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THE DISTRIBUTION OF MAHLER'S MEASURES OF RECIPROCAL POLYNOMIALS

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We study the distribution of Mahler's measures of reciprocal polynomials with complex coefficients and bounded even degree. We discover that the distribution function associated to Mahler's measure restricted to monic reciprocal polynomials is a reciprocal (or antireciprocal) Laurent polynomial on $[1,\infty)$ and identically zero on $[0,1)$. Moreover, the coefficients of this Laurent polynomial are rational numbers times a power of π. We are led to this discovery by the computation of the Mellin transform of the distribution function. This Mellin transform is an even (or odd) rational function with poles at small integers and residues that are rational numbers times a power of π. We also use this Mellin transform to show that the volume of the set of reciprocal polynomials with complex coefficients, bounded degree, and Mahler's measure less than or equal to one is a rational number times a power of π.

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1. Introduction. The Mahler's measure of a polynomial $f(x) \in \mathbb{C}[x]$ is given by the expression

$$\mu(f) = \exp\left\{\int_0^1 \log\left|f\left(e^{2\pi i t}\right)\right| dt\right\}. \tag{1.1}$$

If $f(x)$ has degree M and factors over $\mathbb{C}$ as $f(x) = w_M \prod_{m=1}^{M}(x-\beta_m)$, then by Jensen's formula,

$$\mu(f) = \left|w_M\right| \prod_{m=1}^{M} \max\left\{1, \left|\beta_m\right|\right\}. \tag{1.2}$$

It is readily apparent that Mahler's measure is a multiplicative function on $\mathbb{C}[x]$. In this sense, Mahler's measure forms a natural height function on $\mathbb{C}[x]$. In this paper, we study the distribution of values of Mahler's measure restricted to the set of reciprocal polynomials with bounded even degree and complex coefficients.

$f(x)$ is said to be reciprocal if it satisfies the condition

$$x^M f\left(\frac{1}{x}\right) = f(x). \tag{1.3}$$

If $f(x)$ is reciprocal and $f(x) = \sum_{m=0}^{M} w_m x^m$, then it is easily seen that $w_m = w_{M-m}$ for

$m = 0,\dots,M$. The reciprocal condition also imposes a condition on the roots of $f(x)$: if $f(\alpha) = 0$, then $f(\alpha^{-1}) = 0$. If $M = 2N$, there exists a Laurent polynomial

$$p_{\mathbf{v}}(x) = v_0 + \sum_{n=1}^{N} v_n (x^n + x^{-n}) \tag{1.4}$$

such that $f(x) = x^N p_{\mathbf{v}}(x)$. We call $p_{\mathbf{v}}(x)$ the reciprocal Laurent polynomial with coefficient vector $\mathbf{v}$. The collection of reciprocal Laurent polynomials with complex coefficients forms a graded algebra.

The integral defining Mahler's measure makes sense for reciprocal Laurent polynomials, and it is easily seen that $\mu(p_{\mathbf{v}}) = \mu(f)$. It is convenient to work with reciprocal Laurent polynomials since they form an algebra (the set of reciprocal polynomials is not closed under addition). We define the *reciprocal Mahler's measure* to be the function $\mu_{\text{rec}} : \mathbb{C}^{N+1} \to \mathbb{R}$ given by

$$\mu_{\text{rec}}(\mathbf{v}) = \mu(p_{\mathbf{v}}) = \exp\left\{ \int_0^1 \log \left| v_0 + 2\sum_{n=1}^{N} v_n \cos(2\pi n t) \right| dt \right\}. \tag{1.5}$$

If $\mathbf{v} = (v_0,\dots,v_L,0,\dots,0)$ with $v_L \neq 0$, then there exist $\alpha_1,\dots,\alpha_{2L}$ not necessarily distinct nonzero complex roots of $p_{\mathbf{v}}(x)$. By reordering, if necessary, we may assume $\alpha_{L+n} = \alpha_n^{-1}$, and we may write

$$x^L p_{\mathbf{v}}(x) = v_L \prod_{n=1}^{L} (x - \alpha_n)(x - \alpha_n^{-1}), \tag{1.6}$$

and from Jensen's formula, we have

$$\mu_{\text{rec}}(\mathbf{v}) = |v_L| \prod_{n=1}^{L} \max\{|\alpha_n|, |\alpha_n^{-1}|\}. \tag{1.7}$$

From this expression, we see that for all $\mathbf{v} \in \mathbb{C}^{N+1}$ and $k \in \mathbb{C}$, the reciprocal Mahler's measure is

(i) nonnegative: $\mu_{\text{rec}}(\mathbf{v}) \geq 0$,
(ii) homogeneous: $\mu_{\text{rec}}(k\mathbf{v}) = |k|\mu_{\text{rec}}(\mathbf{v})$,
(iii) positive-definite: $\mu_{\text{rec}}(\mathbf{v}) = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

In addition, μ_{rec} is continuous as originally proved by Mahler [3].

By properties (i), (ii), and continuity, we find that μ_{rec} is a symmetric distance function in the sense of the geometry of numbers (see, for instance, the discussion in [1, Chapter IV]). μ_{rec} satisfies all the properties of a vector norm except the triangle inequality. The "unit ball" is thus not convex. Explicitly,

$$\mathcal{V}_{N+1} = \{\mathbf{v} \in \mathbb{C}^{N+1} : \mu_{\text{rec}}(\mathbf{v}) \leq 1\} \tag{1.8}$$

is a symmetric star body. By property (iii), this star body is bounded. We call $\mathcal{V}_{N+1}$ the degree N star body determined by the reciprocal Mahler's measure. One of the principal results presented here is the computation of the volume (Lebesgue measure) of $\mathcal{V}_{N+1}$.

We introduce the *monic* reciprocal Mahler's measure $\nu_{\text{rec}} : \mathbb{C}^N \to \mathbb{R}$ defined by

$$\nu_{\text{rec}}(\mathbf{b}) = \mu_{\text{rec}}\begin{pmatrix}\mathbf{b}\\ 1\end{pmatrix}. \tag{1.9}$$

Thus $\nu_{\text{rec}}(\mathbf{b})$ is the Mahler's measure of the monic reciprocal Laurent polynomial

$$\tilde{p}_{\mathbf{b}}(x) = (x^N + x^{-N}) + b_0 + \sum_{n=1}^{N-1} b_n(x^n + x^{-n}). \tag{1.10}$$

We denote Lebesgue measure on Borel subsets of $\mathbb{C}^N$ by λ_{2N} and introduce the distribution function associated with the monic reciprocal Mahler's measure $h_N(\xi) : [0,\infty) \to [0,\infty)$ given by

$$h_N(\xi) = \lambda_{2N}\{\mathbf{b} \in \mathbb{C}^N : \nu_{\text{rec}}(\mathbf{b}) \le \xi\}. \tag{1.11}$$

$h_N(\xi)$ encodes statistical information about the distribution of Mahler's measures of reciprocal polynomials with complex coefficients and even degree bounded by $2N$.

The distribution function $h_N(\xi)$ is increasing and continuous from the right. From (1.7), we see $\nu_{\text{rec}}(\mathbf{b}) \ge 1$ for all $\mathbf{b} \in \mathbb{C}^N$, and thus $h_N(\xi)$ is identically zero on $[0,1)$. In fact, $h_N(1) = 0$. To see this, suppose $\mathbf{b} \in \mathbb{C}^N$ with $\nu_{\text{rec}}(\mathbf{b}) = 1$. Then, from (1.7), $\tilde{p}_{\mathbf{b}}(x)$ has all its roots on the unit circle. Thus, if α is a root of $\tilde{p}_{\mathbf{b}}(x)$, then so is $\overline{\alpha} = \alpha^{-1}$. We find that $\mathbf{b} \in \mathbb{R}^N$, and hence the set of $\mathbf{b} \in \mathbb{C}^N$ such that $\nu_{\text{rec}}(\mathbf{b}) = 1$ has λ_{2N}-measure 0. Thus $h_N(1) = 0$, and $h_N(\xi)$ is continuous at $\xi = 1$.

We recall the definition of the Mellin transform. Given a function $g : [0,\infty) \to \mathbb{R}$, the Mellin transform of g is the function of the complex variable s given by

$$\hat{g}(s) = \int_0^\infty \xi^{-2s} g(\xi)\frac{d\xi}{\xi}. \tag{1.12}$$

We will give an explicit formula for $h_N(\xi)$ by computing its Mellin transform. We note that, since $h_N(\xi)$ is identically zero on $[0,1]$, the integral defining $\widehat{h_N}(s)$ can be written with domain of integration $[1,\infty)$.

The integral defining $\widehat{h_N}(s)$ converges in the half plane $\Re(s) > N$. To see this, we use the following consequence of Jensen's inequality:

$$\mu(f) \le \|f(x)\|_2, \tag{1.13}$$

where $\|f(x)\|_2$ is the Euclidean norm of the coefficient vector of $f(x)$. Thus from (1.10), we have

$$\nu_{\text{rec}}(\mathbf{b}) \le \left(2 + |b_0|^2 + 2|b_1|^2 + \cdots + 2|b_{N-1}|^2\right)^{1/2} \le \sqrt{2}\left(1 + |b_0|^2 + \cdots + |b_{N-1}|^2\right)^{1/2}, \tag{1.14}$$

and hence

$$\{\mathbf{b} \in \mathbb{C}^N : \nu_{\text{rec}}(\mathbf{b}) \le \xi\} \subset \left\{\mathbf{b} \in \mathbb{C}^N : \left(1 + |b_0|^2 + \cdots + |b_{N-1}|^2\right)^{1/2} \le \frac{\xi}{\sqrt{2}}\right\}. \tag{1.15}$$

The latter set is a "slice" of a solid sphere of dimension $2N+1$, and is thus a solid sphere of dimension $2N$ with radius less than $\xi/\sqrt{2}$. Thus there exists a constant C such that

$$h_N(\xi) = \lambda_{2N}\{\mathbf{b} \in \mathbb{C}^N : \nu_{\text{rec}}(\mathbf{b}) \le \xi\} \le C\xi^{2N}. \tag{1.16}$$

It follows that

$$\widehat{h_N}(s) = \int_1^\infty \xi^{-2s} h_N(\xi)\frac{d\xi}{\xi} \le C\int_1^\infty \xi^{2N-2s}\frac{d\xi}{\xi}. \tag{1.17}$$

The latter integral converges if $\Re(s) > N$, and hence $\widehat{h_N}(s)$ is defined in the half plane $\Re(s) > N$.

We follow the method introduced by Chern and Vaaler in [2] to express the volume of $\mathcal{V}_{N+1}$ in terms of the Mellin transform of $h_N(\xi)$.

Theorem 1.1. *For each positive integer N,*

$$\lambda_{2N+2}(\mathcal{V}_{N+1}) = 2\pi\widehat{h_N}(N+1). \tag{1.18}$$

Proof. The volume of $\mathcal{V}_{N+1}$ is given by

$$\lambda_{2N+2}(\mathcal{V}_{N+1}) = \int_{\mathbb{C}} \lambda_{2N}\left\{\mathbf{b} \in \mathbb{C}^N : \mu_{\text{rec}}\begin{pmatrix}\mathbf{b}\\ z\end{pmatrix} \le 1\right\} d\lambda_2(z). \tag{1.19}$$

By the homogeneity of μ_{rec}, we see that

$$\begin{aligned}\lambda_{2N}\left\{\mathbf{b} \in \mathbb{C}^N : \mu_{\text{rec}}\begin{pmatrix}\mathbf{b}\\ z\end{pmatrix} \le 1\right\} &= \lambda_{2N}\left\{z\mathbf{c} \in \mathbb{C}^N : \mu_{\text{rec}}\begin{pmatrix}z\mathbf{c}\\ z\end{pmatrix} \le 1\right\}\\ &= |z|^{2N}\lambda_{2N}\left\{\mathbf{c} \in \mathbb{C}^N : \mu_{\text{rec}}\begin{pmatrix}\mathbf{c}\\ 1\end{pmatrix} \le \frac{1}{|z|}\right\}\\ &= |z|^{2N}h_N\left(\frac{1}{|z|}\right)\end{aligned} \tag{1.20}$$

and thus the integral in (1.19) can be written as

$$\int_{\mathbb{C}} |z|^{2N}h_N\left(\frac{1}{|z|}\right)d\lambda_2(z) = 2\pi\int_0^1 r^{2N+1}h_N\left(\frac{1}{r}\right)dr. \tag{1.21}$$

The domain of integration in the latter integral is $[0,1)$ since $h_N(1/r)$ is identically zero on $[1,\infty)$. By the change of variables $r = 1/\xi$, we find

$$\lambda_{2N+2}(\mathcal{V}_{N+1}) = 2\pi\int_1^\infty \xi^{-2(N+1)-1}h_N(\xi)d\xi = 2\pi\widehat{h_N}(N+1). \tag{1.22}$$

□

If we regard the integral defining $\widehat{h_N}(s)$ as a Lebesgue-Stieltjes integral, we may use integration by parts to write

$$\widehat{h_N}(s) = -\frac{\xi^{-2s}h_N(\xi)}{2s}\bigg|_1^\infty + \frac{1}{2s}\int_1^\infty \xi^{-2s}dh_N(\xi). \tag{1.23}$$

Since $h_N(1)=0$ and $h_N(\xi)$ is dominated by $C\xi^{2N}$, the first term vanishes when $\Re(s)>N$. After a change of variables, we can write

$$\widehat{h_N}(s) = \frac{1}{2s}\int_{\mathbb{C}^N} \nu_{\mathrm{rec}}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}). \tag{1.24}$$

The latter integral is interesting enough to name

$$H_N(s) = \int_{\mathbb{C}^N} \nu_{\mathrm{rec}}(\mathbf{b})^{-2s}\, d\lambda_{2N}(\mathbf{b}). \tag{1.25}$$

The bulk of this paper is committed to the discovery that $H_N(s)$ analytically continues to a rational function of s.

Theorem 1.2. *For each positive integer N, the function $H_N(s)$ extends by analytic continuation to an (even or odd) rational function. In particular,*

$$H_N(s) = \prod_{n=1}^{N} \frac{2\pi s}{s^2-n^2}. \tag{1.26}$$

Corollary 1.3. *For each positive integer N,*

$$\lambda_{2N+2}(\mathcal{V}_{N+1}) = \frac{2^N\pi^{N+1}(N+1)^N}{(2N+1)!}. \tag{1.27}$$

Proof. This follows immediately from Theorems 1.1 and 1.2. □

Corollary 1.4. *For each positive integer N, $h_N(\xi)$ is a reciprocal or antireciprocal Laurent polynomial on the domain $[1,\infty)$ and identically zero on $[0,1)$. Explicitly, if $\xi\geq 1$, then*

$$h_N(\xi) = 2^N\pi^N \sum_{n=1}^{N} \frac{(-1)^{N-n}n^N}{(N+n)!(N-n)!}\left(\xi^{-2n}+(-1)^N\xi^{2n}\right). \tag{1.28}$$

Proof. $\widehat{h_N}(s) = H_N(s)/2s$ is a rational function whose denominator is a product of distinct linear factors of the form $s-n$. We use the partial fraction decomposition to write

$$\widehat{h_N}(s) = \sum_{n=1}^{N}\left(\frac{\rho(n)}{s-n}+\frac{\rho(-n)}{s+n}\right), \quad \rho(n) = \operatorname*{Res}_{s=n}\left(\widehat{h_N}(s)\right). \tag{1.29}$$

We compute $\rho(n)$:

$$(s-n)\widehat{h_N}(s) = \frac{\pi}{s+n}\prod_{\substack{m=1\\ m\neq n}}^{N}\frac{2\pi s}{s^2-m^2}, \tag{1.30}$$

and so

$$\begin{aligned}\rho(n) &= \pi^N 2^{N-2} n^{N-2} \prod_{m=1}^{n-1} \frac{1}{(n^2-m^2)} \prod_{m=n+1}^{N} \frac{1}{(n^2-m^2)} \\ &= \pi^N 2^{N-2} n^{N-2} \left(\frac{n!}{(n-1)!(2n-1)!}\right)\left(\frac{(-1)^{N-n}(2n)!}{(N+n)!(N-n)!}\right) \\ &= \pi^N 2^{N-1} n^N (-1)^{N-n} \frac{1}{(N+n)!(N-n)!}.\end{aligned} \tag{1.31}$$

It is clear that $\rho(-n) = (-1)^N \rho(n)$, and so

$$\widehat{h_N}(s) = \sum_{n=1}^{N} \rho(n)\left(\frac{1}{s-n} + \frac{(-1)^N}{s+n}\right). \tag{1.32}$$

A quick calculation shows that, for $s > n$,

$$2\int_1^\infty \xi^{-2s}(\xi^{-2n} \pm \xi^{2n}) \frac{d\xi}{\xi} = \frac{1}{s+n} \pm \frac{1}{s-n}. \tag{1.33}$$

And so, by the uniqueness of the Mellin transform, we find that

$$h_N(\xi) = \sum_{n=1}^{N} 2\rho(n)(\xi^{-2n} + (-1)^N \xi^{2n}) \tag{1.34}$$

for $\xi \in (1,\infty)$. The lemma follows by substituting (1.31) into (1.34). □

We outline the proof of Theorem 1.2. Given $\boldsymbol{\alpha} \in (\mathbb{C}\setminus\{0\})^N$, we can create the unique monic reciprocal Laurent polynomial $\tilde{p}_{\mathbf{a}}(x)$ having $\alpha_1,\dots,\alpha_N$, $\alpha_1^{-1},\dots,\alpha_N^{-1}$ as roots. We will use the change of variables $\boldsymbol{\alpha} \mapsto \mathbf{a}$ to write $H_N(s)$ as an integral over root vectors of reciprocal Laurent polynomials, as opposed to coefficient vectors. This change of variables is useful, since by (1.7), $\nu_{\text{rec}}(\mathbf{a})$ is a simple product in the roots of $\tilde{p}_{\mathbf{a}}(x)$ (i.e., in the coordinates of $\boldsymbol{\alpha}$). Analysis of the Jacobian of this change of variables will allow us to write $H_N(s)$ as the determinant of an $N\times N$ matrix, the entries of which are Mellin transforms which evolve to rational functions of s. Theorem 1.2 will follow from the evaluation of the determinant of this matrix.

Before proceeding to the proof of Theorem 1.2, we present μ_{rec} and $\mathcal{V}_{N+1}$ from another perspective. Given the positive integer M, we define the Mahler's measure function to be $\mu : \mathbb{C}^{M+1} \to \mathbb{R}$, where $\mu(\mathbf{u})$ is the Mahler's measure of the polynomial with coefficient vector $\mathbf{u}$. As was shown in [2], μ is nonnegative, homogeneous, positive-definite, and continuous. Thus μ is a symmetric distance function and the set

$$\mathcal{U}_{M+1} = \{\mathbf{u} \in \mathbb{C}^{M+1} : \mu(\mathbf{u}) \le 1\} \tag{1.35}$$

is a bounded symmetric star body. Let $M = 2N$ and consider the linear map $\Lambda : \mathbb{C}^{N+1} \to \mathbb{C}^{2N+1}$ defined by $\Lambda(\mathbf{v}) = (v_0, v_1, \dots, v_{N-1}, v_N, v_{N-1}, \dots, v_1, v_0)^{\mathrm{T}}$. We define $V = \Lambda(\mathbb{C}^{N+1})$ to be the subspace of reciprocal coefficient vectors. By (1.1), (1.4), and (1.5), we find $\mu_{\text{rec}}(\mathbf{v}) = \mu(\Lambda(\mathbf{v}))$. Thus, the star body formed by the intersection of $\mathcal{U}_{2N+1}$ and V is related to the reciprocal star body. Specifically,

$$\mathcal{V}_{N+1} = \Lambda^{-1}(V \cap \mathcal{U}_{2N+1}). \tag{1.36}$$

Every bounded symmetric star body uniquely determines a symmetric distance function [1, Chapter IV.2, Theorem 1]. Thus, armed with μ and Λ, we could "discover" μ_{rec}. Equation (1.7) can be recovered from the symmetry in the definition of Λ, so we would lose no information if we were to define μ_{rec} in this manner.

The volume of $\mathcal{U}_{M+1}$ as well as the subspace volume of the star body formed by intersecting $\mathcal{U}_{M+1}$ with the subspace of real coefficient vectors were investigated in [2]. Thus the computation of the volume of $\mathcal{V}_{N+1}$ yields subspace volume information of another "slice" of $\mathcal{U}_{2N+1}$.

2. A change of variables. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, and define the map $\mathcal{E}_N : (\mathbb{C}^\times)^N \to \mathbb{C}^N$ by $\mathcal{E}_N(\boldsymbol{\alpha}) = \mathbf{a}$, where

$$x^N \tilde{p}_{\mathbf{a}}(x) = \prod_{n=1}^{N} (x+\alpha_n)(x+\alpha_n^{-1}). \tag{2.1}$$

Thus the nth coordinate function of $\mathcal{E}_N$ is given by $\varepsilon_n(\alpha_1,\dots,\alpha_N,\alpha_1^{-1},\dots,\alpha_N^{-1})$, where ε_n is the nth elementary symmetric function in $2N$ variables. Let $E_N : \mathbb{C}^N \to \mathbb{C}^N$ be the function whose nth coordinate function is e_n, the nth elementary symmetric function in N variables. That is, given $\boldsymbol{\beta} \in \mathbb{C}^N$, if $\mathbf{b} = E_N(\boldsymbol{\beta})$, then

$$\prod_{n=1}^{N} (x+\beta_n) = x^N + \sum_{n=0}^{N-1} b_n x^n. \tag{2.2}$$

It is well known that the (complex) Jacobian of $E_N(\boldsymbol{\beta})$ is given by $|V(\boldsymbol{\beta})|^2$, where

$$V(\boldsymbol{\beta}) = \prod_{1\le m<n\le N} (\beta_n - \beta_m) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_N \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{N-1} & \beta_2^{N-1} & \cdots & \beta_N^{N-1} \end{pmatrix} \tag{2.3}$$

is the Vandermonde determinant. We will relate the Jacobian of $\mathcal{E}_N$ to the Jacobian of E_N.

Lemma 2.1. *For each positive integer N, the Jacobian of $\mathcal{E}_N(\boldsymbol{\alpha})$ is given by*

$$\left| V\left(\alpha_1 + \frac{1}{\alpha_1},\dots,\alpha_N + \frac{1}{\alpha_N}\right)\right|^2 \cdot \prod_{n=1}^{N} \left| \left(\frac{\alpha_n^2 - 1}{\alpha_n^2}\right)\right|^2. \tag{2.4}$$

Proof. By definition, $\varepsilon_n(x_1,\dots,x_N,x_1',\dots,x_N')$ is composed of all monomials of degree n in the variables $x_1,\dots,x_N$, $x_1',\dots,x_N'$. If we impose the relation $x_m x_m' = 1$ for $m = 1,\dots,N$, then $\varepsilon_n(x_1,\dots,x_N,x_1',\dots,x_N')$ is no longer homogeneous. In this situation, it is easy to see that the monomials of degree n of $\varepsilon_n(x_1,\dots,x_N,x_1',\dots,x_N')$ are exactly the monomials which do not contain both x_m and x_m' for $m = 1,\dots,N$. Hence,

$$\begin{aligned}\varepsilon_n(x_1,\dots,x_N,x_1',\dots,x_N') \\ = e_n(x_1+x_1',\dots,x_N+x_N') + (\text{monomials of degree} < n).\end{aligned} \tag{2.5}$$

In general, $\varepsilon_n(x_1,\dots,x_N,x_1',\dots,x_N')$ has monomials of degree $n-2M$: those monomials which contain x_m and x_m', where m runs over a subset of $1,\dots,N$ of cardinality M. By counting the number of times each monomial of degree $n-2M$ appears, we arrive at the identity

$$\begin{aligned}\varepsilon_n\left(\alpha_1,\dots,\alpha_N,\frac{1}{\alpha_1},\dots,\frac{1}{\alpha_N}\right) &= e_n\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_N+\frac{1}{\alpha_N}\right) \\ &\quad + \binom{N-n-2}{1} e_{n-2}\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_N+\frac{1}{\alpha_N}\right) \\ &\quad + \binom{N-n-4}{2} e_{n-4}\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_N+\frac{1}{\alpha_N}\right) + \cdots \\ &= \sum_{M=0}^{[N/2]} \binom{N-n-2M}{M} e_{n-2M}\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_N+\frac{1}{\alpha_N}\right),\end{aligned} \tag{2.6}$$

where $[N/2]$ is the integer part of $N/2$.

Thus

$$\mathscr{E}_N(\boldsymbol{\alpha}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \end{pmatrix} E_N(\boldsymbol{\beta}), \tag{2.7}$$

where

$$\boldsymbol{\beta} = \left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_N+\frac{1}{\alpha_N}\right)^{\mathrm{T}}, \tag{2.8}$$

and $*$ represents entries which are not necessarily 0. The Jacobian of $E_N(\boldsymbol{\beta}) = |V(\boldsymbol{\beta})|^2$, and thus by the chain rule, we arrive at the formula for the Jacobian of $\mathscr{E}_N(\boldsymbol{\alpha})$ given in the statement of the lemma. □

The Jacobian of $\mathscr{E}_N(\boldsymbol{\alpha})$ is nonzero for λ_{2N}-almost all points of $(\mathbb{C}^\times)^N$, and there are $2^N N!$ preimages for λ_{2N}-almost all $\mathbf{a} \in \mathbb{C}^N$. Employing the change-of-variables formula, we find

$$\begin{aligned}H_N(s) &= \int_{\mathbb{C}^N} \nu_{\text{rec}}(\mathbf{a})^{-2s}\, d\lambda_{2N}(\mathbf{a}) \\ &= \frac{1}{2^N N!}\int_{(\mathbb{C}^\times)^N} \left\{ \prod_{n=1}^{N} \max\{|\alpha_n|,|\alpha_n^{-1}|\}^{-2s} \left| \left(\frac{\alpha_n^2-1}{\alpha_n^2}\right)\right|^2 \right\} \\ &\quad \times \left| V\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_n+\frac{1}{\alpha_n}\right)\right|^2 d\lambda_{2N}(\boldsymbol{\alpha}).\end{aligned} \tag{2.9}$$

The latter integral admittedly looks formidable; however, this change of variables is beneficial since it allows us to exploit the multiplicative nature of ν_{rec}.

3. $H_N(s)$ is a determinant.

We first prove a short technical lemma concerning determinants.

Lemma 3.1. *Let N be a positive integer. If $I = I(j,k)$ is an $N\times N$ matrix and S_N is the Nth symmetric group, then*

$$\det(I) = \frac{1}{N!}\sum_{\tau\in S_N}\sum_{\sigma\in S_N}\operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\prod_{n=1}^{N} I(\tau(n),\sigma(n)). \tag{3.1}$$

Proof.

$$\prod_{n=1}^{N} I(\tau(n),\sigma(n)) = \prod_{n=1}^{N} I(n,\sigma\circ\tau^{-1}(n)). \tag{3.2}$$

Thus we can write (3.1) as

$$\begin{aligned}\frac{1}{N!}&\sum_{\tau\in S_N}\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma\circ\tau^{-1})\prod_{n=1}^{N} I(n,\sigma\circ\tau^{-1}(n))\\ &= \frac{1}{N!}\sum_{\tau\in S_N}\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{n=1}^{N} I(n,\sigma(n)) = \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{n=1}^{N} I(n,\sigma(n)),\end{aligned} \tag{3.3}$$

which is the familiar formula for $\det(I)$. □

Using (2.3), we expand the Vandermonde determinant as a sum over the symmetric group to find

$$\left|V\left(\alpha_1+\frac{1}{\alpha_1},\dots,\alpha_n+\frac{1}{\alpha_n}\right)\right|^2 = \left|\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{n=1}^{N}\left(\alpha_n+\frac{1}{\alpha_n}\right)^{\sigma(n)-1}\right|^2, \tag{3.4}$$

which we rewrite as

$$\sum_{\sigma\in S_N}\sum_{\tau\in S_N}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{n=1}^{N}\left(\alpha_n+\frac{1}{\alpha_n}\right)^{\sigma(n)-1}\left(\overline{\alpha}_n+\frac{1}{\overline{\alpha}_n}\right)^{\tau(n)-1}. \tag{3.5}$$

Substituting this expression into (2.9), we can write $H_N(s)$ as

$$\begin{aligned}\frac{1}{2^N N!}\int_{(\mathbb{C}^\times)^N}&\left\{\prod_{n=1}^{N}\max\{|\alpha_n|,|\alpha_n^{-1}|\}^{-2s}\left|\left(\frac{\alpha_n^2-1}{\alpha_n^2}\right)\right|^2\right\}\\ &\times\left(\sum_{\sigma\in S_N}\sum_{\tau\in S_N}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{n=1}^{N}\left(\alpha_n+\frac{1}{\alpha_n}\right)^{\sigma(n)-1}\left(\overline{\alpha}_n+\frac{1}{\overline{\alpha}_n}\right)^{\tau(n)-1}\right)d\lambda_{2N}(\boldsymbol{\alpha}).\end{aligned} \tag{3.6}$$

Exchanging the sums and the integral, and consolidating the products, we find

$$H_N(s) = \sum_{\sigma\in S_N}\sum_{\tau\in S_N} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\frac{1}{2^N N!} \times \int_{(\mathbb{C}^\times)^N}\left\{\prod_{n=1}^{N} \max\{|\alpha_n|,|\alpha_n^{-1}|\}^{-2s} \times \left(\frac{\alpha_n^2-1}{\alpha_n^2}\right)\left(\frac{\overline{\alpha}_n^2-1}{\overline{\alpha}_n^2}\right)\left(\alpha_n+\frac{1}{\alpha_n}\right)^{\sigma(n)-1}\left(\overline{\alpha}_n+\frac{1}{\overline{\alpha}_n}\right)^{\tau(n)-1}\right\} d\lambda_{2N}(\boldsymbol{\alpha}). \tag{3.7}$$

By an application of Fubini's theorem, we find

$$H_N(s) = \frac{1}{N!}\sum_{\sigma\in S_N}\sum_{\tau\in S_N}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{n=1}^{N}\mathcal{I}(\sigma(n),\tau(n)), \tag{3.8}$$

where $\mathcal{I}(J,K)$ is given by

$$\frac{1}{2}\int_{\mathbb{C}^\times}\max\{|\alpha|,|\alpha^{-1}|\}^{-2s}\left(\alpha-\frac{1}{\alpha}\right)\left(\overline{\alpha}-\frac{1}{\overline{\alpha}}\right)\left(\alpha+\frac{1}{\alpha}\right)^{J-1}\left(\overline{\alpha}+\frac{1}{\overline{\alpha}}\right)^{K-1}\frac{d\lambda_2(\alpha)}{|\alpha|^2}. \tag{3.9}$$

Applying Lemma 3.1 to (3.8), we find that $H_N(s)$ is the determinant of the $N\times N$ matrix $\mathcal{I} = \mathcal{I}(J,K)$.

4. The entries of $\mathcal{I}$ are rational functions of s. We will view $\mathcal{I}(J,K)$, not only as an entry in a matrix, but also as a function of s. We note that $\lambda_2(\alpha)/|\alpha|^2$ is normalized Haar measure on $\mathbb{C}^\times$. Thus $\mathcal{I}(J,K;s)$ is invariant under the substitution $\alpha\mapsto\alpha^{-1}$, and we may write

$$\mathcal{I}(J,K;s) = \int_{\mathbb{C}\setminus D}|\alpha|^{-2s}\left(\alpha-\frac{1}{\alpha}\right)\left(\overline{\alpha}-\frac{1}{\overline{\alpha}}\right)\left(\alpha+\frac{1}{\alpha}\right)^{J-1}\left(\overline{\alpha}+\frac{1}{\overline{\alpha}}\right)^{K-1}\frac{d\lambda_2(\alpha)}{|\alpha|^2}, \tag{4.1}$$

where D is the open unit disk. By setting $\alpha = re^{i\theta}$, we may write $\mathcal{I}(J,K;s) = \widehat{h}(J,K;r)$, where $h(J,K;r)$ is given by

$$\int_0^{2\pi}\left(re^{i\theta}-\frac{1}{re^{i\theta}}\right)\left(\frac{r}{e^{i\theta}}-\frac{e^{i\theta}}{r}\right)\left(re^{i\theta}+\frac{1}{re^{i\theta}}\right)^{J-1}\left(\frac{r}{e^{i\theta}}+\frac{e^{i\theta}}{r}\right)^{K-1}d\theta \tag{4.2}$$

for $r\in[1,\infty)$, and identically zero on $[0,1)$.

By the change of variables $\theta\mapsto-\theta$, we see that $h(J,K;r) = h(K,J;r)$. We conclude that $\mathcal{I}$ is a symmetric matrix whose J,K entry is $\widehat{h}(J,K;s)$.

Lemma 4.1. *$\mathcal{I}(J,K;s)$ analytically continues to a rational function. Specifically,*

$$\mathcal{I}(J,K;s) = \pi\sum_{n=1}^{N}c_n(J)c_n(K)\frac{2s}{s^2-n^2}, \tag{4.3}$$

where

$$c_n(J)=\begin{cases}\left[\binom{J-1}{\frac{J+n}{2}}-\binom{J-1}{\frac{J+n}{2}-1}\right] & \text{if } n<J \text{ and } n\equiv J(\operatorname{mod}2),\\ 1 & \text{if } n=J,\\ 0 & \text{otherwise.}\end{cases}\tag{4.4}$$

Proof. Without loss of generality, we assume $K\ge J$. There is a constant $\mathscr{C}$ such that $h(J,K;r)<\mathscr{C}r^{J+K}$ on $[1,\infty)$. Thus the integral defining $\mathscr{I}(J,K;s)$ converges in the half plane $\Re(s)>(J+K)/2$.

Writing $(re^{i\theta}+1/re^{i\theta})^{J-1}$ and $(r/e^{i\theta}+e^{i\theta}/r)^{K-1}$ as sums with binomial coefficients, we may rewrite (4.2) as

$$\begin{aligned}h(J,K;r)=&\sum_{j=0}^{J-1}\sum_{k=0}^{K-1}\binom{J-1}{j}\binom{K-1}{k}r^{J+K-2(j+k)-2}\\&\times\int_0^{2\pi}\left(r^2+\frac{1}{r^2}-\left(e^{2i\theta}+e^{-2i\theta}\right)\right)e^{(J-K-2(j-k))i\theta}d\theta.\end{aligned}\tag{4.5}$$

The integral appearing in this expression can be readily evaluated:

$$\begin{aligned}&\int_0^{2\pi}\left(r^2+\frac{1}{r^2}-\left(e^{2i\theta}+e^{-2i\theta}\right)\right)e^{(J-K-2(j-k))i\theta}d\theta\\&\quad=\begin{cases}2\pi\left(r^2+\frac{1}{r^2}\right), & k=j+\frac{(K-J)}{2},\\ -2\pi, & k=j+1+\frac{(K-J)}{2},\\ -2\pi, & k=j-1+\frac{(K-J)}{2}.\end{cases}\end{aligned}\tag{4.6}$$

If $J\not\equiv K(\operatorname{mod}2)$, we see that $h(J,K;r)$ (and hence $\mathscr{I}(J,K;s)$) is identically zero.

The conditions given in (4.6) allow us to eliminate one of the summations in (4.5). We use the facts that $0\le k\le K-1$ and $0\le j\le J-1$ together with the conditions in (4.6) to find conditions on j. Specifically,

$$\begin{aligned}k=j+\frac{K-J}{2}&\Longrightarrow 0\le j\le J-1,\\ k=j+1+\frac{K-J}{2}&\Longrightarrow 0\le j\le\min\left\{\frac{J+K}{2}-2,J-1\right\},\\ k=j-1+\frac{K-J}{2}&\Longrightarrow \max\left\{\frac{J-K}{2}+1,0\right\}\le j\le J-1.\end{aligned}\tag{4.7}$$

Since $K\ge J$, we can write

$$\max\left\{\frac{J-K}{2}+1,0\right\}=\delta_{JK},\qquad \min\left\{\frac{J+K}{2}-2,J-1\right\}=J-1-\delta_{JK},\tag{4.8}$$

where $\delta_{JK} = 1$ if $J = K$ and is 0 otherwise. From this information, we may write $h(J,K;r)$ as

$$2\pi\left(\sum_{j=0}^{J-1}\binom{J-1}{j}\binom{K-1}{\frac{K-J}{2}+j}r^{2J-4j}+\sum_{j=0}^{J-1}\binom{J-1}{j}\binom{K-1}{\frac{K-J}{2}+j}r^{2J-4j-4}\right.$$
$$\left.-\sum_{j=\delta_{JK}}^{J-1}\binom{J-1}{j}\binom{K-1}{\frac{K-J}{2}+j-1}r^{2J-4j}-\sum_{j=0}^{J-1-\delta_{JK}}\binom{J-1}{j}\binom{K-1}{\frac{K-J}{2}+j+1}r^{2J-4j-4}\right). \tag{4.9}$$

Using the convention that $\binom{K-1}{K} = 0$ and $\binom{K-1}{-1} = 0$, we may eliminate δ_{JK} from the latter two sums. Reindexing each sum based on the powers of r and simplifying the binomial coefficients, we find

$$h(J,K;r) = 2\pi\left(\sum_{l=-J/2+1}^{J/2}\binom{J-1}{\frac{J}{2}-l}\binom{K-1}{\frac{K}{2}-l}r^{4l}+\sum_{l=-J/2}^{J/2-1}\binom{J-1}{\frac{J}{2}+l}\binom{K-1}{\frac{K}{2}+l}r^{4l}\right.$$
$$\left.-\sum_{l=-J/2+1}^{J/2}\binom{J-1}{\frac{J}{2}-l}\binom{K-1}{\frac{K}{2}-l-1}r^{4l}-\sum_{l=-J/2}^{J/2-1}\binom{J-1}{\frac{J}{2}+l}\binom{K-1}{\frac{K}{2}+l-1}r^{4l}\right). \tag{4.10}$$

Note that in the case that J is odd, these sums run over consecutive odd multiples of $1/2$. Reindexing the first and third sum by $l \mapsto -l$, we may combine the first and second sums, and the third and fourth sums. We may then write $h(J,K;r)$ as

$$2\pi\left(\sum_{l=-J/2}^{J/2-1}\binom{J-1}{\frac{J}{2}+l}\binom{K-1}{\frac{K}{2}+l}\left(r^{4l}+r^{-4l}\right)-\sum_{l=-J/2}^{J/2-1}\binom{J-1}{\frac{J}{2}+l}\binom{K-1}{\frac{K}{2}+l-1}\left(r^{4l}+r^{-4l}\right)\right). \tag{4.11}$$

Due to the symmetry in the summands, we may reindex the sums using only positive indices. Let $l_0 = 0$ if J and K are even, and $l_0 = 1/2$ if J and K are odd; then

$$h(J,K;r) = 2\pi\left[\binom{K-1}{\frac{J+K}{2}-1}-\binom{K-1}{\frac{J+K}{2}}\right]\left(r^{2J}+r^{-2J}\right)$$
$$+2\pi\sum_{l=l_0}^{J/2-1}\left[\binom{J-1}{\frac{J}{2}+l}-\binom{J-1}{\frac{J}{2}+l-1}\right]\left[\binom{K-1}{\frac{K}{2}+l}-\binom{K-1}{\frac{K}{2}+l-1}\right]\left(r^{4l}+r^{-4l}\right). \tag{4.12}$$

We are now in position to compute $\widehat{h}(J,K;s)$. There is a correspondence between the coefficients and powers of r which appear in $h(J,K;r)$ and the poles and residues of $\widehat{h}(J,K;s)$. As was demonstrated in the proof of Corollary 1.4, the Mellin transform of $r^{4l}+r^{-4l}$ analytically continues to the rational function $s/(s^2-4l^2)$. Thus $\mathcal{I}(J,K;s)$

extends to a rational function:

$$\mathcal{J}(J,K;s) = 2\pi\left[\binom{K-1}{\frac{J+K}{2}-1}-\binom{K-1}{\frac{J+K}{2}}\right]\frac{s}{s^2-J^2} + 2\pi\sum_{l=l_0}^{J/2-1}\left[\binom{J-1}{\frac{J}{2}+l}-\binom{J-1}{\frac{J}{2}+l-1}\right]\left[\binom{K-1}{\frac{K}{2}+l}-\binom{K-1}{\frac{K}{2}+l-1}\right]\frac{s}{s^2-4l^2}. \tag{4.13}$$

Reindexing this sum by setting $n = 2l$, we find

$$\mathcal{J}(J,K;s) = 2\pi\left[\binom{K-1}{\frac{J+K}{2}-1}-\binom{K-1}{\frac{J+K}{2}}\right]\frac{s}{s^2-J^2} + 2\pi\sum_{n}\left[\binom{J-1}{\frac{J+n}{2}}-\binom{J-1}{\frac{J+n}{2}-1}\right]\left[\binom{K-1}{\frac{K+n}{2}}-\binom{K-1}{\frac{K+n}{2}-1}\right]\frac{s}{s^2-n^2}, \tag{4.14}$$

where the sum is over $n \in \{1,3,\dots,J-2\}$ if J and K are odd, and over $n \in \{2,4,\dots,J-2\}$ if J and K are even. If $J = K$, the leading coefficient is 1. Using (4.4) we may write (4.14) as in the statement of the lemma. It is easy to verify that expression (4.3) is symmetric in J and K, giving $\mathcal{J}(J,K;s) = \mathcal{J}(K,J;s)$ as expected. Additionally, if $J \not\equiv K \pmod 2$, the expression in (4.3) yields $\mathcal{J}(J,K;s) = 0$. This proves the lemma. □

We identify $\mathcal{J}(J,K;s)$ with the rational function it extends to. When J and K are odd, $\mathcal{J}(J,K;s)$ has poles at $\pm 1, \pm 3, \dots, \pm\min\{J,K\}$. When J and K are even, $\mathcal{J}(J,K;s)$ has poles at $\pm 2, \pm 4, \dots, \pm\min\{J,K\}$. $\mathcal{J}(J,K;s)$ has a zero of multiplicity one at 0.

We are now in position to prove the first part of Theorem 1.2. $H_N(s)$ is the determinant of $\mathcal{J}$, and the entries of $\mathcal{J}$ extend to rational functions of s. Since the determinant is a polynomial in the entries of a matrix, $H_N(s)$ itself extends to a rational function of s. In fact, since the determinant is a homogeneous polynomial in the entries of a matrix and the entries of $\mathcal{J}$ analytically continue to odd functions, $H_N(s)$ analytically continues to an even rational function when N is even, and analytically continues to an odd rational function when N is odd. We also see that $H_N(s)$ has a zero of multiplicity N at 0.

5. $H_N(s)$ is a simple product.

In this section, we express $\det(\mathcal{J})$ as a simple product. The structure of the poles and residues of $\mathcal{J}(J,K;s)$ will allow us to find linear dependence relations on the rows of $\mathcal{J}$.

Let B_n be the $N \times N$ matrix whose J,K entry is the integer $c_n(J)c_n(K)$. Then by Lemma 4.1, we have the matrix equation

$$\mathcal{J} = \sum_{n=1}^{N} B_n \frac{2\pi s}{s^2-n^2}. \tag{5.1}$$

Define $\boldsymbol{\omega}_n^{\mathrm{T}} \in \mathbb{Q}^N$ to be the row vector given by $\boldsymbol{\omega}_n^{\mathrm{T}} = (c_n(K))_{K=1}^N$. It follows then that the Jth row vector of B_n is given by $c_n(J)\boldsymbol{\omega}_n^{\mathrm{T}}$, and thus every row of B_n is a scalar multiple of $\boldsymbol{\omega}_n^{\mathrm{T}}$.

We may find a nonzero vector $\boldsymbol{\psi} \in \mathbb{Q}^N$ such that $\boldsymbol{\omega}_n^{\mathrm{T}}\boldsymbol{\psi} = 0$ for $1 \le n \le N-1$. In fact, $B_n\boldsymbol{\psi} = \mathbf{0}$ for $1 \le n \le N-1$, leading us to the vector equation

$$\mathcal{I}\boldsymbol{\psi} = \sum_{n=1}^{N} B_n \boldsymbol{\psi} \frac{2\pi s}{s^2 - n^2} = B_N \boldsymbol{\psi} \frac{2\pi s}{s^2 - N^2}. \tag{5.2}$$

We see that $(\mathcal{I} - B_N(2\pi s/(s^2 - N^2)))\boldsymbol{\psi} = \mathbf{0}$, and so $\det(\mathcal{I} - B_N(2\pi s/(s^2 - N^2))) = 0$. From the definition of B_N and Lemma 4.1, we find

$$\mathcal{I} - B_N \frac{2\pi s}{s^2 - N^2} = \begin{pmatrix} \mathcal{I}(1,1) & \mathcal{I}(1,2) & \cdots & \mathcal{I}(1,N) \\ \mathcal{I}(2,1) & \mathcal{I}(2,2) & \cdots & \mathcal{I}(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}(N,1) & \mathcal{I}(N,2) & \cdots & \mathcal{I}(N,N) - \dfrac{2\pi s}{s^2 - N^2} \end{pmatrix}. \tag{5.3}$$

Taking determinants and exploiting the multilinearity of the determinant, we obtain the following:

$$\det \begin{pmatrix} \mathcal{I}(1,1) & \mathcal{I}(1,2) & \cdots & \mathcal{I}(1,N) \\ \mathcal{I}(2,1) & \mathcal{I}(2,2) & \cdots & \mathcal{I}(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}(N,1) & \mathcal{I}(N,2) & \cdots & \mathcal{I}(N,N) \end{pmatrix} = \det \begin{pmatrix} \mathcal{I}(1,1) & \mathcal{I}(1,2) & \cdots & 0 \\ \mathcal{I}(2,1) & \mathcal{I}(2,2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}(N,1) & \mathcal{I}(N,2) & \cdots & \dfrac{2\pi s}{s^2 - N^2} \end{pmatrix}. \tag{5.4}$$

The left-hand side is $H_N(s)$. By a simple induction argument, we finally arrive at a simple product formulation of $H_N(s)$:

$$H_N(s) = \prod_{n=1}^{N} \frac{2\pi s}{s^2 - n^2}. \tag{5.5}$$

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