beta \big(\beta(i) y \prec_\beta \beta(i) z + \beta(i) y \succ_\beta \beta(i) z + \beta(i) y \bullet_\beta \beta(i) z\big) \\ &= x \prec_\beta \beta(i) \big(y \prec_\beta \beta(i) z + y \succ_\beta \beta(i) z + y \bullet_\beta \beta(i) z\big) \\ &= x \tilde{\prec}_\beta (y \tilde{\star}_\beta z). \end{aligned} \tag{2.6}$$

$\square$

2.2. Dendriform trialgebras. We recall some motivations for the introduction of dendriform trialgebras. Motivated by the K-theory, Loday first introduced a "noncommutative version" of Lie algebras called Leibniz algebras [10]. Such algebras are described by a bracket $[\cdot, z]$ verifying the Leibniz identity:

$$[[x,y],z] = [[x,z],y] + [x,[y,z]]. \tag{2.7}$$

When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and Leibniz algebras turn out to be Lie algebras. A way to construct such Leibniz algebras is to start with associative dialgebras, that is, k-vector spaces D equipped with two associative products $\vdash$ and $\dashv$ and verifying some conditions [11]. The operad Dias associated with associative dialgebras is then Koszul dual to the operad DiDend associated with dendriform dialgebras [11]. A *dendriform dialgebra* is a k-vector space E equipped with two binary operations $\prec, \succ : E^{\otimes 2} \to E$, satisfying the following relations for all $x, y, z \in E$:

$$\begin{aligned}(x \prec y) \prec z &= x \prec (y \star z),\\ (x \succ y) \prec z &= x \succ (y \prec z),\\ (x \star y) \succ z &= x \succ (y \succ z),\end{aligned} \tag{2.8}$$

where, by definition, $x \star y := x \prec y + x \succ y$, for all $x, y \in E$. The dendriform dialgebra $(E, \star)$ is then an associative algebra such that $\star \to \prec + \succ$. Similarly, to propose a "noncommutative version" of Poisson algebras, Loday and Ronco [14] introduced the notion of associative trialgebras. It turns out that Trias, the operad associated with this type of algebras, is Koszul dual to TriDend, the operad associated with dendriform trialgebras.

Definition 2.5 (dendriform trialgebra)**.** A *dendriform trialgebra* is a k-vector space T equipped with three binary operations $\prec, \succ, \circ : T^{\otimes 2} \to T$, satisfying the following relations for all $x, y, z \in T$:

$$\begin{gathered}(x \prec y) \prec z = x \prec (y \star z), \qquad (x \succ y) \prec z = x \succ (y \prec z),\\ (x \star y) \succ z = x \succ (y \succ z), \qquad (x \succ y) \circ z = x \succ (y \circ z),\\ (x \prec y) \circ z = x \circ (y \succ z), \qquad (x \circ y) \prec z = x \circ (y \prec z),\\ (x \circ y) \circ z = x \circ (y \circ z),\end{gathered} \tag{2.9}$$

where, by definition, $x \star y := x \prec y + x \succ y + x \circ y$, for all $x, y \in T$. The k-vector space $(T, \star)$ is then an associative algebra such that $\star \to \prec + \succ + \circ$. There exists a functor $F_2 : \text{TriDend} \to \text{As}$.

Observe that these axioms are globally invariant under the transformations $x \prec^{\text{op}} y = y \succ x$, $x \succ^{\text{op}} y = y \prec x$, and $x \circ^{\text{op}} y = y \circ x$. A dendriform trialgebra is said to be *commutative* if $x \prec y = y \succ x$ and $x \circ y = y \circ x$.

To construct dendriform trialgebras, t-Baxter operators, also called Rota-Baxter operators, have been used in [6, 9] to generalize [2]. Let (A, μ) be an associative algebra and let $t \in k$. A *t-Baxter operator* is a linear map $\xi : A \to A$ verifying

$$\xi(x)\xi(y) = \xi(x\xi(y) + \xi(x)y + txy). \tag{2.10}$$

For $t = 0$, this map is called a Baxter operator. It appears originally in a work of Baxter [3] and the importance of such a map was stressed by Rota in [15]. We present another way to produce dendriform trialgebras. Let (A, μ) be a unital associative algebra with unit i. The linear map $\gamma : A \to A$ is said to be a TD-*operator* if

$$\gamma(x)\gamma(y) = \gamma(\gamma(x)y + x\gamma(y) - x\gamma(i)y), \tag{2.11}$$

for all $x, y \in A$.

Proposition 2.6. *Let A be a unital algebra with unit i. Suppose $\gamma : A \to A$ is a* TD-*operator. Define three binary operations $\prec_\gamma, \succ_\gamma, \circ_\gamma : A^{\otimes 2} \to A$ as follows:*

$$x \prec_\gamma y = x\gamma(y), \quad x \succ_\gamma y = \gamma(x)y, \quad x \circ_\gamma y = -x\gamma(i)y, \quad \forall\, x, y \in A. \tag{2.12}$$

Then, $A^{\gamma} := (A, \prec_{\gamma}, \succ_{\gamma}, \circ_{\gamma})$ is a dendriform trialgebra. The operation $\bar{\star}_{\gamma} : A^{\otimes 2} \to A$ defined by

$$x \bar{\star}_{\gamma} y := x\gamma(y) + \gamma(x)y - x\gamma(i)y \tag{2.13}$$

is associative.

Proof. Straightforward by noticing that $\gamma(i)\gamma(x) = \gamma(x)\gamma(i)$, for all $x \in A$. □

3. Construction of Nijenhuis operators and TD-operators from dendriform-Nijenhuis bialgebras. In [1], Baxter operators are constructed from infinitesimal bialgebras (ϵ-bialgebras). This idea has been used in [2, 9] to produce commuting t-Baxter operators. Recall that a *t-infinitesimal bialgebra* ($\epsilon(t)$-bialgebra) is a triple (A, μ, Δ), where (A, μ) is an associative algebra and (A, Δ) is a coassociative coalgebra such that for all $a, b \in A$,

$$\Delta(ab) = a_{(1)} \otimes a_{(2)}b + ab_{(1)} \otimes b_{(2)} + ta \otimes b. \tag{3.1}$$

If $t = 0$, an $\epsilon(t)$-bialgebra is called an ϵ-bialgebra. Such bialgebras appeared for the first time in the work of Joni and Rota in [8], see also Aguiar [1] for the case $t = 0$ and Loday [12] for the case $t = -1$.

To produce Nijenhuis operators from bialgebras, we replace the term $ta \otimes b$ in the definition of $\epsilon(t)$-bialgebras by the term $-\mu(\Delta(a)) \otimes b$. A physical interpretation of this term can be the following. The two-body system $\mu(a \otimes b) := ab$, for instance, a two-particle system or a string of letters in informatics, made out from a and b, is sounded by a coproduct Δ, representing a physical system. This coproduct "reads sequentially" the system $\mu(a \otimes b)$ giving the systems $\Delta(a)b$ and $a\Delta(b)$. In the case of $\epsilon(t)$-bialgebras, $\Delta(ab)$ studies the behavior between the "sequential reading" $\Delta(a)b$ and $a\Delta(b)$, and $ta \otimes b$ which can be interpreted as the system a decorrelated with system b. In the replacement, $ta \otimes b$ by $\mu(\Delta(a)) \otimes b$, we want to compare the sequential reading $\Delta(a)b$ and $a\Delta(b)$ to the decorrelated system $\mu(\Delta(a)) \otimes b$ made out with b and the system obtained from the recombination of the pieces created by reading the system a.

Definition 3.1 (dendriform-Nijenhuis bialgebra)**.** A *dendriform-Nijenhuis bialgebra* is a triple (A, μ, Δ), where (A, μ) is an associative algebra and (A, Δ) is a coassociative coalgebra such that

$$\Delta(ab) := \Delta(a)b + a\Delta(b) - \mu(\Delta(a)) \otimes b, \quad \forall a, b \in A. \tag{3.2}$$

The k-vector space $\mathrm{End}(A)$ of linear endomorphisms of A is viewed as an associative algebra under composition denoted simply by concatenation TS, for $T, S \in \mathrm{End}(A)$. Another operation called the convolution product $*$ defined by $T * S := \mu(T \otimes S)\Delta$, for all $T, S \in \mathrm{End}(A)$, will be used.

Proposition 3.2. *Let (A, μ, Δ) be a dendriform-Nijenhuis bialgebra. Equip* $\mathrm{End}(A)$ *with the convolution product $*$. Define the operators $\beta, \gamma : \mathrm{End}(A) \to \mathrm{End}(A)$ by $T \mapsto \beta(T) := id * T$ (right shift) and $T \mapsto \gamma(T) := T * id$ (left shift). Then, the right shift β is a Nijenhuis operator and the left shift γ is a* TD*-operator. Moreover, $\beta\gamma = \gamma\beta$.*

Proof. We show that the right shift $\beta : T \mapsto id * T$ is a Nijenhuis operator. Fix $T,S \in \mathrm{End}(A)$ and $a \in A$. On the one hand,

$$\begin{aligned}\beta(T)\beta(S)(a) &= \mu(id\otimes T)\Delta(a_{(1)}S(a_{(2)}))\\ &= \mu(id\otimes T)(a_{(1)(1)}\otimes a_{(1)(2)}S(a_{(2)})+a_{(1)}S(a_{(2)})_{(1)}\otimes S(a_{(2)})_{(2)}\\ &\qquad\qquad - a_{(1)(1)}a_{(1)(2)}\otimes S(a_{(2)}))\\ &= a_{(1)(1)}T(a_{(1)(2)}S(a_{(2)}))+a_{(1)}S(a_{(2)})_{(1)}T(S(a_{(2)})_{(2)})\\ &\quad - a_{(1)(1)}a_{(1)(2)}T(S(a_{(2)})).\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}\beta(\beta(T)S)(a) &= \mu(id\otimes\beta(T)S)(a_{(1)}\otimes a_{(2)})\\ &= a_{(1)}S(a_{(2)})_{(1)}T(S(a_{(2)})_{(2)}),\\ \beta(T\beta(S))(a) &= \mu(id\otimes T\beta(S))(a_{(1)}\otimes a_{(2)})\\ &= a_{(1)}T(a_{(2)(1)}S(a_{(2)(2)}))\\ &= a_{(1)(1)}T(a_{(1)(2)}S(a_{(2)})),\end{aligned} \tag{3.4}$$

since Δ is coassociative. Moreover,

$$\begin{aligned}\beta(\beta(TS))(a) &= \mu(id\otimes\beta(TS))(a_{(1)}\otimes a_{(2)})\\ &= a_{(1)}\beta(TS)(a_{(2)})\\ &= a_{(1)}\mu(id\otimes TS)(a_{(2)(1)}\otimes a_{(2)(2)})\\ &= a_{(1)}a_{(2)(1)}TS(a_{(2)(2)})\\ &= a_{(1)(1)}a_{(1)(2)}TS(a_{(2)}),\end{aligned} \tag{3.5}$$

since Δ is coassociative. Similarly, we can show that the left shift $\gamma : T \mapsto T * id$ is a TD-operator. □

An *L-antidipterous* algebra $(A,\bowtie,\prec_A)$ is an associative algebra $(A,\bowtie)$ equipped with a right module structure on itself, that is, $(x\prec_A y)\prec_A z = x\prec_A(y\bowtie z)$ and such that $\bowtie$ and $\prec_A$ are linked by the relation $(x\bowtie y)\prec_A z = x\bowtie(y\prec_A z)$. This notion has been introduced in [10] and comes from a particular notion of bialgebras. See also [13].

Proposition 3.3. *Let (A,μ,Δ) be a dendriform-Nijenhuis bialgebra. Set $x\bowtie y := \mu(\Delta(x))y$ and $x\prec_A y := x\mu(\Delta(y))$, for all $x,y\in A$. Then, the k-vector space $(A,\bowtie,\prec_A)$ is an L-antidipterous algebra.*

Proof. Straightforward by using $\mu(\Delta(xy)) = x\mu(\Delta(y))$, for all $x,y\in A$. □

Let (A,μ,Δ) be a dendriform-Nijenhuis bialgebra. We end this subsection by showing that A can admit a left counit and that the k-vector space $\mathrm{Der}(A)$ of derivatives from A to A is stable by the right shift β.

Proposition 3.4. *Let (A,μ,Δ) be a dendriform-Nijenhuis bialgebra. As a k-algebra, set $A := k\langle S\rangle/\mathcal{R}$, where S is a nonempty set and $\mathcal{R}$ is a set of relations. Suppose $\eta : A \to k$ is a μ-homomorphism and $(\eta\otimes id)\Delta := id$ on S. Then, η is a left counit, that is, $(\eta\otimes id)\Delta := id$ on A.*

Proof. Fix $a,b\in A$. Suppose $\eta : A\to k$ is a μ-homomorphism and $(\eta\otimes id)\Delta(a) := a$ and $(\eta\otimes id)\Delta(b) := b$. Then,

$$(\eta\otimes id)\Delta(ab) := \eta(a_{(1)})a_{(2)}b + \eta(a)\eta(b_1)b_2 - \eta(a_{(1)})\eta(a_{(2)})b. \tag{3.6}$$

However, $\eta(a_{(1)})a_{(2)}b := ab$, $\eta(a)\eta(b_1)b_2 := \eta(a)b$, and $\eta(a_{(1)})\eta(a_{(2)})b := \eta(\eta(a_{(1)})a_{(2)})b := \eta(a)b$. Therefore, $(\eta\otimes id)\Delta(ab) := ab$. □

Proposition 3.5. *Let (A,μ,Δ) be a dendriform-Nijenhuis bialgebra. If $\partial : A\to k$ is a derivative, that is, $\partial(xy) := \partial(x)y + x\partial(y)$, then so is the linear map $\beta(\partial) : x \mapsto \mu(id\otimes\partial)\Delta(x)$.*

Proof. Straightforward. □

4. Examples

Proposition 4.1. *Let S be a set. Suppose $\Delta : kS \to kS^{\otimes 2}$ is a coassociative coproduct on the free k-vector space spanned by S. Denote by $\mathrm{As}(S)$ the free associative algebra generated by S and extend the coproduct Δ to $\Delta_\sharp : \mathrm{As}(S)\to \mathrm{As}(S)^{\otimes 2}$ as follows:*

$$\begin{aligned} \Delta_\sharp(s) &:= \Delta(s), \quad \forall s\in kS,\\ \Delta_\sharp(ab) &:= \Delta_\sharp(a)b + a\Delta_\sharp(b) - \mu(\Delta_\sharp(a))\otimes b, \end{aligned} \tag{4.1}$$

for all $a,b\in\mathrm{As}(S)$. Then, $(\mathrm{As}(S),\Delta_\sharp)$ is a dendriform-Nijenhuis bialgebra.

Proof. Keep the notation of Proposition 4.1. The cooperation $\Delta_\sharp$ is well defined since it does not depend on the writing of a given element $c\in\mathrm{As}(S)$. Indeed, it is straightforward to show that if $c = ab = a'b' \in \mathrm{As}(S)$, with $a,b,a',b'\in\mathrm{As}(S)$. Then,

$$\begin{aligned} \Delta_\sharp(c) &:= \Delta_\sharp(a)b + a\Delta_\sharp(b) - \mu(\Delta_\sharp(a))\otimes b\\ &= \Delta_\sharp(a')b' + a'\Delta_\sharp(b') - \mu(\Delta_\sharp(a'))\otimes b'. \end{aligned} \tag{4.2}$$

We show that $\Delta_\sharp$ is coassociative. Let $a,b\in\mathrm{As}(S)$. Write $\Delta_\sharp(a) := a_{(1)}\otimes a_{(2)}$ and $\Delta_\sharp(b) := b_{(1)}\otimes b_{(2)}$. Suppose $(id\otimes\Delta_\sharp)\Delta_\sharp(x) = (\Delta_\sharp\otimes id)\Delta_\sharp(x)$, for $x = a,b$. By definition, $\Delta_\sharp(ab) := \Delta_\sharp(a)b + a\Delta_\sharp(b) - \mu(\Delta_\sharp(a))\otimes b$. On the one hand,

$$\begin{aligned} (id\otimes\Delta_\sharp)\Delta_\sharp(ab) &:= a_{(1)}\otimes\Delta_\sharp(a_{(2)}b) + ab_{(1)}\otimes\Delta_\sharp(b_{(2)}) - a_{(1)}a_{(2)}\otimes\Delta_\sharp(b)\\ &= a_{(1)}\otimes\Delta_\sharp(a_{(2)})b + a_{(1)}\otimes a_{(2)}b_{(1)}\otimes b_{(2)}\\ &\quad - a_{(1)}\otimes a_{(2)(1)}a_{(2)(2)}\otimes b\\ &\quad + ab_{(1)}\otimes\Delta_\sharp(b_{(2)}) - a_{(1)}a_{(2)}\otimes\Delta_\sharp(b). \end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned}(\Delta_\sharp\otimes id)\Delta_\sharp(ab) :&= \Delta_\sharp(a_{(1)})\otimes a_{(2)}b+\Delta_\sharp(ab_{(1)})\otimes b_{(2)}-\Delta_\sharp(a_{(1)}a_{(2)})\otimes b\\ &= \Delta_\sharp(a_{(1)})\otimes a_{(2)}b+a_{(1)}\otimes a_{(2)}b_{(1)}\otimes b_{(2)}\\ &\quad + a\Delta_\sharp(b_{(1)})\otimes b_{(2)}-a_{(1)}a_{(2)}\otimes\Delta_\sharp(b)\\ &\quad -\Delta_\sharp(a_{(1)})a_{(2)}\otimes b-a_{(1)}\Delta_\sharp(a_{(2)})\otimes b\\ &\quad + a_{(1)(1)}a_{(1)(2)}\otimes a_{(2)}\otimes b.\end{aligned}\tag{4.4}$$

The two equations are equal since $(id\otimes\Delta_\sharp)\Delta_\sharp(x)=(\Delta_\sharp\otimes id)\Delta_\sharp(x)$, for $x=a,b$. Since $\Delta_\sharp$ is supposed to be coassociative on kS and $\mathrm{As}(S)$ is the free associative algebra generated by S, $\Delta_\sharp$ is coassociative on the whole $\mathrm{As}(S)$. □

Example 4.2 (dendriform-Nijenhuis bialgebras from duplications)**.** Keep the notation of Proposition 4.1. Define the coproduct $\Delta : kS\to kS^{\otimes 2}$ as follows:

$$\Delta(s):=s\otimes s.\tag{4.5}$$

Then, $(\mathrm{As}(S),\Delta_\sharp)$ is a dendriform-Nijenhuis bialgebra. For instance,

$$\Delta(s_1s_2)=s_1\otimes s_1s_2+s_1s_2\otimes s_2-s_1^2\otimes s_2,\quad \forall s_1,s_2\in S.\tag{4.6}$$

5. Dendriform-Nijenhuis algebras

Definition 5.1 (dendriform-Nijenhuis algebra)**.** A *dendriform-Nijenhuis algebra* is a k-vector space DN equipped with nine operations $\nearrow,\searrow,\swarrow,\nwarrow,\uparrow,\downarrow,\tilde{\prec},\tilde{\succ},\tilde{\bullet} : \mathrm{DN}^{\otimes 2}\to\mathrm{DN}$ verifying 28 relations. To ease notation, seven sum operations are introduced:

$$\begin{aligned} x\triangleleft y &= x\nwarrow y+x\swarrow y+x\,\tilde{\prec}\,y,\\ x\triangleright y &= x\nearrow y+x\searrow y+x\,\tilde{\succ}\,y,\\ x\,\bar{\bullet}\,y &= x\uparrow y+x\downarrow y+x\,\tilde{\bullet}\,y,\\ x\wedge y &= x\nwarrow y+x\nearrow y+x\uparrow y,\\ x\vee y &= x\searrow y+x\swarrow y+x\downarrow y,\\ x\,\tilde{\star}\,y &= x\,\tilde{\prec}\,y+x\,\tilde{\succ}\,y+x\,\tilde{\bullet}\,y,\\ x\,\bar{\star}\,y &= x\nearrow y+x\searrow y+x\swarrow y+x\nwarrow y\\ &\quad +x\uparrow y+x\downarrow y+x\,\tilde{\prec}\,y+x\,\tilde{\succ}\,y+x\,\tilde{\bullet}\,y.\end{aligned}\tag{5.1}$$

That is,

$$x\,\bar{\star}\,y=x\triangleleft y+x\triangleright y+x\,\bar{\bullet}\,y=x\wedge y+x\vee y+x\,\tilde{\star}\,y,\tag{5.2}$$

for all $x,y,z\in A$. The 28 relations are presented in two matrices. The first one is a 7×3 matrix denoted by $(M^1_{ij})_{(i:=1,\dots,7;\,j:=1,\dots,3)}$; the second one is a 7×1 matrix denoted by $(M_i^2)_{(i:=1,\dots,7)}$:

$$
\begin{gathered}
(x \nwarrow y) \nwarrow z = x \nwarrow (y \mathbin{\bar\star} z), \qquad (x \nearrow y) \nwarrow z = x \nearrow (y \triangleleft z), \qquad (x \wedge y) \nearrow z = x \nearrow (y \triangleright z), \\
(x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), \qquad (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), \qquad (x \vee y) \nearrow z = x \searrow (y \nearrow z), \\
(x \triangleleft y) \swarrow z = x \swarrow (y \vee z), \qquad (x \triangleright y) \swarrow z = x \searrow (y \swarrow z), \qquad (x \mathbin{\bar\star} y) \searrow z = x \searrow (y \searrow z), \\
(x \swarrow y) \mathbin{\tilde\prec} z = x \swarrow (y \mathbin{\tilde\star} z), \qquad (x \searrow y) \mathbin{\tilde\prec} z = x \searrow (y \mathbin{\tilde\prec} z), \qquad (x \vee y) \mathbin{\tilde\succ} z = x \searrow (y \mathbin{\tilde\succ} z), \\
(x \nwarrow y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\prec} (y \vee z), \qquad (x \nearrow y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\succ} (y \swarrow z), \qquad (x \wedge y) \mathbin{\tilde\succ} z = x \mathbin{\tilde\succ} (y \searrow z), \\
(x \mathbin{\tilde\prec} y) \nwarrow z = x \mathbin{\tilde\prec} (y \wedge z), \qquad (x \mathbin{\tilde\succ} y) \nwarrow z = x \mathbin{\tilde\succ} (y \nwarrow z), \qquad (x \mathbin{\tilde\star} y) \nearrow z = x \mathbin{\tilde\succ} (y \nearrow z), \\
(x \mathbin{\tilde\prec} y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\prec} (y \mathbin{\tilde\star} z), \qquad (x \mathbin{\tilde\succ} y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\succ} (y \mathbin{\tilde\prec} z), \qquad (x \mathbin{\tilde\star} y) \mathbin{\tilde\succ} z = x \mathbin{\tilde\succ} (y \mathbin{\tilde\succ} z), \\
(x \wedge y) \uparrow z + (x \uparrow y) \nwarrow z = x \nearrow (y \mathbin{\bar\bullet} z) + x \uparrow (y \mathbin{\bar\star} z), \\
(x \vee y) \uparrow z + (x \downarrow y) \nwarrow z = x \searrow (y \uparrow z) + x \downarrow (y \wedge z), \\
(x \mathbin{\bar\star} y) \downarrow z + (x \mathbin{\bar\bullet} y) \swarrow z = x \searrow (y \downarrow z) + x \downarrow (y \vee z), \\
(x \vee y) \mathbin{\tilde\bullet} z + (x \downarrow y) \mathbin{\tilde\prec} z = x \searrow (y \mathbin{\tilde\bullet} z) + x \downarrow (y \mathbin{\tilde\star} z), \\
(x \wedge y) \mathbin{\tilde\bullet} z + (x \uparrow y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\succ} (y \downarrow z) + x \mathbin{\tilde\bullet} (y \vee z), \\
(x \mathbin{\tilde\star} y) \uparrow z + (x \mathbin{\tilde\bullet} y) \nwarrow z = x \mathbin{\tilde\succ} (y \uparrow z) + x \mathbin{\tilde\bullet} (y \wedge z), \\
(x \mathbin{\tilde\star} y) \mathbin{\tilde\bullet} z + (x \mathbin{\tilde\bullet} y) \mathbin{\tilde\prec} z = x \mathbin{\tilde\succ} (y \mathbin{\tilde\bullet} z) + x \mathbin{\tilde\bullet} (y \mathbin{\tilde\star} z).
\end{gathered}
\tag{5.3}
$$

The *vertical structure* of the dendriform-Nijenhuis algebra DN is, by definition, the k-vector space $\mathrm{DN}_v := (\mathrm{DN}, \triangleleft, \triangleright, \bar\bullet)$ (which will turn out to be an NS-algebra). Its *horizontal structure* is by definition the k-vector space $\mathrm{DN}_h := (\mathrm{DN}, \wedge, \vee, \tilde\star)$ (which will turn out to be a dendriform trialgebra).

Remark 5.2 (opposite structure of a dendriform-Nijenhuis algebra)**.** There exists a symmetry letting the two matrices of relations of a dendriform-Nijenhuis algebra be globally invariant. This allows the construction of the so-called *opposite structure*, defined as follows:

$$
\begin{gathered}
x \searrow^{\mathrm{op}} y = y \nwarrow x, \qquad x \nearrow^{\mathrm{op}} y = y \swarrow x, \\
x \nwarrow^{\mathrm{op}} y = y \searrow x, \qquad x \swarrow^{\mathrm{op}} y = y \nearrow x, \qquad x \uparrow^{\mathrm{op}} y = y \downarrow x, \\
x \downarrow^{\mathrm{op}} y = y \uparrow x, \qquad x \mathbin{\tilde\succ}^{\mathrm{op}} y = y \mathbin{\tilde\prec} x, \qquad x \mathbin{\tilde\bullet}^{\mathrm{op}} y = y \mathbin{\tilde\bullet} x, \qquad x \mathbin{\tilde\prec}^{\mathrm{op}} y = y \mathbin{\tilde\succ} x.
\end{gathered}
\tag{5.4}
$$

Therefore, $x \triangleleft^{\mathrm{op}} y = y \triangleright x$, $x \triangleright^{\mathrm{op}} y = y \triangleleft x$, $x \vee^{\mathrm{op}} y = y \wedge x$, and $x \wedge^{\mathrm{op}} y = y \vee x$. A dendriform-Nijenhuis algebra is said to be *commutative* when it coincides with its opposite. For any $x, y \in T$, observe that $x \mathbin{\bar\star} y := y \mathbin{\bar\star} x$. Observe that the matrix of relations M^1 has three centers of symmetry. The first one M_{22}^1 corresponds to the first block of three rows; the second one M_{52}^1 to the second block of three rows. The last one is M_{72}^1. There are also three centers of symmetry for the matrix of relations M^2. The first one M_2^2 corresponds to the first block of three rows; the second one, M_5^2, to the second block of three rows. The last one is M_7^2.

Theorem 5.3. *Keep the notation of Definition 5.1. Let* DN *be a dendriform-Nijenhuis algebra. Then, its vertical structure* $\mathrm{DN}_v := (\mathrm{DN}, \triangleleft, \triangleright, \bar{\bullet})$ *is an* NS*-algebra and its horizontal structure* $\mathrm{DN}_h := (\mathrm{DN}, \wedge, \vee, \bar{\star})$ *is a dendriform trialgebra.*

Proof. Let DN be a dendriform-Nijenhuis algebra. The vertical structure $\mathrm{DN}_v := (\mathrm{DN}, \triangleleft, \triangleright, \bar{\bullet})$ is an NS-algebra. Indeed, for all $x, y, z \in \mathrm{DN}$,

$$\begin{aligned}
&\sum_{i=1,2,3,7} M^1_{i1} - \sum_{i=4,5,6} M^1_{i1} \Longleftrightarrow (x \triangleleft y) \triangleleft z = x \triangleleft (y \bar{\star} z),\\
&\sum_{i=1,2,3,7} M^1_{i2} - \sum_{i=4,5,6} M^1_{i2} \Longleftrightarrow (x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z),\\
&\sum_{i=1,2,3,7} M^1_{i3} - \sum_{i=4,5,6} M^1_{i3} \Longleftrightarrow (x \bar{\star} y) \triangleright z = x \triangleright (y \triangleright z),\\
&\sum_{i=1,2,3,7} M^2_{i} - \sum_{i=4,5,6} M^2_{i} \Longleftrightarrow (x \bar{\star} y) \bar{\bullet} z + (x \bar{\bullet} y) \triangleleft z = x \triangleright (y \bar{\bullet} z) + x \bar{\bullet} (y \bar{\star} z).
\end{aligned} \tag{5.5}$$

The horizontal structure $\mathrm{DN}_h := (\mathrm{DN}, \wedge, \vee, \bar{\star})$ is a dendriform trialgebra. Indeed, for all $x, y, z \in \mathrm{DN}$,

$$\begin{aligned}
&\sum_{j=1,2,3} M^1_{1j} - M^2_1 \Longleftrightarrow (x \wedge y) \wedge z = x \wedge (y \bar{\star} z),\\
&\sum_{j=1,2,3} M^1_{2j} - M^2_2 \Longleftrightarrow (x \vee y) \wedge z = x \vee (y \wedge z),\\
&\sum_{j=1,2,3} M^1_{3j} - M^2_3 \Longleftrightarrow (x \bar{\star} y) \vee z = x \vee (y \vee z),\\
&\sum_{j=1,2,3} M^1_{4j} - M^2_4 \Longleftrightarrow (x \vee y) \tilde{\star} z = x \vee (y \tilde{\star} z),\\
&\sum_{j=1,2,3} M^1_{5j} - M^2_5 \Longleftrightarrow (x \wedge y) \tilde{\star} z = x \tilde{\star} (y \vee z),\\
&\sum_{j=1,2,3} M^1_{6j} - M^2_6 \Longleftrightarrow (x \tilde{\star} y) \wedge z = x \tilde{\star} (y \wedge z),\\
&\sum_{j=1,2,3} M^1_{7j} - M^2_7 \Longleftrightarrow (x \tilde{\star} y) \tilde{\star} z = x \tilde{\star} (y \tilde{\star} z).
\end{aligned} \tag{5.6}$$

□

Remark 5.4. From a categorical point of view, Theorem 5.3 gives two functors F_v and F_h represented in the following diagram:

$$\begin{tikzcd}
\text{Dend Nijenhuis} \arrow[r, "F_v"] \arrow[d, "F_h"'] \arrow[dr, "f"] & \text{NS} \arrow[d, "F_1"] \\
\text{TriDend} \arrow[r, "F_2"] & \text{As}
\end{tikzcd} \tag{5.7}$$

This diagram commutes, that is, $F_2F_h = f = F_1F_v$.

Definition 5.5. Let $(\mathcal{N}, \prec, \succ, \bullet)$ be an NS-algebra. Set $N(2) := k\{\prec, \succ, \bullet\}$. A TD-operator γ on $\mathcal{N}$ is a linear map $\gamma : \mathcal{N} \to \mathcal{N}$ such that

(1) there exist $i \in \mathcal{N}$ and two linear maps

$$\begin{aligned} \cdot_1 : \mathcal{N} \otimes k\langle \gamma(i) \rangle &\longrightarrow \mathcal{N}, & x \otimes \lambda\gamma(i) &\longmapsto \lambda x \cdot_1 \gamma(i), \\ \cdot_2 : k\langle \gamma(i) \rangle \otimes \mathcal{N} &\longrightarrow \mathcal{N}, & \lambda\gamma(i) \otimes x &\longmapsto \lambda\gamma(i) \cdot_2 x, \end{aligned} \tag{5.8}$$

such that $x \diamond \gamma(i) \cdot_2 y = x \cdot_1 \gamma(i) \diamond y = -x \tilde{\diamond} y$, for all $\diamond \in N(2)$ and $x, y \in \mathcal{N}$;

(2) in addition,

$$\gamma(x) \diamond \gamma(y) = \gamma\big(\gamma(x) \diamond y + x \diamond \gamma(y) + x \tilde{\diamond} y\big), \tag{5.9}$$

for all $x, y \in \mathcal{N}$ and $\gamma(x) \cdot_1 \gamma(i) = \gamma(i) \cdot_2 \gamma(x)$;

(3) for all $\diamond \in N(2)$, $x, y \in \mathcal{N}$, $\gamma(i) \cdot_2 \gamma(x) \diamond y = \gamma(i) \cdot_2 (\gamma(x) \diamond y)$ and $x \diamond \gamma(y) \cdot_1 \gamma(i) = (x \diamond \gamma(y)) \cdot_1 \gamma(i)$.

Proposition 5.6. *Let (A, μ) be a unital associative algebra with unit i. Suppose there exists a Nijenhuis operator $\beta : (A, \mu) \to (A, \mu)$ which commutes with a* TD*-operator $\gamma : (A, \mu) \to (A, \mu)$. Suppose $\gamma(i) = \beta(i)$. Then, γ is a* TD*-operator on the* NS*-algebra A^β.*

Proof. By Proposition 2.3, A^β is an NS-algebra. By applying Proposition 2.4, Definition 5.5(1) and (3) hold since $\gamma(i) = \beta(i)$ and $\gamma(i)\gamma(x) = \gamma(x)\gamma(i)$, for all $x \in A$. Fix $x, y \in A$. Then

$$\begin{aligned} \gamma(x) \prec_\beta \gamma(y) &= \gamma(x)\beta(\gamma(y)) = \gamma(x)\gamma(\beta(y)) \\ &= \gamma\big(\gamma(x)\beta(y) + x\gamma(\beta(y)) - x\gamma(i)\beta(y)\big) \\ &= \gamma\big(\gamma(x) \prec_\beta y + x \prec_\beta \gamma(y) + x \tilde{\prec}_\beta y\big). \end{aligned} \tag{5.10}$$

Checking the three other equations is straightforward. □

Remark 5.7. Observe that $\gamma(x) \star_\beta \gamma(y) = \gamma(\gamma(x) \star_\beta y + x \star_\beta \gamma(y) + x \tilde{\star}_\beta y)$ and thus $x \bar{\star} y := \gamma(x) \star_\beta y + x \star_\beta \gamma(y) + x \tilde{\star}_\beta y$ is an associative product, or that $\gamma : (A, \bar{\star}) \to (A, \star_\beta)$ is a morphism of associative algebras.

Proposition 5.8. *Let $(\mathcal{N}, \prec, \succ, \bullet)$ be an* NS*-algebra, with $\star \to \prec + \succ + \bullet$, and let γ be a* TD*-operator on $\mathcal{N}$. Denote by $i \in \mathcal{N}$ the element which verifies items (1), (2) and (3) of Definition 5.5(1), (2), and (3). For all $x, y \in \mathcal{N}$, define nine operations as follows:*

$$\begin{gathered} x \searrow_\gamma y = \gamma(x) \succ y, \qquad x \nwarrow_\gamma y = x \prec \gamma(y), \qquad x \nearrow_\gamma y = x \succ \gamma(y), \\ x \swarrow_\gamma y = \gamma(x) \prec y, \qquad x \uparrow_\gamma y = x \bullet \gamma(y), \qquad x \downarrow_\gamma y = \gamma(x) \bullet y, \\ x \tilde{\prec}_\gamma y = -x\gamma(i) \prec y\,(= -x \prec \gamma(i) y), \\ x \tilde{\succ}_\gamma y = -x \succ \gamma(i) y\,(= -x\gamma(i) \succ y), \\ x \tilde{\bullet}_\gamma y = -x\gamma(i) \bullet y\,(= -x \bullet \gamma(i) y), \end{gathered} \tag{5.11}$$

for all $x, y \in A$. Define also sum operations as follows:

$$
\begin{gathered}
x \triangleright_\gamma y = x \searrow_\gamma y + x \nearrow_\gamma y + x \tilde{\succ}_\gamma y = \gamma(x) \succ y + x \succ \gamma(y) + x \tilde{\succ}_\gamma y,\\
x \triangleleft_\gamma y = x \nwarrow_\gamma y + x \swarrow_\gamma y + x \tilde{\prec}_\gamma y = \gamma(x) \prec y + x \prec \gamma(y) + x \tilde{\prec}_\gamma y,\\
x \bar{\bullet}_\gamma y = x \uparrow_\gamma y + x \downarrow_\gamma y + x \tilde{\bullet}_\gamma y = x \bullet \gamma(y) + \gamma(x) \bullet y + x \tilde{\bullet}_\gamma y,\\
x \vee_\gamma y = x \searrow_\gamma y + x \swarrow_\gamma y + x \downarrow_\gamma y = \gamma(x) \star y,\\
x \wedge_\gamma y = x \nearrow_\gamma y + x \nwarrow_\gamma y + x \uparrow_\gamma y = x \star \gamma(y),\\
x \tilde{\star}_\gamma y = x \tilde{\prec}_\gamma y + x \tilde{\succ}_\gamma y + x \tilde{\bullet}_\gamma y,\\
x \bar{\star}_\gamma y = x \vee_\gamma y + x \wedge_\gamma y + x \tilde{\star}_\gamma y = x \triangleright_\gamma y + x \triangleleft_\gamma y + x \bar{\bullet}_\gamma y.
\end{gathered}
\tag{5.12}
$$

Equipped with these nine operations, $\mathcal{N}$ is a dendriform-Nijenhuis algebra.

Proof. We check the relation $M_{11}^1 := (x \nwarrow_\gamma y) \nwarrow_\gamma z = x \nwarrow_\gamma (y \bar{\star}_\gamma z)$ of Definition 5.1:

$$
\begin{aligned}
(x \nwarrow_\gamma y) \nwarrow_\gamma z &= (x \prec \gamma(y)) \prec \gamma(z)\\
&= x \prec (\gamma(y) \star \gamma(z))\\
&= x \prec \gamma(y \bar{\star}_\gamma z)\\
&= x \nwarrow_\gamma (y \bar{\star}_\gamma z).
\end{aligned}
\tag{5.13}
$$

Similarly, we check the relation $M_{51}^1 := (x \nwarrow_\gamma y) \tilde{\prec}_\gamma z = x \tilde{\prec}_\gamma (y \vee_\gamma z)$:

$$
\begin{aligned}
(x \nwarrow_\gamma y) \tilde{\prec}_\gamma z &= (x \prec \gamma(y)) \prec \gamma(i) \cdot_2 z\\
&= x \prec (\gamma(y) \star \gamma(i) \cdot_2 z)\\
&= x \prec (\gamma(y) \cdot_1 \gamma(i) \star z)\\
&= x \prec (\gamma(i) \cdot_2 \gamma(y) \star z)\\
&= x \prec \gamma(i) \cdot_2 (\gamma(y) \star z)\\
&= x \tilde{\prec}_\gamma (y \vee_\gamma z).
\end{aligned}
\tag{5.14}
$$

Checking the 26 other axioms does not present any difficulties. □

6. Transpose of a dendriform-Nijenhuis algebra

Definition 6.1 (transpose of a dendriform-Nijenhuis algebra)**.** A dendriform-Nijenhuis algebra $\mathrm{DN}_1 := (\mathrm{DN}, \searrow_1, \nwarrow_1, \nearrow_1, \swarrow_1, \uparrow_1, \downarrow_1, \tilde{\prec}_1, \tilde{\succ}_1, \tilde{\bullet}_1)$ is said to be the *transpose* of a dendriform-Nijenhuis algebra $\mathrm{DN}_2 := (\mathrm{DN}, \searrow_2, \nwarrow_2, \nearrow_2, \swarrow_2, \uparrow_2, \downarrow_2, \tilde{\prec}_2, \tilde{\succ}_2, \tilde{\bullet}_2)$ if for all $x, y \in \mathrm{DN}$,

$$
\begin{gathered}
x \searrow_1 y = x \searrow_2 y, \qquad x \nwarrow_1 y = x \nwarrow_2 y, \qquad x \nearrow_1 y = x \swarrow_2 y,\\
x \swarrow_1 y = x \nearrow_2 y, \qquad x \uparrow_1 y = x \tilde{\prec}_2 y,\\
x \downarrow_1 y = x \tilde{\succ}_2 y, \qquad x \tilde{\prec}_1 y = x \uparrow_2 y, \qquad x \tilde{\succ}_1 y = x \downarrow_2 y,\\
x \tilde{\bullet}_1 y = x \tilde{\bullet}_2 y, \qquad x \triangleright_1 y = x \vee_2 y,\\
x \triangleleft_1 y = x \wedge_2 y, \qquad x \vee_1 y = x \triangleright_2 y, \qquad x \wedge_1 y = x \triangleleft_2 y,\\
x \bar{\bullet}_1 y = x \tilde{\star}_2 y, \qquad x \tilde{\star}_1 y = x \bar{\bullet}_2 y.
\end{gathered}
\tag{6.1}
$$

Definition 6.2. A *Nijenhuis operator on a dendriform trialgebra* $(\mathrm{TD}, \prec, \succ, \circ)$ is a linear map $\beta : \mathrm{TD} \to \mathrm{TD}$ such that for all $x, y \in \mathrm{TD}$ and $\diamond \in \{\prec, \succ, \circ\}$,

$$\beta(x) \diamond \beta(y) = \beta\big(\beta(x) \diamond y + x \diamond \beta(y) - \beta(x \diamond y)\big). \tag{6.2}$$

Remark 6.3. If $\star$ denotes the associative operation of a dendriform trialgebra, then $\beta(x) \star \beta(y) = \beta(\beta(x) \star y + x \star \beta(y) - \beta(x \star y))$. Therefore, the map β is a Nijenhuis operator on the associative algebra $(\mathrm{TD}, \star)$ and a morphism of associative algebras $(A, \bar{\star}) \to (A, \star)$, where the associative operation $\bar{\star}$ is defined by $x \bar{\star} y := \beta(x) \star y + x \star \beta(y) - \beta(x \star y)$, for all $x, y \in \mathrm{TD}$.

Proposition 6.4. *Let A be a unital associative algebra. Suppose $\beta : A \to A$ is a Nijenhuis operator which commutes with a* TD*-operator $\gamma : A \to A$. Then, β is a Nijenhuis operator on the dendriform trialgebra A^{γ}.*

Proof. By Proposition 2.6, A^{γ} is a dendriform trialgebra. Fix $x, y \in A$. For instance,

$$\begin{aligned}
\beta(x) \prec_{\gamma} \beta(y) &:= \beta(x)\gamma(\beta(y)) \\
&= \beta(x)\beta(\gamma(y)) \\
&= \beta\big(\beta(x)\gamma(y) + x\beta(\gamma(y)) - \beta(x\gamma(y))\big) \\
&= \beta\big(\beta(x)\gamma(y) + x\gamma(\beta(y)) - \beta(x\gamma(y))\big) \\
&= \beta\big(\beta(x) \prec_{\gamma} y + x \prec_{\gamma} \beta(y) - \beta(x \prec_{\gamma} y)\big).
\end{aligned} \tag{6.3}$$

□

Proposition 6.5. *Let* $(\mathrm{TD}, \prec, \succ, \circ)$ *be a dendriform trialgebra and β a Nijenhuis operator. For all $x, y \in \mathrm{TD}$, define nine operations as follows:*

$$\begin{gathered}
x \searrow_{\beta} y = \beta(x) \succ y, \qquad x \nwarrow_{\beta} y = x \prec \beta(y), \qquad x \nearrow_{\beta} y = x \succ \beta(y), \\
x \swarrow_{\beta} y = \beta(x) \prec y, \qquad x \uparrow_{\beta} y = x \circ \beta(y), \qquad x \downarrow_{\beta} y = \beta(x) \circ y, \\
x \tilde{\prec}_{\beta} y = -\beta(x \prec y), \quad x \tilde{\succ}_{\beta} y = -\beta(x \succ y), \quad x \tilde{\bullet}_{\beta} y = -\beta(x \circ y), \quad \forall x, y \in A.
\end{gathered} \tag{6.4}$$

The sum operations are defined as in Definition 5.1. Then, for all $x, y, z \in \mathrm{TD}$ (the label β being omitted),

$$\begin{array}{lll}
(x \nwarrow y) \nwarrow z = x \nwarrow (y \bar{\star} z), & (x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), & (x \lhd y) \swarrow z = x \swarrow (y \vee z), \\
(x \nearrow y) \nwarrow z = x \nearrow (y \lhd z), & (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), & (x \rhd y) \swarrow z = x \searrow (y \swarrow z), \\
(x \wedge y) \nearrow z = x \nearrow (y \rhd z), & (x \vee y) \nearrow z = x \searrow (y \nearrow z), & (x \bar{\star} y) \searrow z = x \searrow (y \searrow z), \\
(x \nearrow y) \uparrow z = x \nearrow (y \bar{\bullet} z), & (x \searrow y) \uparrow z = x \searrow (y \uparrow z), & (x \rhd y) \downarrow z = x \searrow (y \downarrow z), \\
(x \nwarrow y) \uparrow z = x \uparrow (y \rhd z), & (x \swarrow y) \uparrow z = x \downarrow (y \nearrow z), & (x \lhd y) \downarrow z = x \downarrow (y \searrow z), \\
(x \uparrow y) \nwarrow z = x \uparrow (y \lhd z), & (x \downarrow y) \nwarrow z = x \downarrow (y \nwarrow z), & (x \bar{\bullet} y) \swarrow z = x \downarrow (y \swarrow z),
\end{array}$$

$$
\begin{gathered}
(x \uparrow y) \uparrow z = x \uparrow (y \bar{\bullet} z), \qquad (x \downarrow y) \uparrow z = x \downarrow (y \uparrow z), \qquad (x \bar{\bullet} y) \downarrow z = x \downarrow (y \downarrow z),\\
(x \triangleleft y) \tilde{\prec} z + (x \tilde{\prec} y) \nwarrow z = x \swarrow (y \tilde{\star} z) + x \tilde{\prec} (y \bar{\star} z),\\
(x \triangleright y) \tilde{\prec} z + (x \tilde{\succ} y) \nwarrow z = x \searrow (y \tilde{\prec} z) + x \tilde{\succ} (y \triangleleft z),\\
(x \bar{\star} y) \tilde{\succ} z + (x \tilde{\star} y) \nearrow z = x \searrow (y \tilde{\succ} z) + x \tilde{\succ} (y \triangleright z),\\
(x \triangleright y) \tilde{\bullet} z + (x \tilde{\succ} y) \uparrow z = x \searrow (y \tilde{\bullet} z) + x \tilde{\succ} (y \bar{\bullet} z),\\
(x \triangleleft y) \tilde{\bullet} z + (x \tilde{\prec} y) \uparrow z = x \downarrow (y \tilde{\succ} z) + x \tilde{\bullet} (y \triangleright z),\\
(x \bar{\bullet} y) \tilde{\prec} z + (x \tilde{\bullet} y) \nwarrow z = x \downarrow (y \tilde{\prec} z) + x \tilde{\bullet} (y \triangleleft z),\\
(x \bar{\bullet} y) \tilde{\bullet} z + (x \tilde{\bullet} y) \uparrow z = x \downarrow (y \tilde{\bullet} z) + x \tilde{\bullet} (y \bar{\bullet} z).
\end{gathered}
\tag{6.5}
$$

Otherwise stated, $(\mathrm{TD}, \searrow_\beta, \nwarrow_\beta, \nearrow_\beta, \swarrow_\beta, \uparrow_\beta, \downarrow_\beta, \tilde{\prec}_\beta, \tilde{\succ}_\beta, \tilde{\bullet}_\beta)$ *is a dendriform-Nijenhuis algebra, where* $\searrow_\beta$ *plays the role of* $\searrow$, $\nwarrow_\beta \equiv \nwarrow$, $\nearrow_\beta \equiv \swarrow$, $\swarrow_\beta \equiv \nearrow$, $\uparrow_\beta \equiv \tilde{\prec}$, $\downarrow_\beta \equiv \tilde{\succ}$, $\tilde{\prec}_\beta \equiv \uparrow$, $\tilde{\succ}_\beta \equiv \downarrow$, $\tilde{\bullet}_\beta \equiv \tilde{\bullet}$, $\triangleright_\beta \equiv \vee$, $\triangleleft_\beta \equiv \wedge$, $\vee_\beta \equiv \triangleright$, $\wedge_\beta \equiv \triangleleft$, $\bar{\bullet}_\beta \equiv \tilde{\star}$, *and* $\tilde{\star}_\beta \equiv \bar{\bullet}$.

Proof. Keep the notation of Proposition 6.5. Fix $x, y, z \in \mathrm{TD}$. We check, for instance,

$$(x \uparrow y) \uparrow z = x \uparrow (y \bar{\bullet} z), \tag{6.6}$$

the label β being omitted on the operations:

$$
\begin{aligned}
(x \uparrow y) \uparrow z &= \big(x \circ \gamma(y)\big) \circ \gamma(z)\\
&= x \circ \big(\gamma(y) \circ \gamma(z)\big)\\
&= x \circ \gamma(y \bar{\bullet} z)\\
&= x \uparrow (y \bar{\bullet} z).
\end{aligned}
\tag{6.7}
$$

□

Proposition 6.6. *Let* (A, μ) *be a unital associative algebra with unit* i. *Suppose* $\beta : A \to A$ *is a Nijenhuis operator which commutes with a* TD*-operator* $\gamma : A \to A$ *and* $\gamma(i) = \beta(i)$. *Then, the dendriform-Nijenhuis algebra obtained by the action of* β *on the dendriform trialgebra* A^γ *is the transpose of the dendriform-Nijenhuis algebra obtained by action of* γ *on the* NS*-algebra* A^β.

Proof. By Proposition 2.6, the action of the TD-operator γ on A yields a dendriform trialgebra A^γ. By Proposition 6.4, β which commutes with γ is a Nijenhuis operator on A^γ. By Proposition 6.5, A has a dendriform-Nijenhuis algebra structure. Conversely, by Proposition 2.3, the action of the Nijenhuis operator β on A yields a Nijenhuis operator A^β. By Proposition 5.6, γ is a TD-operator on A^β. The k-vector space A has then another dendriform-Nijenhuis algebra structure by Proposition 5.8. Fix $x, y \in A$ and keep the notation of Propositions 5.8 and 6.5. Observe that $x \nearrow_\gamma y := x \succ_\beta \gamma(y) := \beta(x)\gamma(y)$ and $x \swarrow_\beta y := \beta(x) \prec_\gamma y := \beta(x)\gamma(y)$. Therefore, $x \nearrow_\gamma y = x \swarrow_\beta y$ as expected in Definition 6.1. Similarly, $x \downarrow_\gamma y := \gamma(x) \bullet_\beta y := -\beta(\gamma(x)y)$ and $x \tilde{\succ}_\beta y := -\beta(x \succ_\gamma y) := -\beta(\gamma(x)y)$. Therefore, $x \downarrow_\gamma y := x \tilde{\succ}_\beta y$. Similarly, $x \tilde{\prec}_\gamma y := -x\gamma(i) \prec_\beta y := -x\gamma(i)\beta(y) := x \circ_\gamma \beta(y) := x \uparrow_\beta y$, and so forth. □

7. Dendriform-Nijenhuis algebras from dendriform-Nijenhuis bialgebras

Theorem 7.1. *Let (A,μ,Δ) be a dendriform-Nijenhuis bialgebra and consider* $\mathrm{End}(A)$ *as an associative algebra under composition, equipped with the convolution product $*$. Then, there exist two dendriform-Nijenhuis algebra structures on* $\mathrm{End}(A)$*, the one being the transpose of the other.*

Proof. By Proposition 3.2, the right shift β is a Nijenhuis operator which commutes with the left shift γ which is a TD-operator. If $id : A \to A$ denotes the identity map, then observe that $\beta(id) := id * id =: \gamma(id)$. Use Proposition 6.6 to conclude. □

Remark 7.2. Let $T,S \in \mathrm{End}(A)$. By applying Proposition 5.8, the nine operations are given by

$$\begin{aligned}
T \searrow S &= \beta\gamma(T)S = (id * T * id)S,\\
T \nwarrow S &= T\beta\gamma(S) = T(id * S * id),\\
T \nearrow S &= \beta(T)\gamma(S) = (id * T)(S * id),\\
T \swarrow S &= \gamma(T)\beta(S) = (T * id)(id * S),\\
T \uparrow S &= -\beta(T\gamma(S)) = -id * (T(S * id)),\\
T \downarrow S &= -\beta(\gamma(T)S) = -id * ((T * id)S),\\
T \tilde{\prec} S &= -T\gamma(id)\beta(S) = -T(id * id)(id * S),\\
T \tilde{\succ} S &= -\beta(T)\gamma(id)S = -(id * T)(id * id)S,\\
T \tilde{\bullet} S &= \beta(T\gamma(id)S) = id * (T(id * id)S).
\end{aligned} \tag{7.1}$$

The horizontal structure is a dendriform trialgebra given by

$$\begin{aligned}
T \wedge S = T \nearrow S + T \nwarrow S + T \uparrow S &= (id * T)(S * id)\\
&\quad + T(id * S * id) - id * (T(S * id)),\\
T \vee S = T \searrow S + T \swarrow S + T \downarrow S &= (id * T * id)S\\
&\quad + (T * id)(id * S) - id * ((T * id)S),\\
T \tilde{\star} S = T \tilde{\prec} S + T \tilde{\succ} S + T \tilde{\bullet} S &= -T(id * id)(id * S)\\
&\quad - (id * T)(id * id)S + id * (T(id * id)S).
\end{aligned} \tag{7.2}$$

The vertical structure is an NS-algebra given by

$$\begin{aligned}
T \rhd S &= T \nearrow S + T \searrow S + T \tilde{\succ} S\\
&= (id * T)(S * id) + (id * T * id)S - (id * T)(id * id)S,\\
T \lhd S &= T \nwarrow S + T \swarrow S + T \tilde{\prec} S\\
&= T(id * S * id) + (T * id)(id * S) - T(id * id)(id * S),\\
T \bullet S &= T \uparrow S + T \downarrow S + T \tilde{\bullet} S\\
&= -id * (T(S * id)) - id * ((T * id)S) + id * (T(id * id)S).
\end{aligned} \tag{7.3}$$

The associative operation $\bar{\star}$ is the sum of the nine operations, that is,

$$\begin{aligned} T\bar{\star}S = {} & (id*T)(S*id)+(id*T*id)S-(id*T)(id*id)S \\ & +T(id*S*id)+(T*id)(id*S)-T(id*id)(id*S) \\ & -id*T(S*id)-id*((T*id)S)+id*(T(id*id)S). \end{aligned} \tag{7.4}$$

8. Free NS-algebra and free dendriform-Nijenhuis algebra. We recall what an operad is; see [7, 12], for instance.

Let P be a type of algebras, for instance, the NS-algebras, and $P(V)$ the free P-algebra on the k-vector space V. Suppose $P(V) := \oplus_{n\geq 1}P(n)\otimes_{S_n} V^{\otimes n}$, where $P(n)$ are right S_n-modules. Consider P as an endofunctor on the category of k-vector spaces. The structure of the free P-algebra of $P(V)$ induces a natural transformation $\pi : P\tilde{\circ}P \to P$ as well as $u : Id \to P$ verifying usual associativity and unitarity axioms. An algebraic operad is then a triple (P,π,u). A P-algebra is then a k-vector space V together with a linear map $\pi_A : P(A) \to A$ such that $\pi_A\tilde{\circ}\pi(A) = \pi_A\tilde{\circ}P(\pi_A)$ and $\pi_A\tilde{\circ}u(A) = Id_A$. The k-vector space $P(n)$ is the space of n-ary operations for P-algebras. We will always suppose there is, up to homotheties, a unique 1-ary operation, the identity, that is, $P(1) := kId$ and that all possible operations are generated by composition from $P(2)$. The operad is said to be *binary*. It is said to be *quadratic* if all the relations between operations are consequences of relations described exclusively with the help of monomials with two operations. An operad is said to be *regular* if, in the relations, the variables x, y, z appear in the same order. The k-vector space $P(n)$ can be written as $P(n) := P'(n)\otimes k[S_n]$, where $P'(n)$ is also a k-vector space and S_n the symmetric group on n elements. In this case, the free P-algebra is entirely induced by the free P-algebra on one generator $P(k) := \oplus_{n\geq 1}P'(n)$. The generating function of the operad P is given by

$$\begin{aligned} f^P(x) &:= \sum(-1)^n\frac{\dim P(n)}{n!}x^n \\ &:= \sum(-1)^n\dim P'(n)x^n. \end{aligned} \tag{8.1}$$

Below, we will indicate the sequence $(\dim P'(n))_{n\geq 1}$.

8.1. On the free NS-algebra. Let V be a k-vector space. The *free* NS-*algebra* $\mathcal{N}(V)$ on V is, by definition, an NS-algebra equipped with a map $i : V \mapsto \mathcal{N}(V)$ which satisfies the following universal property: for any linear map $f : V \to A$, where A is an NS-algebra, there exists a unique NS-algebra morphism $\bar{f} : \mathcal{N}(V) \to A$ such that $\bar{f}\circ i = f$. The same definition holds for the *free dendriform-Nijenhuis algebra*.

Since the three operations of an NS-algebra have no symmetry and since compatibility axioms involve only monomials where x, y, and z stay in the same order, the free NS-algebra is of the form

$$\mathcal{N}(V) := \bigoplus_{n\geq 1}\mathcal{N}_n\otimes V^{\otimes n}. \tag{8.2}$$

In particular, the free NS-algebra on one generator x is $\mathcal{N}(k) := \bigoplus_{n\geq 1}\mathcal{N}_n$, where $\mathcal{N}_1 := kx$, $\mathcal{N}_2 := k(x\prec x)\oplus k(x\succ x)\oplus k(x\bullet x)$. The space of three variables made out of three operations is of dimension $2\times 3^2 = 18$. As we have four relations, the space $\mathcal{N}_3$ has a dimension equal to $18-4=14$. Therefore, the sequence associated with the dimensions of $(\mathcal{N}_n)_{n\in\mathbb{N}}$ starts with $1,3,14,\dots$. Finding the free NS-algebra on one generator is an open problem. However, we will show that the augmented free NS-algebra over a k-vector space V has a connected Hopf algebra structure. Before, some preparations are needed. To be as self-contained as possible, we introduce some notation to expose a theorem due to Loday [12].

Recall that a bialgebra $(H,\mu,\Delta,\eta,\kappa)$ is a unital associative algebra (H,μ,η) together with counital coassociative coalgebra (H,Δ,κ). Moreover, it is required that the coproduct Δ and the counit κ are morphisms of unital algebras. A bialgebra is connected if there exists a filtration $(F_rH)_r$ such that $H=\bigcup_r F_rH$, where $F_0H := k1_H$ and, for all r,

$$F_rH := \{x\in H;\ \Delta(x)-1_H\otimes x-x\otimes 1_H\in F_{r-1}H\otimes F_{r-1}H\}. \tag{8.3}$$

Such a bialgebra admits an antipode. Consequently, connected bialgebras are connected Hopf algebras.

Let P be a binary quadratic operad. By a *unit action* [12], we mean the choice of two linear applications

$$\upsilon : P(2)\longrightarrow P(1), \qquad \varpi : P(2)\longrightarrow P(1), \tag{8.4}$$

giving sense, when possible, to $x\diamond 1$ and $1\diamond x$, for all operations $\diamond\in P(2)$ and for all x in the P-algebra A, that is, $x\diamond 1=\upsilon(\diamond)(x)$ and $1\diamond x=\varpi(\diamond)(x)$. If $P(2)$ contains an associative operation, say $\bar{\star}$, then we require that $x\,\bar{\star}\,1 := x := 1\,\bar{\star}\,x$, that is, $\upsilon(\bar{\star}) := Id := \varpi(\bar{\star})$. We say that the unit action, or the couple (υ,ϖ) is *compatible* with the relations of the P-algebra A if they still hold on $A_+ := k1\oplus A$ as far as the terms are defined. Let A, B be two P-algebras such that $P(2)$ contains an associative operation $\bar{\star}$. Using the couple (υ,ϖ), we extend binary operations $\diamond\in P(2)$ to the k-vector space $A\otimes 1\cdot k\oplus k\cdot 1\otimes B\oplus A\otimes B$ by requiring

$$\begin{aligned} (a\otimes b)\diamond(a'\otimes b') &:= (a\,\bar{\star}\,a')\otimes(b\diamond b') \quad \text{if } b\otimes b'\neq 1\otimes 1,\\ (a\otimes 1)\diamond(a'\otimes 1) &:= (a\diamond a')\otimes 1 \quad \text{otherwise.}\end{aligned} \tag{8.5}$$

The unit action or the couple (υ,ϖ) is said to be *coherent* with the relations of P if $A\otimes 1\cdot k\oplus k\cdot 1\otimes B\oplus A\otimes B$, equipped with these operations, is still a P-algebra. Observe that a necessary condition for having coherence is compatibility.

One of the main interests of these two concepts is the construction of a connected Hopf algebra on the augmented free P-algebra.

Theorem 8.1 (Loday [12]). *Let P be a binary quadratic operad. Suppose there exists an associative operation in $P(2)$. Then, any unit action coherent with the relations of P equips the augmented free P-algebra $P(V)_+$ on a k-vector space V with a coassociative coproduct $\Delta : P(V)_+\to P(V)_+\otimes P(V)_+$, which is a P-algebra morphism. Moreover, $P(V)_+$ is a connected Hopf algebra.*

Proof. See [12] for the proof. However, we reproduce it to make things clearer. Let V be a k-vector space and $P(V)$ the free P-algebra on V. Since the unit action is coherent, $P(V)_+\otimes P(V)_+$ is a P-algebra. Consider the linear map $\delta : V\to P(V)_+\otimes P(V)_+$, given by $v \mapsto 1 \otimes v + v \otimes 1$. Since $P(V)$ is the free P-algebra on V, there exists a unique extension of δ to a morphism of augmented P-algebra $\Delta : P(V)_+ \to P(V)_+ \otimes P(V)_+$. Now, Δ is coassociative since the morphisms $(\Delta \otimes id)\Delta$ and $(id \otimes \Delta)\Delta$ extend the linear map $V \to P(V)_+^{\otimes 3}$ which maps v to $1 \otimes 1 \otimes v + 1 \otimes v \otimes 1 + v \otimes 1 \otimes 1$. By unicity of the extension, the coproduct Δ is coassociative. The bialgebra we have just obtained is connected. Indeed, by definition, the free P-algebra $P(V)$ can be written as $P(V) := \oplus_{n\geq 1} P(V)_n$, where $P(V)_n$ is the k-vector space of products of n elements of V. Moreover, we have $\Delta(x \diamond y) := 1 \otimes (x \diamond y) + (x \diamond y) \otimes 1 + x \otimes (1 \diamond y) + y \otimes (x \diamond 1)$, for all $x, y \in P(V)$ and $\diamond \in P(2)$. The filtration of $P(V)_+$ is then $FrP(V)_+ = k \cdot 1 \oplus \bigoplus_{1\leq n\leq r} P(V)_n$. Therefore, $P(V)_+ := \cup_r FrP(V)_+$ and $P(V)_+$ is a connected bialgebra. $\square$

We will use this theorem to show that there exists a connected Hopf algebra structure on the augmented free NS-algebra as well as on the augmented free commutative NS-algebra.

Theorem 8.2. *Let $\mathcal{N}(V)$ be the free* NS*-algebra on a k-vector space V. Extend the binary operations $\prec$, $\succ$, and $\bullet$ to $\mathcal{N}(V)_+$ as follows:*

$$x \succ 1 := 0, \qquad 1 \succ x := x, \qquad 1 \prec x := 0, \qquad x \prec 1 := x, \qquad x \bullet 1 := 0 := 1 \bullet x, \tag{8.6}$$

so that $x \star 1 = x = 1 \star x$ for all $x \in \mathcal{N}(V)$. This choice is coherent. Therefore, there exists a connected Hopf algebra structure on the augmented free NS*-algebra as well as on the augmented free commutative* NS*-algebra.*

Remark 8.3. We cannot extend the operations $\succ$ and $\prec$ to k, that is, $1 \succ 1$ and $1 \prec 1$ are not defined.

Proof of Theorem 8.2. Keep the notation introduced in Section 8.1. Firstly, we show that this choice is compatible. Let $x, y, z \in \mathcal{N}(V)_+$. We have to show, for instance, that the relation $(x \prec y) \prec z = x \prec (y \star z)$ holds in $\mathcal{N}(V)_+$. Indeed, for $x = 1$, we get $0 = 0$. For $y = 1$, we get $x \prec z = x \prec z$ and for $z = 1$, we get $x \prec y = x \prec y$. The same checking can be done for the three other equations. The augmented NS-algebra $\mathcal{N}(V)_+$ is then an NS-algebra.

Secondly, we show that this choice is coherent. Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{N}(V)_+$. We have to show that, for instance,

$$((x_1 \otimes y_1) \prec (x_2 \otimes y_2)) \prec (x_3 \otimes y_3) = (x_1 \otimes y_1) \prec ((x_2 \otimes y_2) \star (x_3 \otimes y_3)). \tag{8.7}$$

Indeed, if there exists a unique $y_i = 1$, the other variables belonging to $\mathcal{N}(V)$, then by definition we get

$$(x_1 \star x_2 \star x_3) \otimes ((y_1 \prec y_2) \prec y_3) = (x_1 \star x_2 \star x_3) \otimes (y_1 \prec (y_2 \star y_3)), \tag{8.8}$$

which always holds since our choice of the unit action is compatible. If $y_1 = y_2 = y_3 = 1$, then the equality holds since our choice is compatible. If $y_2 = 1$, $y_1 = 1$, and $y_3 \in \mathcal{N}(V)$, we get $0 = 0$, and similarly if $y_1 = 1$, $y_3 = 1$, and $y_2 \in \mathcal{N}(V)$. If $y_1 \in \mathcal{N}(V)$, $y_2 = 1$, and $y_3 = 1$, the two hand sides are equal to $(x_1 \bar{\star} x_2 \bar{\star} x_3) \otimes y_1$. Therefore, this equation holds in $\mathcal{N}(V) \otimes 1 \cdot k \oplus k \cdot 1 \otimes \mathcal{N}(V) \oplus \mathcal{N}(V) \otimes \mathcal{N}(V)$. Checking the same thing with the three other relations shows that our choice of the unit action is coherent. As our choice is coherent and from Theorem 8.1, we obtain a connected Hopf algebra structure on the augmented free NS-algebra. For the last claim, observe that our choice is in agreement with the symmetry relations defining a commutative NS-algebra since, for instance, $x \prec^{\mathrm{op}} 1 := 1 \succ x := x$ and $1 \succ^{\mathrm{op}} x := x \prec 1 := x$, for all $x \in \mathcal{N}(V)$. □

8.2. On the free dendriform-Nijenhuis algebra. The same claims hold for the free dendriform-Nijenhuis algebra. The associated operad is binary, quadratic, and regular. The free dendriform-Nijenhuis algebra on a k-vector space V is of the form

$$\mathcal{DN}(V) := \bigoplus_{n \geq 1} \mathcal{DN}_n \otimes V^{\otimes n}. \tag{8.9}$$

In particular, on one generator x,

$$\mathcal{DN}(k) := \bigoplus_{n \geq 1} \mathcal{N}_n, \tag{8.10}$$

where $\mathcal{N}_1 := kx$, $\mathcal{N}_2 := k(x \uparrow x) \oplus k(x \downarrow x) \oplus k(x \searrow x) \oplus k(x \nearrow x) \oplus k(x \swarrow x) \oplus k(x \nwarrow x) \oplus k(x \tilde{\precsim} x) \oplus k(x \tilde{\succsim} x) \oplus k(x \bullet x)$. The space of three variables made out of nine operations is of dimension $2 \times 9^2 = 162$. As we have 28 relations, the space $\mathcal{N}_3$ has a dimension equal to $162 - 28 = 134$. Therefore, the sequence associated with the dimensions of $(\mathcal{DN}_n)_{n \in \mathbb{N}}$ starts with $1, 9, 134, \ldots$. Finding the free dendriform-Nijenhuis algebra on one generator is an open problem. However, there exists a connected Hopf algebra structure on the augmented free dendriform-Nijenhuis algebra as well as on the augmented free commutative dendriform-Nijenhuis algebra.

Theorem 8.4. *Let $\mathcal{DN}(V)$ be the free dendriform-Nijenhuis algebra on a k-vector space V. Extend the binary operations $\nwarrow$ and $\searrow$ to $\mathcal{DN}(V)_+$ as follows:*

$$x \nwarrow 1 := x, \quad 1 \nwarrow x := 0, \quad 1 \searrow x := x, \quad x \searrow 1 := 0, \quad \forall x \in \mathcal{DN}(V). \tag{8.11}$$

In addition, for any other operation $\diamond \in \{\nearrow, \swarrow, \uparrow, \downarrow, \tilde{\precsim}, \tilde{\succsim}, \tilde{\bullet}\}$, choose $x \diamond 1 := 0 = 1 \diamond x$, for all $x \in \mathcal{DN}(V)$. Then,

$$x \lhd 1 = x, \qquad 1 \rhd x = x, \qquad 1 \vee x := x, \qquad x \wedge 1 := x, \qquad x \bar{\star} 1 = x = 1 \bar{\star} x. \tag{8.12}$$

Moreover, this choice is coherent. Therefore, there exists a connected Hopf algebra structure on the augmented free dendriform-Nijenhuis algebra as well as on the augmented free commutative dendriform-Nijenhuis algebra.

Remark 8.5. We cannot extend the operations $\searrow$ and $\nwarrow$ to k, that is, $1 \searrow 1$ and $1 \nwarrow 1$ are not defined.

Proof of Theorem 8.4. Keep the notation introduced in Section 8.1. Firstly, we show that this choice is compatible. Let $x, y, z \in \mathcal{DN}(V)_+$. We have to show, for instance, that the relation $(x \nwarrow y) \nwarrow z = x \nwarrow (y \bar{\star} z)$ holds in $\mathcal{DN}(V)_+$. Indeed, for $x = 1$, we get $0 = 0$; for $y = 1$ we get $x \nwarrow z = x \nwarrow z$; and for $z = 1$ we get $x \nwarrow y = x \nwarrow y$. We do the same thing with the 27 others and quickly find that the augmented dendriform-Nijenhuis algebra $\mathcal{DN}(V)_+$ is still a dendriform-Nijenhuis algebra.

Secondly, we show that this choice is coherent. Let $x_1,x_2,x_3,y_1,y_2,y_3 \in \mathcal{DN}(V)_+$. We have to show that, for instance,

$$((x_1 \otimes y_1) \nwarrow (x_2 \otimes y_2)) \nwarrow (x_3 \otimes y_3) = (x_1 \otimes y_1) \nwarrow ((x_2 \otimes y_2)\,\bar{\star}\,(x_3 \otimes y_3)). \tag{8.13}$$

Indeed, if there exists a unique $y_i = 1$, the other variables belonging to $\mathcal{DN}(V)$, then by definition we get

$$(x_1 \,\bar{\star}\, x_2 \,\bar{\star}\, x_3) \otimes (y_1 \nwarrow y_2) \nwarrow y_3 = (x_1 \,\bar{\star}\, x_2 \,\bar{\star}\, x_3) \otimes y_1 \nwarrow (y_2 \,\bar{\star}\, y_3), \tag{8.14}$$

which always holds since our choice of the unit action is compatible. The same holds for $y_1 = y_2 = y_3 = 1$. If $y_1 = 1 = y_2$ and $y_3 \in \mathcal{DN}(V)$, we get $0 = 0$ and similarly if $y_1 = 1 = y_3$ and $y_2 \in \mathcal{DN}(V)$. If $y_1 \in \mathcal{DN}(V)$ and $y_2 = 1 = y_3$, the two hand sides of (8.13) are equal to $(x_1 \,\bar{\star}\, x_2 \,\bar{\star}\, x_3) \otimes y_1$. Therefore, (8.13) holds in $\mathcal{DN}(V) \otimes 1 \cdot k \oplus k \cdot 1 \otimes \mathcal{DN}(V) \oplus \mathcal{DN}(V) \otimes \mathcal{DN}(V)$. Checking the same thing with the 27 other relations shows that our choice of the unit action is coherent. As our choice is coherent and from Theorem 8.1, we obtain a connected Hopf algebra structure on the augmented free dendriform-Nijenhuis algebra. For the last claim, observe that our choice is in agreement with the symmetry relations defining a commutative dendriform-Nijenhuis algebra since, for instance, $x \nwarrow^{\mathrm{op}} 1 := 1 \searrow x := x$ and $1 \searrow^{\mathrm{op}} x := x \nwarrow 1 := x$, for all $x \in \mathcal{DN}(V)$. □

9. Conclusion. There exists another way to produce Nijenhuis operators by defining another type of bialgebras. Instead of defining the coproduct Δ by $\Delta(ab) := \Delta(a)b + a\Delta(b) - \mu(\Delta(a)) \otimes b$ on an associative algebra (A,μ), the following definition can be chosen: $\Delta(ab) := \Delta(a)b + a\Delta(b) - a \otimes \mu(\Delta(b))$. The generalization of what was written is straightforward by observing that the right shift β becomes a TD-operator, whereas the left shift γ becomes a Nijenhuis operator.

We have linked via a bialgebraic framework dendriform trialgebras, closely related to rooted planar trees with Nijenhuis operators, closely related to differential geometry through two shift operators β and γ which place a given linear map to the right- or the left-hand side of the convolution product associated with the underling bialgebra. Studying certain combinations of these placing operators leads to defining exotic associative algebras whose associative product splits into several (nine) operations and whose sum operations present particular symmetries. In particular, these lead to the construction of a connected Hopf algebra on the augmented free algebra described by the placing operators β and γ.

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Philippe Leroux: Institut de Recherche Mathématique de Rennes, Université de Rennes I and CNRS UMR 6625, Campus de Beaulieu, 35042 Rennes Cedex, France

E-mail address: pleroux@univ-rennes1.fr