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PURE IMAGINARY SOLUTIONS OF THE SECOND PAINLEVÉ EQUATION

ALISHER S. ABDULLAYEV

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We prove the existence and obtain local asymptotic formulas for pure imaginary solutions of the general second Painlevé equation.

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1. Introduction. The Painlevé equations were discovered by Painlevé and Gambier at the beginning of our century. They studied equations of the form

$$y'' = F(z, y, y') \tag{1.1}$$

without moveable singularities except poles. Here F is rational in y' and y with coefficients locally analytic in z. Among fifty different types of equations of the form (1.1), there are six distinguished Painlevé equations (PI–PVI) which are not reducible to linear equations, Riccati equations, or equations whose solutions are elliptic functions. The detailed discussion can be found in [16], [12], or [15]. During the last thirty years there has been considerable interest in the Painlevé equations because of their importance in physical applications, for example, fluid dynamics, quantum spin systems, relativity, and so forth; see the monograph of Ablowitz and Segur [5]. For the survey on the Painlevé equations, see the article of Kruskal and Clarkson [26].

The general second Painlevé (PII) equation is given by

$$\frac{d^2y}{dz^2} = 2y^3 + zy + b \tag{1.2}$$

with b an arbitrary complex constant.

Equation (1.2) is connected with the Korteweg-de Vries equation, which is one of the well-known equations solvable by inverse scattering methods [6, 7, 19, 24, 27].

One of the main achievements in the theory of Painlevé equations is connection formulas, which were formally derived using the isomonodromy method [10, 20, 21, 22, 23, 28]. There are three approaches to justification of connection formulas. The first one is based on the independent local asymptotic analysis of the Painlevé equations [19]. The second approach, proposed by Kitaev [25], is based on the Brouwer fixed point theorem, and utilizes the solvability of the inverse monodromy problem. The third approach which has been developed by Deift and Zhou [8] is based on the direct asymptotic analysis of the relevant oscillatory Riemann-Hilbert (RH) problems. The rigorous methodology for studying the RH problems associated with certain nonlinear ordinary differential equations in the case of Painlevé II and Painlevé IV and was introduced earlier by Fokas and Zhou [11]. The detailed discussion of all three approaches is given in the paper by Its et al. [18].

A remarkable recent survey by Its [17] gives a detailed discussion of the RH approach to different applications, including integrable systems, where nonlinear differential equations, including Painlevé equations, arise as a compatibility condition.

The method developed in this article allows to prove existence of pure imaginary solutions for the general second Painlevé equation and the differentiability of the obtained asymptotics, which is very important for applications. If one can prove existence of local asymptotics and obtain those, then the proof of global asymptotics and connection formulas can be simplified dramatically [19]. This proof does not use the isomonodromy method. Moreover, a method of the article can be extended to a more general (nonintegrable) type of nonlinearity [1, 2, 3].

The structure of the paper is as follows. In the introduction we formulate the main result. Section 2.1 discusses a phase of WKB solution. Section 2.2 is devoted to the proof of the main result. All the auxiliary results can be found in the appendix.

We will consider a case of (1.2) with $b = i\alpha$, $\alpha \in \mathbb{R}$, for the real values of x:

$$\frac{d^2y}{dx^2} = 2y^3 + xy + i\alpha. \tag{1.3}$$

The following is the main theorem of the paper.

Theorem 1.1. *There exists a two-parametric family $y(x;\beta,c_0)$ of solutions of (1.3) such that the following asymptotics hold as $x \to \infty$:*

$$y(x;\beta,c_0) = \pm i\sqrt{\frac{x}{2}} + i\beta(2x)^{-1/4}\sin\varphi_0(x) + O(x^{-1}), \tag{1.4}$$

$$\varphi_0(x) = \frac{2\sqrt{2}x^{3/2}}{3} + \frac{3}{2}\alpha\ln x - \frac{3}{2}\beta^2\ln x + c_0 + O(x^{-3/2}) \tag{1.5}$$

(c_0 and β are arbitrary constants). These asymptotics are infinitely differentiable, for example,

$$y'(x;\beta,c_0) = i\beta(2x)^{1/4}\cos\varphi_0(x) + O(x^{-1/2}). \tag{1.6}$$

Remark 1.2. There are two values of β for which the remainder in formula (1.4) will have the order $O(x^{-1}\ln x)$. See Lemmas 2.1 and 2.2.

Remark 1.3. The structure of the asymptotic formula (1.4) is as follows. The term $\pm i\sqrt{x/2}$ is the asymptotic behavior of two roots of the polynomial equation

$$2y^3 + xy + i\alpha = 0 \tag{1.7}$$

as $x \to \infty$. The phase $\varphi_0(x)$ of the second term $i\beta(2x)^{-1/4}\sin\varphi_0(x)$ consists of two parts: $2\sqrt{2}x^{3/2}/3 + (3/2)\alpha\ln x$ and $-(3/2)\beta^2\ln x$. The first part of the phase $\varphi_0(x)$,

$$\frac{2\sqrt{2}x^{3/2}}{3} + \frac{3}{2}\alpha\ln x, \tag{1.8}$$

results from WKB approximation for linear equation

$$u''(x) + Q(x)u = 0, \tag{1.9}$$

where $Q(x)$ is given by formula (2.8) as $x \to \infty$. The second part of the phase $\varphi_0(x)$,

$$-\frac{3}{2}\beta^2 \ln x, \tag{1.10}$$

is obtained as a result of nonlinearity of (2.7).

Remark 1.4. Formulas (1.4) and (1.5) are the special case of [24, formula (26)] (the proof is not published) and coincide with [18, formula (1.12)] when $\alpha = 0$. Much stronger results concerning global behavior of solutions of PII were obtained by Its et al. [18], and by Deift and Zhou [8], using the isomonodromy method and RH method, respectively.

Remark 1.5. See also an excellent paper of Hastings and McLeod [14], which has simple and elegant proof of the asymptotic formulas for real solutions of the second Painlevé equation.

The method used in this paper is a further development of the method used in [1, 2, 3, 4] for the second, third, and fourth Painlevé equations, respectively.

2. Main part

2.1. Heuristic considerations. Stability near parabola $w = i\sqrt{x/2}$. We consider a second Painlevé equation given by

$$y'' = xy + 2y^3 + i\alpha \tag{2.1}$$

as $x \to \infty$. The substitution

$$y(x) = -i\frac{\alpha}{x} + w(x) \tag{2.2}$$

leads to

$$w'' = P(w, x), \tag{2.3}$$

where

$$P(x, w) = 2w^3 - 6i\frac{\alpha}{x}w^2 + \left(x - 6\frac{\alpha^2}{x^2}\right)w + 2i\frac{\alpha}{x^3}\left(1 + \alpha^2\right). \tag{2.4}$$

The right-hand side of (2.3) is a polynomial in w with three roots:

$$\begin{gathered} w_1(x) \longrightarrow 0, \qquad w_2(x) \longrightarrow i\sqrt{\frac{x}{2}} + \frac{3}{2}i\alpha x^{-1} + O\left(x^{-5/2}\right), \\ w_3(x) = -i\sqrt{\frac{x}{2}} + \frac{3}{2}i\alpha x^{-1} + O\left(x^{-5/2}\right) \end{gathered} \tag{2.5}$$

as $x \to \infty$.

We will study stability of solutions only near parabola $w = i\sqrt{x/2}$; the stability of solutions near parabola $w = -i\sqrt{x/2}$ can be studied exactly the same way. The main theorem is formulated for both cases.

After the substitution

$$w(x) = w_2(x) + u(x), \tag{2.6}$$

equation (2.3) becomes

$$u''(x) + Q(x)u = g_0(x) + g_1(x)u^2 + g_2(x)u^3, \tag{2.7}$$

where

$$\begin{aligned} Q(x) &= 2x + \frac{6\alpha}{\sqrt{2}}x^{-1/2} + O(x^{-2}), \qquad g_0(x) = \frac{i}{4\sqrt{2}}x^{-3/2} + O(x^{-3}), \\ g_1(x) &= \frac{6i}{\sqrt{2}}x^{1/2} + 3i\alpha x^{-1} + O(x^{-5/2}), \quad g_2(x) = 2, \quad \text{as } x \longrightarrow \infty. \end{aligned} \tag{2.8}$$

Below we will seek a solution of (2.7) in the form of (2.9) and (2.10):

$$w(x) = Ax^{-1/4}\sin\phi(x) + Bx^{-1}\cos 2\phi(x) + Cx^{-1}, \tag{2.9}$$

where $A = 2^{-1/4}i\beta$, B, C, and β are some constants,

$$\phi(x) = \frac{2\sqrt{2}}{3}x^{3/2} + \frac{3\alpha}{2}\ln x + \psi(x), \tag{2.10}$$

where

$$\psi'(x) = \frac{f(\phi(x))}{x^{\gamma}}, \qquad \frac{1}{2} + \gamma > 0, \tag{2.11}$$

γ is a constant and $f(\phi(x))$ is some function differentiable and bounded on (x_0, ∞). After substituting (2.9) and (2.10) into (2.7) and taking into account (2.11), we get

$$\begin{aligned} &\frac{5}{16}Ax^{-9/4}\sin\phi + \sqrt{2}A(f'(\phi)\cos\phi - 2f(\phi)\sin\phi)x^{1/4-\gamma} \\ &\quad + Af(f'(\phi)\cos\phi - f(\phi)\sin\phi)x^{-1/4-2\gamma} - \left(\frac{1}{2}+\gamma\right)Af(\phi)\cos\phi x^{-5/4-\gamma} \\ &\quad + \frac{3}{2}\cdot\sqrt{2}B\sin 2\phi x^{-3/2} - 2\sqrt{2}B\{f'_{\phi}\sin 2\phi + 4f(\phi)\cos 2\phi\}x^{-1/2-\gamma} \\ &\quad - 6B\cos 2\phi + 2C + \frac{6\alpha}{\sqrt{2}}Bx^{-3/2}\cos 2\phi + \frac{6\alpha}{\sqrt{2}}Cx^{-3/2} \\ &= \frac{6i}{\sqrt{2}}A^2\sin^2\phi + \frac{12i}{\sqrt{2}}ABx^{-3/4}\sin\phi\cos 2\phi + \frac{12i}{\sqrt{2}}ACx^{-3/4}\sin\phi \\ &\quad + 2A^3x^{-3/4}\sin^3\phi + O(x^{-3/2}). \end{aligned} \tag{2.12}$$

Equalizing constant terms will get

$$2C-6B\cos 2\phi=\frac{6i}{\sqrt{2}}A^2\sin^2\phi, \tag{2.13}$$

or

$$C=\frac{3i}{2\sqrt{2}}A^2, \qquad B=\frac{i}{2\sqrt{2}}A^2. \tag{2.14}$$

Since the highest power of x in the left-hand side is equal to $(1/4-\gamma)$ and the power of x in the right-hand side equals $-3/4$, we have $1/4-\gamma=-3/4$ whence $\gamma=1$. Therefore, discarding the terms of the order $O(x^{-3/2})$ and higher, and replacing ϕ by φ_0, we obtain the equation

$$f'_{\varphi_0}\cos\varphi_0-2f\sin\varphi_0=6iB\sin\varphi_0\cos 2\varphi_0+6iC\sin\varphi_0+\sqrt{2}A^2\sin^3\varphi_0 \tag{2.15}$$

or

$$f'_{\varphi_0}\cos\varphi_0-2f\sin\varphi_0=C_1\sin\varphi_0+C_2\sin^3\varphi_0, \tag{2.16}$$

where

$$C_1=-\frac{12}{\sqrt{2}}A^2=6\beta^2, \qquad C_2=\frac{8}{\sqrt{2}}A^2=-4\beta^2, \tag{2.17}$$

whose all bounded solutions according to Lemma A.1 are given by the formula

$$f(\varphi_0)=-\frac{3}{2}\beta^2-\frac{1}{2}\beta^2\cos 2\varphi_0. \tag{2.18}$$

From (2.18), (2.10), and (2.11), we obtain the integral equation

$$\varphi_0(x)=\frac{2\sqrt{2}}{3}x^{3/2}+\frac{3}{2}\alpha\ln x-\frac{3}{2}\beta^2\ln x+c_0-\frac{\beta^2}{2}\int_\infty^x\frac{\cos 2\varphi_0(t)}{t}dt, \tag{2.19}$$

where c_0 is a some constant. According to Lemma A.3 there exists the unique smooth solution of the integral equation (2.19) and that solution has the form

$$\varphi_0(x)=\frac{2\sqrt{2}}{3}x^{3/2}+\frac{3}{2}\alpha\ln x-\frac{3}{2}\beta^2\ln x+c_0+O(x^{-3/2}), \tag{2.20}$$

$$\varphi_0'(x)=\sqrt{2}x^{1/2}+O(x^{-1}). \tag{2.21}$$

2.2. Asymptotics of solutions of the second Painlevé equation. We will seek the solutions of (2.7) in the form

$$u(x)=u_0(x)+w(x), \tag{2.22}$$

where $u_0(x) = i(2x)^{-1/4}\beta\sin\varphi_0(x)$ and $\varphi_0(x)$ is the solution of the integral equation (2.18). Then (2.6) takes the form

$$w''(x)+F(x)w = f_0(x)+f_1(x)w^2+f_2(x)w^3, \tag{2.23}$$

where

$$\begin{aligned} F(x) &= 2x+6\sqrt[4]{2}\beta x^{1/4}\sin\varphi_0(x)+\frac{3\sqrt{2}}{2}\beta^2(1-\cos 2\varphi_0(x))x^{-1/2}, \\ f_0(x) &= -\frac{3i\beta^2}{2}(1-\cos 2\varphi_0(x))+O(x^{-3/2}), \\ f_1(x) &= 6i\sqrt{\frac{x}{2}}+6i(2x)^{-1/4}\beta\sin\varphi_0(x), \\ f_2(x) &= 2 \quad \text{as } x \longrightarrow \infty. \end{aligned} \tag{2.24}$$

We need the following two lemmas.

Lemma 2.1. *The equation*

$$y''+F(x)y = 0, \tag{2.25}$$

where $F(x)$ is defined by (2.24), has a fundamental system of solutions $\{y_1(x), y_2(x)\}$ of the form

$$\begin{aligned} y_1(x) &= x^{-1/4+\rho}[-\cos(t+2\psi-\theta)+\sin t+O(x^{-3/2})], \\ y_1'(x) &= x^{-1/4+\rho}[\sin(t+2\psi-\theta)+\cos t+O(x^{-3/2})], \\ y_2(x) &= x^{-1/4-\rho}[\cos(t+2\psi+\theta)+\sin t+O(x^{-3/2})], \\ y_2'(x) &= x^{-1/4-\rho}[-\sin(t+2\psi+\theta)+\cos t+O(x^{-3/2})] \end{aligned} \tag{2.26}$$

as $x \to \infty$. Here

$$\begin{aligned} t &= \frac{2\sqrt{2}}{3}x^{3/2}+a_0\beta^2\ln x+c_0+O(x^{-3/4}), \\ t+2\psi &= \frac{2\sqrt{2}}{3}x^{3/2}+a_1\beta^2\ln x+c_1+O(x^{-3/4}), \\ \rho &= a^2\frac{\sqrt{5}}{24} > 0, \qquad \theta = \arctan\frac{2}{\sqrt{5}}, \quad 0<\theta<\frac{\pi}{2}, \end{aligned} \tag{2.27}$$

a_0, a_1, c_0, c_1 are some constants.

Lemma 2.2. *There exists a solution $y_0(x)$ of the equation*

$$y''+F(x)y = f_0(x), \tag{2.28}$$

where $F(x)$ and $f_0(x)$ are defined by (2.24), such that $y_0(x) = O(x^{-1})$, $y_0'(x) = O(x^{-1})$ when $\rho \neq 3/4$ and $y_0(x) = O(x^{-1}\ln x)$, $y_0'(x) = O(x^{-1}\ln x)$ when $\rho = 3/4$ as $x \to \infty$.

Proof of Theorem 1.1. The integral equation that corresponds to the differential equation (2.23) has the form

$$\begin{aligned} w(x) = y_0(x) + c y_1(x) \int_\infty^x y_2(\tau)[f_1(\tau)w^2 + f_2(\tau)w^3]d\tau \\ - c y_2(x)\bigg[\int_{x_1}^x y_1(\tau)f_1(\tau)w^2 d\tau + \int_{x_2}^x y_1(\tau)f_2(\tau)w^3 d\tau\bigg], \end{aligned} \tag{2.29}$$

where c is some constant. By the substitution $w(x) = x^{-1}z(x)$, we get

$$\begin{aligned} z(x) = z_0(x) + cxy_1(x)\int_\infty^x y_2(\tau)[f_1(\tau)\tau^{-2}z^2 + f_2(\tau)\tau^{-3}z^3]d\tau \\ - cxy_2(x)\bigg[\int_{x_1}^x y_1(\tau)f_1(\tau)\tau^{-2}z^2 d\tau + \int_{x_2}^x y_1(\tau)f_2(\tau)\tau^{-3}z^3 d\tau\bigg]. \end{aligned} \tag{2.30}$$

Here $z_0(x) = xy_0(x)$. From now on we will assume that ρ does not equal $3/4$. Due to Lemma 2.2 it follows that if $x_1 \geq x_0 > 0$, then $|z_0| \leq b_0$ and $|z_0'| \leq b_0$, where b_0 is some positive constant. We are going to show the existence of a solution of (2.30) by the method of successive approximations. We rewrite (2.30) in the form

$$z(x) = z_0(x) + K(x,z), \tag{2.31}$$

where $K(x,z)$ is the integral operator

$$\begin{aligned} K(x,z) = cxy_1(x)\int_\infty^x y_2(\tau)[f_1(\tau)\tau^{-2}z^2 + f_2(\tau)\tau^{-3}z^3]d\tau \\ - cxy_2(x)\bigg[\int_{x_1}^x y_1(\tau)f_1(\tau)\tau^{-2}z^2 d\tau + \int_{x_2}^x y_1(\tau)f_2(\tau)\tau^{-3}z^3 d\tau\bigg]. \end{aligned} \tag{2.32}$$

According to the method of successive approximations, we define

$$\begin{aligned} z_{-1}(x) &= 0, \\ z_0(x) &= z_0(x), \\ z_n(x) &= z_0(x) + K\big(x, z_{n-1}(x)\big), \quad n = 1,2,\ldots. \end{aligned} \tag{2.33}$$

Then

$$z_1(x) - z_0(x) = K\big(x, z_0(x)\big), \tag{2.34}$$

where

$$\begin{aligned} K\big(x,z_0(x)\big) = cxy_1(x)\int_\infty^x y_2(\tau)[f_1(\tau)\tau^{-2}z_0^2 + f_2(\tau)\tau^{-3}z_0^3]d\tau \\ - cxy_2(x)\bigg[\int_{x_1}^x y_1(\tau)f_1(\tau)\tau^{-2}z_0^2 d\tau + \int_{x_2}^x y_1(\tau)f_2(\tau)\tau^{-3}z_0^3 d\tau\bigg]. \end{aligned} \tag{2.35}$$

We note that the order of the integral operator $K(x,z)$ is equal to the order of its first term. To find the order of the first term we use integration by parts, taking into account that

$$f_1'(x) = 6\sqrt[4]{2}\, i x^{1/4}\beta\cos\varphi_0(x) + O(x^{-1/2}) \tag{2.36}$$

and, utilizing formulas (2.24) and (2.26), we get

$$\begin{aligned}
& cxy_1(x)\int_{\infty}^{x} y_2(\tau)f_1(\tau)\tau^{-2}z_0^2\,d\tau \\
&= cxy_1(x)\int_{\infty}^{x} \tau^{-9/4-\rho}\Big[\cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\Big) \\
&\qquad\qquad + \sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\Big)\Big]f_1(\tau)\tau^{-2}z_0^2\,d\tau \\
&= O\big(x^{-3/2}\big) \\
&\quad - cxy_1(x)\int_{\infty}^{x}\frac{1}{\sqrt{2}}\Big[\sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\Big) \\
&\qquad\qquad - \cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\Big)\Big]\frac{d}{d\tau}\big[f_1(\tau)\tau^{-11/4-\rho}z_0^2(\tau)\big]d\tau \\
&= O\big(x^{-3/2}\big) \\
&\quad - cxy_1(x)\int_{\infty}^{x}\Big(-\frac{11}{4}-\rho\Big)\tau^{-15/4-\rho}f_1(\tau)z_0^2(\tau) \\
&\qquad\qquad \times\frac{1}{\sqrt{2}}\Big[\sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\Big) \\
&\qquad\qquad\qquad - \cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\Big)\Big]d\tau \\
&\quad - cxy_1(x)\int_{\infty}^{x}\tau^{-11/4-\rho} \\
&\qquad\qquad \times\frac{1}{\sqrt{2}}\Big[\sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\Big) \\
&\qquad\qquad\qquad - \cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\Big)\Big]\frac{d}{d\tau}\big[f_1(\tau)z_0^2(\tau)\big]d\tau.
\end{aligned} \tag{2.37}$$

Since

$$\begin{aligned}
\Big|\frac{1}{\sqrt{2}}\frac{d}{d\tau}\big[f_1(\tau)z_0^2(\tau)\big]\Big| &= \Big|\frac{1}{\sqrt{2}}\big[f_1'(\tau)z_0^2(\tau)+f_1(\tau)2z_0(\tau)z_0'(\tau)\big]\Big| \\
&\le 6\big[O\big(\tau^{-1/4}\big)+1\big]b_0^2\tau^{1/2} < 7b_0^2\tau^{1/2},
\end{aligned} \tag{2.38}$$

then the absolute value of the above term is less than

$$\frac{56c}{5+4\rho}b_0^2x^{-1/2}. \tag{2.39}$$

Similarly

$$\Big|cxy_2(x)\int_{x_1}^{x} y_1(\tau)f_1(\tau)\tau^{-2}z_0^2\,d\tau\Big| \le \frac{56c}{5+4\rho}b_0^2x^{-1/2}. \tag{2.40}$$

Here $x_1 = \infty$ if the corresponding integral $\int_\infty^x y_1(\tau) f_1(\tau)\tau^{-2} d\tau < \infty$ and $x_1 = x_0$ if this integral is divergent. The orders of the other terms are higher than $O(x^{-1/2})$. Therefore

$$|z_1(x) - z_0(x)| \le qx^{-1/2}, \quad |z_1(x)| \le b_0 + qx^{-1/2}, \quad q \equiv \frac{56c}{5+4\rho} b_0^2. \tag{2.41a}$$

Similarly, one can show that

$$|z_1'(x) - z_0'(x)| \le qx^{-1/2}, \quad |z_1'(x)| \le b_0 + qx^{-1/2} \tag{2.41b}$$

with the same q. We show by induction that

$$|z_n(x) - z_{n-1}(x)| \le \frac{k}{\sqrt{x}} \{ |z_{n-1}(x) - z_{n-2}(x)| + |z_{n-1}'(x) - z_{n-2}'(x)| \}, \tag{2.42a}$$

$$|z_n'(x) - z_{n-1}'(x)| \le \frac{k}{\sqrt{x}} \{ |z_{n-1}(x) - z_{n-2}(x)| + |z_{n-1}'(x) - z_{n-2}'(x)| \}, \tag{2.42b}$$

and also

$$|z_n(x) - z_{n-1}(x)| \le \frac{q}{\sqrt{x}} \left(\frac{2k}{\sqrt{x}} \right)^{n-1}, \qquad |z_n'(x) - z_{n-1}'(x)| \le \frac{q}{\sqrt{x}} \left(\frac{2k}{\sqrt{x}} \right)^{n-1}, \tag{2.43a}$$

$$|z_n(x)| \le b_0 + \frac{q}{\sqrt{x}} \frac{1-(2k/\sqrt{x})^n}{1-2k/\sqrt{x}}, \qquad |z_n'(x)| \le b_0 + \frac{q}{\sqrt{x}} \frac{1-(2k/\sqrt{x})^n}{1-2k/\sqrt{x}}, \tag{2.43b}$$

for $n \ge 1$ with $k \equiv (72/(5+4\rho))b_0 c$. Formulas (2.43) follow from (2.42) using (2.41). Formulas (2.42) and (2.43) are true for $n = 1$. We assume that they are true for n and will prove them for $(n+1)$. Indeed,

$$|z_{n+1}(x) - z_n(x)| = K(x, z_n(x)) - K(x, z_{n-1}(x)), \tag{2.44}$$

where

$$\begin{aligned} &K(x, z_n(x)) - K(x, z_{n-1}(x)) \\ &\quad = cxy_1(x) \int_\infty^x y_2(\tau) f_1(\tau)\tau^{-2} (z_n(\tau) - z_{n-1}(\tau))(z_n(\tau) + z_{n-1}(\tau)) d\tau \\ &\qquad - cxy_2(x) \int_{x_1}^x y_1(\tau) f_1(\tau)\tau^{-2} (z_n(\tau) - z_{n-1}(\tau))(z_n(\tau) + z_{n-1}(\tau)) d\tau + O(x^{-3/2}). \end{aligned} \tag{2.45}$$

To find the order of the first term on the right-hand side of (2.45) we integrate by parts and see that the term equals

$$
\begin{aligned}
&cxy_1(x)\int_\infty^x y_2(\tau)f_1(\tau)\tau^{-2}\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)d\tau\\
&=cxy_1(x)\int_\infty^x \tau^{-1/4-\rho}\left[\cos\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\right)\right.\\
&\qquad\left.+\sin\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\right)\right]f_1(\tau)\tau^{-2}\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)d\tau\\
&=O\left(x^{-3/2}\right)\left(z_n(x)-z_{n-1}(x)\right)\left(z_n(x)+z_{n-1}(x)\right)\\
&\quad+cxy_1(x)\int_\infty^x \frac{1}{\sqrt{2}}\sin\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\right)\frac{d}{d\tau}\\
&\qquad\times\left[\tau^{-11/4-\rho}f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)\right]d\tau\\
&\quad-cxy_1(x)\int_\infty^x \frac{1}{\sqrt{2}}\cos\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\right)\frac{d}{d\tau}\\
&\qquad\times\left[\tau^{-11/4-\rho}f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)\right]d\tau\\
&=O\left(x^{-3/2}\right)\left(z_n(x)-z_{n-1}(x)\right)\left(z_n(x)+z_{n-1}(x)\right)\\
&\quad+cxy_1(x)\int_\infty^x \frac{(-11/4-\rho)}{\sqrt{2}}\tau^{-15/4-\rho}\\
&\qquad\times\sin\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\right)f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)d\tau\\
&\quad-cxy_1(x)\int_\infty^x \frac{(-11/4-\rho)}{\sqrt{2}}\tau^{-15/4-\rho}\\
&\qquad\times\cos\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\right)f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)d\tau\\
&\quad+cxy_1(x)\int_\infty^x \frac{1}{\sqrt{2}}\tau^{-11/4-\rho}\\
&\qquad\times\sin\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_1\beta^2\ln\tau+c_1\right)\frac{d}{d\tau}\left[f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)\right]d\tau\\
&\quad-cxy_1(x)\int_\infty^x \frac{1}{\sqrt{2}}\tau^{-11/4-\rho}\\
&\qquad\times\cos\left(\frac{2\sqrt{2}}{3}\tau^{3/2}+\alpha_0\beta^2\ln\tau+c_0\right)\frac{d}{d\tau}\left[f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)\right]d\tau.
\end{aligned}
\tag{2.46}
$$

Since

$$
\begin{aligned}
&\frac{d}{d\tau}\left[f_1(\tau)\left(z_n(\tau)-z_{n-1}(\tau)\right)\left(z_n(\tau)+z_{n-1}(\tau)\right)\right]\\
&\quad=f_1'(\tau)\left(z_n(\tau)-z_{n-1}(\tau)\right)\left(z_n(\tau)+z_{n-1}(\tau)\right)\\
&\qquad+f_1(\tau)\left(z_n'(\tau)-z_{n-1}'(\tau)\right)\left(z_n(\tau)+z_{n-1}(\tau)\right)\\
&\qquad+f_1(\tau)\left(z_n(\tau)-z_{n-1}(\tau)\right)\left(z_n'(\tau)+z_{n-1}'(\tau)\right),
\end{aligned}
\tag{2.47}
$$

then

$$\begin{aligned}&\left|\frac{1}{\sqrt{2}}\frac{d}{d\tau}\left[f_1(\tau)\left(z_n^2(\tau)-z_{n-1}^2(\tau)\right)\right]\right|\\&\qquad\leq 18b_0\tau^{1/2}\left\{\left|z_n(\tau)-z_{n-1}(\tau)\right|+\left|\left(z_n'(\tau)-z_{n-1}'(\tau)\right)\right|\right\}.\end{aligned}\tag{2.48}$$

Therefore the absolute value of the whole expression (2.45) is less than

$$\frac{k}{\sqrt{x}}\left(\left|z_n(x)-z_{n-1}(x)\right|+\left|\left(z_n'(x)-z_{n-1}'(x)\right)\right|\right),\tag{2.49}$$

and also taking into account (2.43a) we get

$$\begin{aligned}\left|z_{n+1}(x)\right|&\leq\left|z_{n+1}(x)-z_n(x)\right|+\left|z_n(x)\right|\\&\leq\frac{q}{\sqrt{x}}\left(\frac{2k}{\sqrt{x}}\right)^n+b_0+\frac{q}{\sqrt{x}}\frac{1-(2k/\sqrt{x})^n}{1-2k/\sqrt{x}}\\&=b_0+\frac{q}{\sqrt{x}}\frac{1-(2k/\sqrt{x})^{n+1}}{1-2k/\sqrt{x}}.\end{aligned}\tag{2.50}$$

Similarly,

$$\begin{aligned}&z_{n+1}'(x)-z_n'(x)\\&\quad=cxy_1'(x)\int_\infty^x y_2(\tau)f_1(\tau)\tau^{-2}\left(z_n(\tau)-z_{n-1}(\tau)\right)\left(z_n(\tau)+z_{n-1}(\tau)\right)d\tau\\&\qquad-cxy_2'(x)\int_{x_1}^x y_1(\tau)f_1(\tau)\tau^{-2}\left(z_n(\tau)-z_{n-1}(\tau)\right)\left(z_n(\tau)+z_{n-1}(\tau)\right)d\tau+O\left(x^{-3/2}\right).\end{aligned}\tag{2.51}$$

So

$$\left|z_{n+1}'(x)-z_n'(x)\right|\leq\frac{k}{\sqrt{x}}\left(\left|z_n(x)-z_{n-1}(x)\right|+\left|\left(z_n'(x)-z_{n-1}'(x)\right)\right|\right),\tag{2.52}$$

and also taking into account (2.43b) we get

$$\left|z_{n+1}'(x)\right|\leq\left|z_{n+1}'(x)-z_n'(x)\right|+\left|z_n'(x)\right|\leq b_0+\frac{q}{\sqrt{x}}\frac{1-(2k/\sqrt{x})^{n+1}}{1-2k/\sqrt{x}}.\tag{2.53}$$

This completes the proof of (2.42) and (2.43) by induction. Now one can show that

$$\begin{aligned}\left|z(x)\right|&=\left|z_0(x)+\left(z_1(x)-z_0(x)\right)+\cdots+\left(z_n(x)-z_{n-1}(x)\right)+\cdots\right|\\&\leq b_0+\frac{q}{\sqrt{x}}\sum_{n=0}^{\infty}(2k/\sqrt{x})^n=b_0+\frac{q}{\sqrt{x}}\frac{1}{1-2k/\sqrt{x}}<2b_0,\end{aligned}\tag{2.54}$$

and similarly

$$\begin{aligned}\left|z'(x)\right|&=\left|z_0'(x)+\left(z_1'(x)-z_0'(x)\right)+\cdots+\left(z_n'(x)-z_{n-1}'(x)\right)+\cdots\right|\\&\leq b_0+\frac{q}{\sqrt{x}}\sum_{n=0}^{\infty}(2k/\sqrt{x})^n=b_0+\frac{q}{\sqrt{x}}\frac{1}{1-2k/\sqrt{x}}<2b_0,\end{aligned}\tag{2.55}$$

therefore $\{z_n(x)\}$ is a fundamental in $C^1[x_0,\infty)$ and partial sums $\{z_n(x)\}$ and $\{z_n'(x)\}$ uniformly converge to $z(x)$ and $z'(x)$, respectively. Thus we proved that $z(x)=z_0(x)+O(x^{-1/2})$ and $z'(x)=z_0'(x)+O(x^{-1/2})$, or $w(x)=O(x^{-1})$ and $w'(x)=O(x^{-1})$. Now from formulas (2.22), (2.19), and (2.20) we will get the statement of the main theorem. □

Remark 2.3. If $\rho=3/4$, then $w(x)=O(x^{-1}\ln x)$ and $w'(x)=O(x^{-1}\ln x)$.

Appendix. We consider the following first-order linear ordinary differential equation:

$$\frac{df}{d\varphi}\cos\varphi-2f\sin\varphi=a_0d^{k-1}\sin^k\varphi, \tag{A.1}$$

where $0\le\varphi<\infty$, a_0, d are some constants and integer $k\ge 1$.

Lemma A.1. *Equation (A.1) has bounded solutions if and only if k is odd, $k=2s+1$, $s=1,2,\dots$. All bounded solutions are given by the formulas*

$$f(\varphi)=TA_s+T\sum_{j=1}^{s}c_j\cos(2j\varphi),\quad T\equiv\frac{a_0d^{2s}}{2s+2}, \tag{A.2}$$

where

$$\begin{aligned}A_s&=\sum_{m=1}^{s}(-1)^{m+1}2^{-2m}C(s+1,m+1)C(2m,m)-C(s+1,1),\\ c_j&=\sum_{m=j}^{s}(-1)^{m+1}2^{-2m+1}C(s+1,m+1)C(2m,m-j).\end{aligned} \tag{A.3}$$

Proof. The general solution of (A.1) has the form

$$f(\varphi)=C\cos^{-2}\varphi+\frac{a_0d^{k-1}}{k+1}\sin^{k+1}\varphi\cos^{-2}\varphi. \tag{A.4}$$

The function $f(\varphi)$ is bounded for all φ if and only if $k=2s+1$, $s=1,2,\dots$, and $C+T=0$. In this case,

$$f(\varphi)=(C+T)\cos^{-2}\varphi+T\sum_{n=1}^{s+1}(-1)^nC(s+1,n)\cos^{2(n-1)}\varphi. \tag{A.5}$$

Therefore

$$f(\varphi)=-C(s+1,1)T+T\sum_{m=1}^{s}(-1)^{m+1}C(s+1,m+1)\cos^{2m}\varphi. \tag{A.6}$$

Since

$$\cos^{2m}\varphi=2^{-2m+1}\sum_{j=0}^{m-1}\cos 2(m-j)\varphi+2^{-2m}C(2m,m), \tag{A.7}$$

then formulas (A.2) and (A.3) follow. □

We consider the following initial value problem on $R_+ = (0,\infty)$:

$$\frac{d\varphi}{dx} = f'(x) + b\frac{\cos\varphi(x)}{x^{\gamma}}, \tag{A.8}$$

$$\varphi(x_0) = \varphi_0, \tag{A.9}$$

where $f(x)$ is from the $C^\infty[x_0,\infty)$ class and b, γ are constants.

Lemma A.2. *If the function $f(x)$ in (A.8) meets the asymptotics*

$$f^{(m)}(x) \sim \left(x^{\alpha/2+1}\right)^{(m)} \quad \text{as } x \longrightarrow \infty \tag{A.10}$$

for $m = 0,1,2,\dots$, where α is a constant such that $\alpha/2+1 > 0$, $\alpha/2+\gamma > 0$, $\gamma > 1/2-\alpha/4$, then for $x_0 > 0$ there exists the unique solution $\varphi(x) \in C^\infty[x_0,\infty)$ of the initial value problem (A.8)–(A.9).

Proof. After the following change of variables:

$$t = \frac{1}{\alpha/2+1}x^{\alpha/2+1}, \qquad h = x^{-\alpha/2}\varphi, \tag{A.11}$$

the problem (A.8)–(A.9) takes the form

$$h'(t) = F(t,h(t)), \qquad h(t_0) = h_0. \tag{A.12}$$

Since $f'(x) \sim (\alpha/2+1)x^{\alpha/2}$ for large x, then

$$F(t,h) = -\beta\frac{h}{t} + b\frac{\cos\left[\left((\alpha/2+1)t\right)^{\beta}h\right]}{\left((\alpha/2+1)t\right)^{v}} + O\left(t^{-\beta}\right), \tag{A.13}$$

where

$$\beta = \frac{\alpha}{\alpha+2}, \qquad v = \frac{2\alpha+2\gamma}{\alpha+2}. \tag{A.14}$$

The derivative of $F(t,h)$ with respect to h is

$$\frac{\partial F}{\partial h} \sim -\frac{\beta}{t} - b\frac{\sin\left[\left((\alpha/2+1)t\right)^{\beta}h\right]}{\left((\alpha/2+1)t\right)^{\delta}}, \tag{A.15}$$

where

$$\delta = \frac{\alpha+2\gamma}{\alpha+2}. \tag{A.16}$$

Since $\alpha/2+1 > 0$ and $\alpha/2+\gamma > 0$, then $|\partial F/\partial h| \le C$ for $0 < t_0 \le t$. So there exists a unique smooth solution $h(t)$ for the initial value problem (A.12). The corresponding $\varphi(x)$ is the unique smooth solution of (A.8)–(A.9). □

We reduce the problem (A.8)–(A.9) to the equivalent integral equation. Integrating (A.8), we get that

$$\varphi(x) = f(x) - f(x_0) + \varphi(x_0) + b\int_{x_0}^{x} t^{-\gamma}\cos\varphi(t)\,dt, \tag{A.17}$$

where $x_0 > 0$. Further,

$$\int_{x_0}^{x} t^{-\gamma}\cos\varphi(t)\,dt = \int_{x_0}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt + \int_{\infty}^{x} t^{-\gamma}\cos\varphi(t)\,dt. \tag{A.18}$$

The first integral in the right-hand side equals

$$\int_{x_0}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt = \int_{x_0}^{x_1} t^{-\gamma}\cos\varphi(t)\,dt + \int_{x_1}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt. \tag{A.19}$$

The integral $\int_{x_0}^{x_1} t^{-\gamma}\cos\varphi(t)\,dt$ clearly converges. To prove that the second integral also converges, we observe that from (A.8) we have

$$\varphi'(x) = f'(x) + O(x^{-\gamma}). \tag{A.20}$$

Since $\alpha/2+\gamma > 0$, then for $x > x_1 \gg 1$, one has

$$0 < C_1 x^{\alpha/2} \le |\varphi'(x)| \le C_2 x^{\alpha/2}. \tag{A.21}$$

Using integration by parts, we see that

$$\int_{x_1}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt = \frac{\sin\varphi(t)}{t^{\gamma}\varphi'(t)}\bigg|_{x_1}^{\infty} - \int_{x_1}^{\infty}\sin\varphi(t)\,d\left(t^{\gamma}\varphi'(t)\right)^{-1}. \tag{A.22}$$

Since $\alpha/2+\gamma > 0$, then

$$\lim_{t\to\infty}\frac{\sin\varphi(t)}{t^{\gamma}\varphi'(t)} = 0. \tag{A.23}$$

Using (A.8) we can get that

$$\left|\frac{d}{dt}\left[\frac{1}{t^{\gamma}\varphi'(t)}\right]\right| \le C_3 t^{-\eta}, \tag{A.24}$$

where $\eta = \min\{\alpha/2+\gamma+1, \alpha/2+2\gamma\}$ and

$$\left|\int_{x_1}^{\infty}\sin\varphi(t)\,d\left(t^{\gamma}\varphi'(t)\right)^{-1}\right| \le C_3\int_{x_1}^{\infty} t^{-\eta}\,dt < \infty. \tag{A.25}$$

Therefore $\int_{x_1}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt$ converges, as well as the integral $\int_{x_0}^{\infty} t^{-\gamma}\cos\varphi(t)\,dt$. Finally we get

$$\varphi(x) = f(x) + c + b\int_{\infty}^{x} t^{-\gamma}\cos\varphi(t)\,dt, \tag{A.26}$$

where c is some constant. The following lemma is true.

Lemma A.3. *Under the conditions of Lemma A.2 there exists a unique smooth solution $\varphi(x)$ of the integral equation (A.26) such that*

$$\varphi(x)=f(x)+c+O\left(x^{-\alpha/2-\gamma}\right),\qquad \varphi'(x)=f'(x)+O\left(x^{-\gamma}\right). \tag{A.27}$$

We need the following.

Theorem A.4. *The differential equation*

$$\frac{d^2x}{dt^2}+\left(1+a\frac{\sin\varphi(t)}{\sqrt{t}}\right)x=0, \tag{A.28}$$

where $\varphi(t)=t+\psi(t)$, $\psi(t)=\alpha\ln t$, and α and a are some constants, has a fundamental system of solutions $\{x_1(t),x_2(t)\}$ satisfying the following asymptotics as $t\to\infty$:

$$\begin{aligned}
x_1(t)&=t^{\rho}\left[-\cos(t+2\psi-\theta)+\sin t\right](1+\varepsilon_1(t)),\\
x_1'(t)&=t^{\rho}\left[\sin(t+2\psi-\theta)+\cos t\right](1+\varepsilon_1(t)),\\
x_2(t)&=t^{-\rho}\left[\cos(t+2\psi+\theta)+\sin t\right](1+\varepsilon_2(t)),\\
x_2'(t)&=t^{-\rho}\left[-\sin(t+2\psi+\theta)+\cos t\right](1+\varepsilon_2(t)).
\end{aligned} \tag{A.29}$$

Here $\rho=a^2(\sqrt{5}/24)$, $\theta=\arctan 2/\sqrt{5}$, $0<\theta<\pi/2$, and $\varepsilon_j(t)=o(1)$ for $j=1,2$.

Proof. The proof of Theorem A.4 is similar to the one given by Harris and Lutz [13, page 579] for (A.28) with $\varphi(t)=t$ and $a=1$. We cannot use their result because the integral $\int_t^\infty \tau^{-1/2}\sin\psi(\tau)\,d\tau$ is not conditionally integrable. But we can still utilize their technique. The differential equation (A.28) is transformed by means of the substitution

$$\vec{x}=\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{bmatrix}\exp(it) & 0\\ 0 & \exp(-it)\end{bmatrix}\vec{y}, \tag{A.30}$$

where $\vec{x}=\binom{x}{x'}$ and $\vec{y}=\binom{y}{y'}$ map into the two-dimensional system

$$\frac{d\vec{y}}{dt}=a\frac{\sin\varphi(t)}{2\sqrt{t}}\begin{bmatrix}i & -\exp(-2it)\\ -\exp(2it) & -i\end{bmatrix}\vec{y}. \tag{A.31}$$

Letting $Q=(q_{ij})$, where $i,j=\overline{1,2}$, $q_{11}=q_{22}=0$,

$$q_{12}(t)=-a\int_\infty^t\frac{\sin\varphi(\tau)}{2\sqrt{\tau}}\exp(-2i\tau)\,d\tau,\qquad q_{21}(t)=-a\int_\infty^t\frac{\sin\varphi(\tau)}{2\sqrt{\tau}}\exp(2i\tau)\,d\tau, \tag{A.32}$$

we observe that

$$\begin{aligned}
q_{12}(t)&=-\frac{1}{4\sqrt{t}}\left(e^{-i(t-\psi(t))}-\frac{1}{3}e^{-i(3t+\psi(t))}\right)+O\left(t^{-3/2}\right),\\
q_{12}(t)&=\frac{1}{4\sqrt{t}}\left(\frac{1}{3}e^{i(3t+\psi(t))}-e^{i(t-\psi(t))}\right)+O\left(t^{-3/2}\right).
\end{aligned} \tag{A.33}$$

Therefore $Q(t) = O(t^{-1/2})$ as $t \to \infty$, and the transformation $\vec{y} = [I + Q(t)]\vec{u}$, where $\vec{u} = \binom{u}{u'}$, leads to the differential system

$$\frac{d\vec{u}}{dt} = \left[\frac{ia\sin\varphi(t)}{2\sqrt{t}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \Lambda(t) + V(t) + R(t)\right]\vec{u}, \tag{A.34}$$

where

$$\begin{aligned}\Lambda(t) &= \frac{ia^2}{16t}\operatorname{diag}\left(-\frac{4}{3}+\frac{1}{3}e^{2i\varphi}+e^{-2i\varphi}, \frac{4}{3}-e^{2i\varphi}-\frac{1}{3}e^{-2i\varphi}\right),\\ V(t) &= \frac{a^2}{8t}\begin{bmatrix} 0 & -e^{2i\psi}+\frac{4}{3}e^{-2it}-\frac{1}{3}e^{-i(4t+2\psi)}\\ -e^{-2i\psi}+\frac{4}{3}e^{2it}-\frac{1}{3}e^{i(4t+2\psi)} & 0\end{bmatrix},\end{aligned} \tag{A.35}$$

and the absolutely integrable matrix-function $R(t)$ has the form

$$\begin{aligned}R(t) = \frac{ia^3}{64t^{3/2}}(e^{i\varphi}-e^{-i\varphi})&\begin{bmatrix} -i\left(-\frac{10}{9}+\frac{1}{3}e^{2i\varphi}+\frac{1}{3}e^{-2i\varphi}\right) & e^{2i\varphi}-\frac{2}{3}e^{-2it}+\frac{1}{9}e^{-i(4t+2\psi)}\\ e^{-2i\varphi}-\frac{2}{3}e^{2it}+\frac{1}{9}e^{i(4t+2\psi)} & i\left(-\frac{10}{9}+\frac{1}{3}e^{2i\varphi}+\frac{1}{3}e^{-2i\varphi}\right)\end{bmatrix}\\ &+O(t^{-5/2}).\end{aligned} \tag{A.36}$$

Since

$$\exp\left(\frac{ia\cos\varphi}{2\sqrt{t}}\right) = 1 + \frac{ia\cos\varphi}{2\sqrt{t}} + O(t^{-1}) \quad \text{as } t \longrightarrow \infty, \tag{A.37}$$

then the transformations

$$\begin{aligned}\vec{u} &= \exp\left[\operatorname{diag}\left(\frac{1}{2}\int_\infty^t \frac{ia\sin\varphi(\tau)}{\sqrt{\tau}}d\tau, -\frac{1}{2}\int_\infty^t \frac{ia\sin\varphi(\tau)}{\sqrt{\tau}}d\tau\right)\right]\vec{\omega},\\ \vec{\omega} &= \exp\left[\operatorname{diag}\left(\frac{ia^2}{16}\int_\infty^t \tau^{-1}\left(\frac{1}{3}e^{2i\varphi}+e^{-2i\varphi}\right)d\tau, -\frac{ia^2}{16}\int_\infty^t \tau^{-1}\left(e^{2i\varphi}+\frac{1}{3}e^{-2i\varphi}\right)d\tau\right)\right]\vec{z},\end{aligned} \tag{A.38}$$

where $\vec{z} = (I + S(t))\vec{v}$ with $S(t) = (s_{ij})$, $s_{11} = s_{22} = 0$,

$$s_{12} = \frac{a^2}{8}\int_\infty^t \tau^{-1}\left(\frac{4}{3}e^{-2i\tau}-\frac{1}{3}e^{-i(4\tau+2\psi)}\right)d\tau, \qquad s_{21} = \frac{a^2}{8}\int_\infty^t \tau^{-1}\left(\frac{4}{3}e^{2i\tau}-\frac{1}{3}e^{i(4\tau+2\psi)}\right)d\tau, \tag{A.39}$$

lead to the differential system

$$\frac{d\vec{v}}{dt} = \left[\frac{a^2}{4t}\begin{pmatrix} -\frac{i}{3} & -\frac{1}{2}e^{2i\psi}\\ -\frac{1}{2}e^{-2i\psi} & \frac{i}{3}\end{pmatrix} - \frac{a^4}{64t^2}\begin{pmatrix} 0 & -e^{2i\psi}\\ -e^{-2i\psi} & 0\end{pmatrix} + O(t^{-5/2})\right]\vec{v}. \tag{A.40}$$

The matrix

$$\begin{bmatrix} -\dfrac{i}{3} & -\dfrac{1}{2}e^{2i\psi} \\ -\dfrac{1}{2}e^{-2i\psi} & \dfrac{i}{3} \end{bmatrix} \tag{A.41}$$

has eigenvalues $\pm\sqrt{5}/6$ with the corresponding eigenvectors

$$\begin{bmatrix} \dfrac{2}{3}\left(i-\dfrac{\sqrt{5}}{2}\right)e^{2i\psi} \\ 1 \end{bmatrix}, \begin{bmatrix} \dfrac{2}{3}\left(i+\dfrac{\sqrt{5}}{2}\right)e^{2i\psi} \\ 1 \end{bmatrix}. \tag{A.42}$$

Hence the substitution

$$\vec{v} = P\vec{\omega} = \begin{bmatrix} \dfrac{2}{3}\left(i-\dfrac{\sqrt{5}}{2}\right)e^{2i\psi} & \dfrac{2}{3}\left(i+\dfrac{\sqrt{5}}{2}\right)e^{2i\psi} \\ 1 & 1 \end{bmatrix}\vec{\omega} \tag{A.43}$$

leads to an L-diagonal system

$$\frac{d\vec{\omega}}{dt} = \left[\frac{a^2}{4t}\begin{pmatrix} \dfrac{\sqrt{5}}{6} & 0 \\ 0 & -\dfrac{\sqrt{5}}{6} \end{pmatrix} - \frac{a^4}{64t^2}\begin{pmatrix} \dfrac{3\sqrt{5}}{6} & \dfrac{4\sqrt{5}}{15}-\dfrac{2}{3}i \\ -\dfrac{4\sqrt{5}}{15}-\dfrac{2}{3}i & -\dfrac{3\sqrt{5}}{6} \end{pmatrix} + O\left(t^{-5/2}\right)\right]\vec{\omega} \tag{A.44}$$

to which Levinson's fundamental theorem [9] can be applied. Thus for the system corresponding to the original differential equation, we obtain a solution matrix $X(t)$ which satisfies as $t \to \infty$ the asymptotics

$$X(t) = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}(I+o(1))P\begin{pmatrix} t^{\rho} & 0 \\ 0 & t^{-\rho} \end{pmatrix}. \tag{A.45}$$

That is, we got a fundamental system of solutions $\{x_1(t),\ x_2(t)\}$ of (A.28) which satisfies (A.29). □

Amplification A.5. *Under the conditions of Theorem A.4 for* $j = 1,2$, $\varepsilon_j(t) = O(t^{-1})$ *as* $t \to \infty$.

Proof. For the proof, we need the following result (see [9]). We consider a linear system of two differential equations

$$\frac{d\vec{y}}{dt} = (\Lambda(t)+R(t))\vec{y}, \tag{A.46}$$

where $t \ge t_0 > 0$ and $\Lambda(t) = \operatorname{diag}\{\lambda_1(t),\lambda_2(t)\}$ satisfies the following conditions:

(1) $\lambda_1(t)-\lambda_2(t) \ge 0$;
(2) $\int_{t_0}^{\infty}(\lambda_1(t)-\lambda_2(t))dt = \infty$;
(3) $R(t)$ is absolutely integrable.

Then there exist solutions $y_1(t)$, $y_2(t)$ of (A.46) such that

$$\vec{y}_j(t) = \exp\left(\int_{t_0}^{t} \lambda_j(t)\,dt\right)(\vec{e}_j + \vec{\varepsilon}), \quad j = 1,2, \tag{A.47}$$

where $\vec{\varepsilon} = \binom{\varepsilon_1(t)}{\varepsilon_2(t)}$, $\vec{e}_1 = \binom{1}{0}$, $\vec{e}_2 = \binom{0}{1}$, and $\varepsilon_1(t)$, $\varepsilon_2(t)$ meet the following estimates:

$$\begin{gathered} |\varepsilon_2(t)| \le C\int_{\infty}^{t} \|R(\tau)\|\,d\tau, \\ |\varepsilon_1(t)| \le C\left[\int_{t_0}^{t} \exp\left(\int_{\tau}^{t} (\lambda_1(s) - \lambda_2(s))\,ds\right)\|R(\tau)\|\,d\tau + \int_{\infty}^{t} \|R(\tau)\|\,d\tau\right], \end{gathered} \tag{A.48}$$

where C is some constant. In our case $\lambda_1(t) = a^2\sqrt{5}/24t$, $\lambda_2(t) = -a^2\sqrt{5}/24t$, and all the elements of the matrix $R(t)$ are of order $O(t^{-2})$. So

$$\begin{aligned} |\varepsilon_2(t)| &\le C\int_{\infty}^{t} O(\tau^{-2})\,d\tau = O(t^{-1}), \\ |\varepsilon_1(t)| &\le C\left[\int_{t_0}^{t} \exp\left(\int_{\tau}^{t} \frac{a^2\sqrt{5}}{12s}\,ds\right)O(\tau^{-2})\,d\tau + O(t^{-1})\right] \\ &= C\left[t^{a^2\sqrt{5}/12}\int_{t_0}^{t} \tau^{-a^2\sqrt{5}/12-2}\,d\tau + O(t^{-1})\right] \\ &= O(t^{-1}). \end{aligned} \tag{A.49}$$

The amplification is proved. □

PROOF OF LEMMA 2.1. The Liouville-Green transformation,

$$t = \int_{x_0}^{x} \sqrt{F(\tau)}\,d\tau, \qquad z = \sqrt[4]{F(x)}\,y, \tag{A.50}$$

takes (2.25) into

$$\frac{d^2z}{dt^2} + (1 + q(t))z = 0, \tag{A.51}$$

where

$$q(t) = a\frac{\sin(t + \psi(t))}{\sqrt{t}} + O(t^{-1}), \quad \psi(t) = \alpha \ln t, \ a = c_3\beta, \ \alpha = c_4\beta^2, \tag{A.52}$$

and c_3, c_4 are some constants. Now applying Theorem A.4 and Amplification A.5 we get formulas (2.26). □

PROOF OF LEMMA 2.2. Using Lemma 2.1 one can show that the solution $y_0(x)$ of (2.28) is representable in the form

$$y_0(x) = y_1(x)\int_{\infty}^{x} y_2(\tau)f_0(\tau)\,d\tau - y_2(x)\int_{b}^{x} y_1(\tau)f_0(\tau)\,d\tau, \tag{A.53}$$

where $\{y_1(x), y_2(x)\}$ is a fundamental system of solution of (2.25). Using formulas (2.26), we will find that the order of the first term in the right-hand side of (A.53) equals

$$\begin{aligned}
&y_1(x)\int_\infty^x y_2(\tau)f_0(\tau)d\tau\\
&\quad = O(x^{-1/4+\rho})\int_\infty^x \tau^{-1/4-\rho}\Big[\cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_1\beta^2\ln\tau+c_1\Big)\\
&\qquad\qquad + \sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_0\beta^2\ln\tau+c_0\Big)\Big]O(1)(1-\cos 2\varphi_0(\tau))d\tau\\
&\qquad + O(x^{-1/4+\rho})\int_\infty^x \tau^{-1/4-\rho}O(\tau^{-3/2})O(1)d\tau\\
&\quad = O(x^{-1}).
\end{aligned}\tag{A.54}$$

To estimate the second term we should consider three cases and take into account that $\rho > 0$ and the remainders in the formulas (2.26) have the order of $O(x^{-3/2})$.

Case 1. Let $\rho < 3/4$. Then we will take $b = \infty$ and the order of the second term can be estimated as

$$\begin{aligned}
&y_2(x)\int_b^x y_1(\tau)f_0(\tau)d\tau\\
&\quad = O(x^{-1/4-\rho})\int_\infty^x \tau^{-1/4+\rho}\Big[\cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_1\beta^2\ln\tau+c_1\Big)\\
&\qquad\qquad + \sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_0\beta^2\ln\tau+c_0\Big)\Big]O(1)(1-\cos 2\varphi_0(\tau))d\tau\\
&\qquad + O(x^{-1/4-\rho})\int_\infty^x \tau^{-1/4+\rho}O(\tau^{-3/2})O(1)d\tau\\
&\quad = O(x^{-1}).
\end{aligned}\tag{A.55}$$

Here the first integral in the right-hand side is estimated using integration by parts, and the second integral, which involves the remainder from the formula (2.26), is estimated directly.

Case 2. Let $\rho > 3/4$. In this case we will take $b = x_0$ and

$$\begin{aligned}
&y_2(x)\int_b^x y_1(\tau)f_0(\tau)d\tau\\
&\quad = O(x^{-1/4-\rho})\int_{x_0}^x \tau^{-1/4+\rho}\Big[\cos\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_1\beta^2\ln\tau+c_1\Big)\\
&\qquad\qquad + \sin\Big(\frac{2\sqrt{2}}{3}\tau^{3/2}+a_0\beta^2\ln\tau+c_0\Big)\Big]O(1)(1-\cos 2\varphi_0(\tau))d\tau\\
&\qquad + O(x^{-1/4-\rho})\int_{x_0}^x \tau^{-1/4+\rho}O(\tau^{-3/2})O(1)d\tau\\
&\quad = O(x^{-1}).
\end{aligned}\tag{A.56}$$

Case 3. Let $\rho = 3/4$. We will take again $b = x_0$. Then

$$y_2(x)\int_b^x y_1(\tau)f_0(\tau)d\tau = O(x^{-1}\ln x). \tag{A.57}$$

Similarly one can show that $y_0'(x) = O(x^{-1})$ when $\rho \neq 3/4$ and $y_0'(x) = O(x^{-1}\ln x)$ when $\rho = 3/4$. Lemma 2.2 is proved. □

References

[1] A. S. Abdullayev, *On the theory of Painlevé's second equation*, Soviet Math. Dokl. **28** (1983), no. 3, 726–729.

[2] ———, *The second Painlevé equation and asymptotic behavior of solutions of certain second-order nonlinear differential equations*, Dokl. Akad. Nauk USSR **1984** (1984), no. 7, 7–8 (Russian).

[3] ———, *Asymptotics of solutions of the generalized sine-Gordon equation, Painlevé's third equation and d'Alambert's equation*, Soviet Math. Dokl. **31** (1985), no. 1, 45–47.

[4] ———, *Justification of asymptotic formulas for the fourth Painlevé equation*, Stud. Appl. Math. **99** (1997), no. 3, 255–283.

[5] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM Studies in Applied Mathematics, vol. 4, Society for Industrial and Applied Mathematics (SIAM), Pennsylvania, 1981.

[6] P. A. Clarkson and J. B. McLeod, *A connection formula for the second Painlevé transcendent*, Arch. Ration. Mech. Anal. **103** (1988), no. 2, 97–138.

[7] ———, *Integral equations and connection formulae for the Painlevé equations*, Painlevé Transcendents: Their Asymptotics and Physical Applications (Sainte-Adèle, PQ, 1990) (D. Levi and P. Winternitz, eds.), NATO Adv. Sci. Inst. Ser. B Phys., vol. 278, Plenum Press, New York, 1992, pp. 1–31.

[8] P. A. Deift and X. Zhou, *Asymptotics for the Painlevé II equation*, Comm. Pure Appl. Math. **48** (1995), no. 3, 277–337.

[9] M. V. Fedoryuk, *Asymptotic Analysis. Linear Ordinary Differential Equations*, Springer-Verlag, Berlin, 1993.

[10] H. Flaschka and A. C. Newell, *Monodromy- and spectrum-preserving deformations. I*, Comm. Math. Phys. **76** (1980), no. 1, 65–116.

[11] A. S. Fokas and X. Zhou, *On the solvability of Painlevé II and IV*, Comm. Math. Phys. **144** (1992), no. 3, 601–622.

[12] V. V. Golubev, *Lectures on Analytical Theory of Differential Equations*, 2nd ed., State Publishing House, Moscow, 1950.

[13] W. A. Harris Jr. and D. A. Lutz, *A unified theory of asymptotic integration*, J. Math. Anal. Appl. **57** (1977), no. 3, 571–586.

[14] S. P. Hastings and J. B. McLeod, *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation*, Arch. Ration. Mech. Anal. **73** (1980), no. 1, 31–51.

[15] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Wiley-Interscience, New York, 1976.

[16] E. L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.

[17] A. R. Its, *The Riemann-Hilbert problem and integrable systems*, Notices Amer. Math. Soc. **50** (2003), no. 11, 1389–1400.

[18] A. R. Its, A. S. Fokas, and A. A. Kapaev, *On the asymptotic analysis of the Painlevé equations via the isomonodromy method*, Nonlinearity **7** (1994), no. 5, 1291–1325.

[19] A. R. Its and V. Yu. Novokshenov, *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Lecture Notes in Mathematics, vol. 1191, Springer-Verlag, Berlin, 1986.

[20] M. Jimbo, *An inverse scattering problem and the fifth Painlevé transcendent*, Progr. Theoret. Phys. **61** (1979), no. 1, 359–360.

[21] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Phys. D **2** (1981), no. 3, 407–448.

[22] ______, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*, Phys. D **4** (1981), no. 1, 26–46.

[23] M. Jimbo, T. Miwa, and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function*, Phys. D **2** (1981), no. 2, 306–352.

[24] A. A. Kapaev, *Global asymptotics of the second Painlevé transcendent*, Phys. Lett. A **167** (1992), no. 4, 356–362.

[25] A. V. Kitaev, *The justification of asymptotic formulas that can be obtained by the method of isomonodromic deformations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **179** (1989), no. 19, 101–109, 189–190 (Russian).

[26] M. D. Kruskal and P. A. Clarkson, *The Painlevé-Kowalevski and poly-Painlevé tests for integrability*, Stud. Appl. Math. **86** (1992), no. 2, 87–165.

[27] H. Segur and M. J. Ablowitz, *Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent*, Phys. D **3** (1981), no. 1-2, 165–184.

[28] K. Ueno, *Monodromy preserving deformation of linear differential equations with irregular singular points*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 3, 97–102.

Alisher S. Abdullayev: Department of Mathematics, American River College, 4700 College Oak Drive, Sacramento, CA 95841, USA

E-mail address: abdula@arc.losrios.edu