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ON STABILITY AND BIFURCATION OF SOLUTIONS OF AN SEIR EPIDEMIC MODEL WITH VERTICAL TRANSMISSION

M. M. A. EL-SHEIKH and S. A. A. EL-MAROUF

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A four-dimensional SEIR epidemic model is considered. The stability of the equilibria is established. Hopf bifurcation and center manifold theories are applied for a reduced three-dimensional epidemic model. The boundedness, dissipativity, persistence, global stability, and Hopf-Andronov-Poincaré bifurcation for the four-dimensional epidemic model are studied.

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1. Introduction. Many infectious diseases in nature transmit through both horizontal and vertical models. These include such human diseases as Rubella, Herpes Simplex, Hepatitis B, Chagas, and the most notorious AIDS (see [8, 9]). For human and animal diseases, horizontal transmission typically occurs through direct or indirect physical contact with hosts, or through a disease vector such as mosquitos, ticks, or other biting insects. Vertical transmission can be accomplished through transplacental transfer of disease agents. Li et al. [10] discussed vertical and horizontal models. In standard SIR compartmental models the vertical transmission can be incorporated by assuming that the fraction q of the offspring from the infectious I class is infectious at birth, and hence birth flux, qbI, enters the I class and the remaining birth $b-qbI$ enters the susceptible S class.

In this paper we study an SEIR model in which vertical transmission is incorporated based on the above assumption. The total host population is partitioned into *susceptible, exposed, infectious*, and recovered with densities denoted, respectively, by $S(t)$, $E(t)$, $I(t)$, and $R(t)$. The natural birth, and death rates are assumed to be identical and denoted by b. The horizontal transmission is assumed to take the form of direct contact between infectious and susceptible hosts. The incidence rate term $H(I,S)$ is assumed to be differentiable, $\partial H/\partial I$ and $\partial H/\partial S$ are nonnegative and finite for all I and S. For special forms of the incidence rate $H(I,S)$, see [3, 10, 11, 13, 14, 16]. Here, b is the natural birth rate of the host population which is assumed to have a constant density 1. For the vertical transmission, we assume that a fraction p and a fraction q of the offspring from the exposed and infectious classes, respectively, are born into the exposed class E. Consequently, the birth flux into the exposed class is given by $pbE+qbI$ and the birth flux into the susceptible class is given by $b-pbE-qbI$, naturally $0 \le p \le 1$ and $0 \le q \le 1$. The above assumptions lead to the following system of differential equations:

$$\begin{aligned}
S' &= b - IH(I,S) - pbE - qbI - bS,\\
E' &= IH(I,S) + pbE + qbI - (\mu + b)E,\\
I' &= \mu E - (\gamma + b)I,\\
R' &= \gamma I - bR,
\end{aligned} \tag{1.1}$$

where $(S,E,I,R) \in \mathbb{R}_+^4$. The parameter $\mu \geq 0$ is the rate at which the exposed individuals become infectious and $\gamma \geq 0$ is the rate at which the infectious individuals become recovered. Therefore $1/\mu$ is the mean latent period and $1/\gamma$ is the mean infectious period.

The model (1.1) is a more general and epidemiological model than those discussed in [3, 12]. This paper is organized as follows. In Section 2, we discuss the stability properties of the reduced three-dimensional epidemic model, with a technique different from the method of [10, 11, 12, 13]. Also, we discuss the bifurcation of periodic solutions using Hopf bifurcation theory with a technique similar to that of [2, 6]. We also use a technique similar to that of [2] to apply center manifold theorem. Our results in Section 2 are consistent with those in [12], in the special case when the incidence rate is $H(I,S) = \beta S$. In Section 3, we study the boundedness, dissipativity, persistence, and global stability of solutions of the four-dimensional model (1.1). Our technique in Section 3 is similar to the technique used in [15]. The paper ends with a brief discussion in Section 4.

2. Three-dimensional reduced epidemic model. As in [10, 11, 12, 13] we suppose that $S(t)+E(t)+I(t)+R(t) = 1$, and so we use the relation $R(t) = 1-S(t)-E(t)-I(t)$ and obtain the following three-dimensional system:

$$\begin{aligned}
S' &= b - IH(I,S) - pbE - qbI - bS,\\
E' &= IH(I,S) + pbE + qbI - (\mu + b)E,\\
I' &= \mu E - (\gamma + b)I
\end{aligned} \tag{2.1}$$

on the closed, positively invariant set

$$\Gamma = \{(S,E,I) \in \mathbb{R}_+^3 : S+E+I \leq 1\}. \tag{2.2}$$

The linearized problem corresponding to (2.1) is

$$Y' = MY, \quad \text{where } Y = \begin{pmatrix} y_1\\ y_2\\ y_3 \end{pmatrix}, \quad (y_1,y_2,y_3) \in \mathbb{R}_+^3, \tag{2.3}$$

where M is the variational matrix

$$M = \begin{pmatrix} -I\dfrac{\partial H}{\partial S} - b & -pb & -H - I\dfrac{\partial H}{\partial I} - qb \\ I\dfrac{\partial H}{\partial S} & bp - (\mu + b) & H + I\dfrac{\partial H}{\partial I} + qb \\ 0 & \mu & -(\gamma + b) \end{pmatrix}. \tag{2.4}$$

It is clear that (2.1) has $\bar{P}_\circ = (1,0,0)$ as a trivial equilibrium (a disease-free equilibrium). The Jacobian matrix of (2.1) at $\bar{P}_\circ$ is

$$M_{\bar{P}_\circ} = \begin{pmatrix} -b & -pb & -H(0,1) - qb \\ 0 & bp - (\mu + b) & H(0,1) + qb \\ 0 & \mu & -(\gamma + b) \end{pmatrix}. \tag{2.5}$$

One of the eigenvalues of $M_{\bar{P}_\circ}$ is $\lambda_1 = -b < 0$, and the other two eigenvalues $\lambda_{2,3}$ satisfy the following quadratic equation:

$$\lambda^2 - \xi_1 \lambda - \xi_2 = 0, \tag{2.6}$$

where $\xi_1 = (pb - (\mu + \gamma + 2b))$ and $\xi_2 = ((\gamma + b)(pb - (\mu + b)) + \mu(H(0,1) + qb))$.

Therefore,

$$\begin{aligned} \lambda_2 + \lambda_3 &= \xi_1 < 0, \\ \lambda_2 \lambda_3 &= -\xi_2. \end{aligned} \tag{2.7}$$

The following result gives sufficient conditions for asymptotic stability of the point $\bar{P}_\circ$.

Theorem 2.1. *Assume that the following condition holds:*

(A1) $\xi_2 < 0$,

where ξ_1 and ξ_2 are as defined in (2.7). The disease-free equilibrium $\bar{P}_\circ = (1,0,0)$ is locally asymptotically stable.

Proof. The proof is by inspection of the eigenvalues of the Jacobian matrix for $\bar{P}_\circ = (1,0,0)$, and the qualitative theory of differential equations. □

Remark 2.2. (i) Theorem 2.1 completely determines the local dynamics of (2.1) in Γ when condition (A1) is satisfied. Its epidemiological implication is that the infected population (the sum of the latent and the infectious population) vanishes in time, so the disease dies out.

(ii) In the above theorem, our results are consistent with those in [12], in the special case $H(I,S) = \beta S$. Also our technique is different from that of [12]. For our model, the basic reproduction number is $R_\circ(p,q) = \mu H(0,1)/((\gamma + b)(\mu + b) - pb(\gamma + b) - \mu qb)$, where $R_\circ(p,q) > 0$ for $0 < p$, $q < 1$. Here, condition (A1) reads $R_\circ(p,q) \leq 1$ in terms of the notation of [12].

Now we consider the nontrivial equilibrium $P = (S_\circ, E_\circ, I_\circ)$ of system (2.1), where

$$S_\circ = 1 - \left(\frac{H_\circ}{b} + \frac{p(\gamma + b)}{\mu} + q\right) I_\circ, \qquad E_\circ = \frac{\gamma + b}{\mu} I_\circ. \tag{2.8}$$

The Jacobian matrix at P is

$$M_P = \begin{pmatrix} -I_\circ H_{S_\circ} - b & -pb & -H_\circ - I_\circ H_{I_\circ} - qb \\ I_\circ H_{S_\circ} & pb - (\mu + b) & H_\circ + I_\circ H_{I_\circ} + qb \\ 0 & \mu & -(\gamma + b) \end{pmatrix}, \tag{2.9}$$

where

$$H_{S_\circ} = \left.\frac{\partial H}{\partial S}\right|_{S=S_\circ}, \qquad H_{I_\circ} = \left.\frac{\partial H}{\partial I}\right|_{I=I_\circ}, \qquad H_\circ = H(I_\circ, S_\circ). \tag{2.10}$$

We assume that $H_{S_\circ}$, $H_{I_\circ}$, and $H_\circ$ are positive. The characteristic equation of M at P is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{2.11}$$

where

$$\begin{aligned} a_1 &= (\mu + \gamma + I_\circ H_{S_\circ} + 3b - pb), \\ a_2 &= ((I_\circ H_{S_\circ} + b)((b + \mu - pb) + (\gamma + b)) + (b + \mu - pb)(\gamma + b) \\ &\quad + pbI_\circ H_{S_\circ} - \mu(H_\circ + I_\circ H_{I_\circ} + qb)), \\ a_3 &= ((I_\circ H_{S_\circ} + b)(b + \mu - pb)(\gamma + b) + pbI_\circ H_{S_\circ}(\gamma + b) \\ &\quad + \mu I_\circ H_{S_\circ}(H_\circ + I_\circ H_{I_\circ} + qb) - \mu(I_\circ H_{S_\circ} + b)(H_\circ + I_\circ H_{I_\circ} + qb)). \end{aligned} \tag{2.12}$$

Since the Routh-Hurwitz criterion says that $P = (S_\circ, E_\circ, I_\circ)$ is asymptotically stable if $a_1 > 0$, $a_3 > 0$, and $a_1a_2 - a_3 > 0$, then we have the following theorem.

Theorem 2.3. *Let the following two conditions be satisfied:*
(A2) $(b + \mu - pb) + (\gamma + b) > \mu(H_\circ + I_\circ H_{I_\circ} + qb)$,
(A3) $pb(I_\circ H_{S_\circ} + \mu + 2b - pb) > (H_\circ + I_\circ H_{I_\circ} + qb)$.
Then the equilibrium point $P = (S_\circ, E_\circ, I_\circ)$ is locally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 2.1, so it is omitted. □

Now choose μ as a bifurcation parameter for system (2.1). Let μ_c be the value of μ at which the characteristic equation (2.11) has two pure imaginary roots $\lambda_{1,2}$. Thus we have the following result.

Theorem 2.4. *If the assumption (2.19) holds, then at $\mu = \mu_c$, there exists a one-parameter family of periodic solutions bifurcating from the critical point $P = (S_\circ, E_\circ, I_\circ)$ with period T, where $T \to T_\circ$ as $\mu \to \mu_c$ and where $T_\circ = 2\pi/\omega_\circ = 2\pi/\sqrt{a_2}$ and a_2 is given in (2.12).*

Proof. Since there exists at least one real root of the cubic equation (2.11), λ_3, say, we have the following factorization:

$$(\lambda - \lambda_3)[\lambda^2 + (\lambda_3 + a_1)\lambda + (\lambda_3^2 + a_1\lambda_3 + a_2)] = 0. \tag{2.13}$$

Since, by (2.11),

$$\lambda_1+\lambda_2+\lambda_3=-a_1, \tag{2.14}$$

also at $\mu=\mu_c$, we obtain

$$\begin{gathered}\lambda_3=-a_1, \qquad \lambda_1=\bar{\lambda}_2,\\ \lambda_{1,2}=-\frac{1}{2}\left\{(\lambda_3+a_1)\pm\sqrt{(\lambda_3+a_1)^2-4(\lambda_3^2+a_1\lambda_3+a_2)}\right\}.\end{gathered} \tag{2.15}$$

Thus, at $\mu=\mu_c$, (2.11) can be written in the following form:

$$D_\mu(a_1)=a_1a_2-a_3. \tag{2.16}$$

Hence, since $a_3>0$ and $a_2>0$ at $\mu=\mu_c$, we should have $\lambda_3=-a_1<0$. Also, the critical value $\mu=\mu_c>0$ is the solution of (2.16) which can be seen by (2.12) to be the quadratic equation in μ as follows:

$$-c_1\mu^2-c_2\mu+c_3=0, \tag{2.17}$$

where

$$\begin{aligned}c_1&=(H_\circ+I_\circ H_{I_\circ}+qb)-(\gamma+I_\circ H_{S_\circ}+2b),\\ c_2&=((H_\circ+I_\circ H_{I_\circ}+qb)(\gamma+I_\circ H_{S_\circ}+2b-pb)\\ &\qquad-(I_\circ H_{S_\circ}+4b-2pb)(\gamma+I_\circ H_{S_\circ}+2b)-pbI_\circ H_{S_\circ}),\\ c_3&=((I_\circ H_{S_\circ}+2b-pb)(\gamma+I_\circ H_{S_\circ}+2b)(\gamma+2b-pb)+pbI_\circ H_{S_\circ}).\end{aligned} \tag{2.18}$$

Conversely, knowing that $a_3>0$, $a_1>0$, and $\mu>0$, then we can solve (2.17) for $\mu_c>0$, and we then know that $a_2>0$, $\lambda_3=-a_1<0$, and $\lambda_{1,2}$ are conjugate imaginary.

Now, choosing both γ and $H_{S_\circ}$ to be sufficiently small and $H_\circ$ sufficiently large, then we get

$$(H_\circ+I_\circ H_{I_\circ}+qb)>(\gamma+I_\circ H_{S_\circ}+2b). \tag{2.19}$$

But since by (2.17), (2.19), $c_1>0$, $c_2>0$, and $c_3>0$,

$$D_0(a_1)=c_3>0, \qquad \lim_{\mu\to\pm\infty}D_\mu(a_1)=\infty. \tag{2.20}$$

Thus μ_c is uniquely determined (Figure 2.1).

Now, since by (2.11), $\lambda_3=-a_1<0$ and

$$\begin{aligned}D_\mu(a_1)&=a_1a_2-a_3\\ &=(a_1+\lambda_3)(\lambda_1\lambda_2-a_1\lambda_1),\\ \operatorname{sgn}D_\mu(a_1)&=\operatorname{sgn}(a_1+\lambda_3),\end{aligned} \tag{2.21}$$

consequently we have

$$\begin{gathered}\operatorname{Re}\lambda_{1,2}=\frac{1}{2}(a_1+\lambda_3)<0 \quad \text{for } \mu>\mu_c,\\ \operatorname{Re}\lambda_{1,2}>0 \quad \text{for } \mu<\mu_c.\end{gathered} \tag{2.22}$$

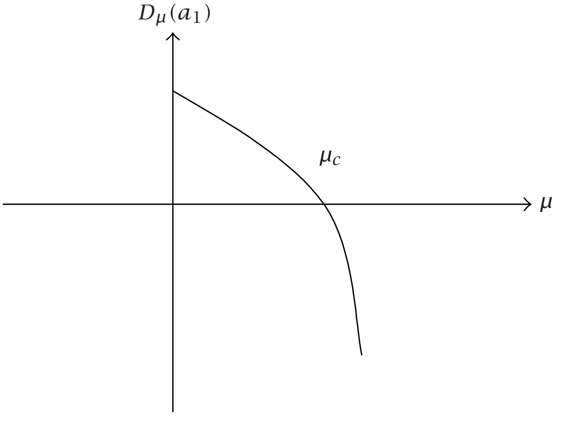

FIGURE 2.1. The uniqueness of the bifurcation parameter μ_c.

By the above discussion, we see that as μ is increased through μ_c, there exists a pair of complex conjugate imaginary eigenvalues $\lambda_{1,2}$ of the Jacobian matrix M_P. Since at $\mu = \mu_c$, $\lambda_3 = -a_1$, $\lambda_{1,2} = \pm i\sqrt{a_2} = \pm i\omega_\circ$, where it is clear that $\omega_\circ > 0$. Now, since for $\lambda_1 = \bar{\lambda}_2$,

$$\operatorname{Re}\lambda_2 = \frac{1}{2}(\lambda_2 + \bar{\lambda}_2) = 0 \quad \text{at } \mu = \mu_c, \tag{2.23}$$

and by the above discussion we see that

$$\begin{aligned} &\operatorname{Re}\lambda_2 > 0 \quad \text{for } \mu < \mu_c, \\ &\operatorname{Re}\lambda_2 < 0 \quad \text{for } \mu > \mu_c, \end{aligned} \tag{2.24}$$

thus

$$\begin{aligned} \frac{d}{d\mu}(\operatorname{Re}\lambda_2)\big|_{\mu=\mu_c} &= \frac{-1}{2}\frac{d}{d\mu}(\lambda_3 + a_1)\big|_{\mu=\mu_c} \\ &= \operatorname{Re}\left(\frac{d}{d\mu}\lambda_2\right)\bigg|_{\mu=\mu_c} < 0. \end{aligned} \tag{2.25}$$

This completes the proof. □

REMARK 2.5. In a similar manner as in [2, page 449] the Jacobian matrix M_P can be diagonalized to be

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & a_2 \end{pmatrix}, \tag{2.26}$$

where $\alpha_{12} = -\alpha_{21} = -\sqrt{a_2}$.

Thus, putting $\eta = \mu - \mu_c$, at $\eta = 0$, $\alpha_{11} = \alpha_{22} = 0$, we can write (2.1) in the form

$$\begin{aligned} y' &= \phi(y, \theta), \\ \theta' &= 0. \end{aligned} \tag{2.27}$$

Then a center manifold C exists for this canonical suspended system at $(y, \theta) = (0, 0) \in \mathbb{R}^3 \times \mathbb{R}^1$ (see [1, page 52]).

3. The four-dimensional epidemic model. In this section we will show that system (1.1) is bounded, positively invariant, with respect to a region in $\mathbb{R}_+^4$, and dissipative.

Definition 3.1 [7, page 394]. A differential equation $X' = f(X)$ is said to be dissipative if there is a bounded subset B of $\mathbb{R}^2$ such that for any $X^\circ \in \mathbb{R}^2$ there is a time $t_\circ$, which depends on X° and B, so that the solution $\phi(t, X^\circ)$ through X° satisfies $\phi(t, X^\circ) \in B$ for $t \geq t_\circ$.

Theorem 3.2. *Let Γ be the region defined by*

$$\Gamma = \{(S, E, I, R) \in \mathbb{R}_+^4 : S + E + I + R = 1\}. \tag{3.1}$$

Then

(i) Γ *is positively invariant,*
(ii) *all the solutions of system (1.1) are uniformly bounded,*
(iii) *system (1.1) is dissipative.*

Proof. Let $S(t_\circ) = \bar{S}_\circ > 0$. Since

$$\begin{aligned} S' &= b - IH(I, S) - pbE - qbI - bS \\ &< b - bS - IH(I, S) \\ &< b - bS - S \min_{S \in \Gamma} \tilde{H}(I, S), \end{aligned} \tag{3.2}$$

where $IH(I, S) = S\tilde{H}(I, S)$, letting $\delta = -(b + \min_{S \in \Gamma} \tilde{H}(I, S))$, thus

$$S' < b + \delta S, \quad \delta < 0. \tag{3.3}$$

Then

$$S \leq \frac{-b}{\delta} + \bar{S}_\circ e^{\delta t} \tag{3.4}$$

so that

$$S \leq \max\left(\frac{-b}{\delta} + \bar{S}_\circ\right), \tag{3.5}$$

thus

$$\limsup_{t \to \infty} S \leq \frac{-b}{\delta}, \quad \delta < 0, \ \bar{S}_\circ \geq 0. \tag{3.6}$$

Hence $S(t)$ is uniformly bounded. Since $S(t) = 1-E(t)-I(t)-R(t)$ and $S(t)$ is uniformly bounded, the solutions of (1.1) are uniformly bounded. Dissipativity of system (1.1) follows by Definition 3.1. Thus the proof is completed. □

Now, we discuss the existence and global stability of the equilibria of (1.1). The equilibria points of (1.1) are obtained by solving the system of isocline equations

$$\begin{gathered} b-IH(I,S)-pbE-qbI-bS=0, \\ IH(I,S)+pbE+qbI-(\mu+b)E=0, \\ \mu E-(\gamma+b)I=0, \\ \gamma I-bR=0. \end{gathered} \tag{3.7}$$

The possible equilibria points of (1.1) are $P_\circ = (1,0,0,0)$ and $P = (S^*,E^*,I^*,R^*)$. The Jacobian matrix due to linearization of (1.1) at the equilibrium point $P_\circ = (1,0,0,0)$ is

$$J_{P_\circ=(1,0,0,0)} = \begin{pmatrix} -b & -pb & -H(0,1)-qb & 0 \\ 0 & pb-(\mu+b) & H(0,1)+qb & 0 \\ 0 & \mu & -(\gamma+b) & 0 \\ 0 & 0 & \gamma & -b \end{pmatrix}. \tag{3.8}$$

The eigenvalues of $P_\circ = (1,0,0,0)$ are given by $\lambda_1 = \lambda_2 = -b < 0$, and the other tow eigenvalues $\lambda_{2,3}$ satisfy the following quadratic equation:

$$\lambda^2-(pb-(\mu+\gamma+2b))\lambda-((\gamma+b)(pb-(\mu+b))+\mu(H(0,1)+qb))=0. \tag{3.9}$$

Thus

$$\begin{gathered} \lambda_3+\lambda_4 = (pb-(\mu+\gamma+2b))<0, \\ \lambda_3\lambda_4 = ((\gamma+b)(pb-(\mu+b))-\mu(H(0,1)+qb)). \end{gathered} \tag{3.10}$$

The above discussion leads to the following results.

Theorem 3.3. (i) *If $(\gamma+b)(\mu+b-pb) > \mu(H(1,0)+qb)$ holds, then $P_\circ(1,0,0,0)$ is locally asymptotically stable.*

(ii) *If $(\gamma+b)(\mu+b-pb) < \mu(H(1,0)+qb)$ holds, then the equilibrium point $P_\circ = (1,0,0,0)$ is a hyperbolic saddle and is repelling in both directions of I and R. In particular, the dimensions of the stable manifold W^+ and unstable manifold W^- are given by*

$$\operatorname{Dim} W^+(P_\circ=(1,0,0,0))=1, \qquad \operatorname{Dim} W^-(P_\circ=(1,0,0,0))=3, \tag{3.11}$$

respectively.

Proof. The proof of (i) is similar to the proof of Theorem 2.1, so it is omitted. The proof of (ii) follows directly from inspection of the eigenvalues of the Jacobian matrix at $P_\circ = (1,0,0,0)$ and examples by Freedman and Mathsen [4]. □

Now, to give sufficient conditions for the existence of a positive interior equilibrium $P = (S^*, E^*, I^*, R^*)$, we discuss the uniform persistence of (1.1). To show a uniform persistence in the set

$$\mathbb{R}^+_{\mathrm{SEIR}} = \{(S,E,I,R) : S > 0,\ E > 0,\ I > 0,\ R > 0\}, \tag{3.12}$$

we assume the following hypotheses for system (1.1).

(h1) All dynamics are trivial on $\partial\mathbb{R}^+_{\mathrm{SEIR}}$ (the boundary of the set $\mathbb{R}^+_{\mathrm{SEIR}}$).

(h2) All invariant sets (equilibrium points) are hyperbolic and isolated.

(h3) No invariant sets on $\partial\mathbb{R}^+_{\mathrm{SEIR}}$ are asymptotically stable.

(h4) If an equilibrium point exists in the interior of any three-dimensional subspace of $\mathbb{R}^+_{\mathrm{SEIR}}$, it must be globally asymptotically stable with respect to orbits initiating in that interior.

(h5) If M is an invariant set on $\partial\mathbb{R}^+_{\mathrm{SEIR}}$ and $W^+(M)$ and it is a strong stable manifold, then $W^+(M) \cap \partial\mathbb{R}^+_{\mathrm{SEIR}} = \phi$.

(h6) All invariant sets are cyclic.

Here, we drive criteria for the global stability hypothesis (h4) to be valid.

Now, we discuss the global stability of $\bar{P}_\circ = (1,0,0)$. In $\mathbb{R}^4_+$ consider the Lyapunov function

$$V = \mu E + (\mu + b - pb)I. \tag{3.13}$$

Thus

$$\begin{aligned} V^{\cdot} &= [\mu H(I,S) - ((\gamma+b)(\mu+b-pb) - qb)]I \\ &= [((\gamma+b)(\mu+b-pb)-qb)]\left[\frac{\mu H(I,S)}{((\gamma+b)(\mu+b-pb)-qb)} - 1\right]I \\ &\le 0. \end{aligned} \tag{3.14}$$

Now we give the following result.

Theorem 3.4. *If*

$$\frac{\mu H(I,S)}{((\gamma+b)(\mu+b-pb)-qb)} \le 1, \tag{3.15}$$

then the equilibrium point $\bar{P}_\circ = (1,0,0)$ is globally asymptotically stable with respect to solution trajectories initiating from $\operatorname{int}\mathbb{R}^+_S$ *(the interior of the set $\mathbb{R}^+_S$).*

Proof. The proof is similar to the proof of [10, Theorem 3.1, page 197], so it is omitted. □

Also, we discuss the global stability of the point $P = (S_\circ, E_\circ, I_\circ)$. In $\mathbb{R}^+_{\text{SEI}}$ we choose the Lyapunov function

$$V = \frac{1}{2}k_1(S - S_\circ)^2 + \frac{1}{2}k_2(E - E_\circ)^2 + I - I_\circ - I_\circ \ln\frac{I}{I_\circ}, \tag{3.16}$$

where $k_i \in \mathbb{R}^+$, $i = 1, 2$.

The derivative of V along the solutions curve in $\mathbb{R}^+_{\text{SEI}}$ is given by the expression

$$\begin{aligned} V^{\cdot} &= k_1(S - S_\circ)(b - IH(I,S) - pbE - qbI - bS) \\ &\quad + k_2(E - E_\circ)(IH(I,S) + pbE + qbI - (\mu + b)E) \\ &\quad + \left(1 - \frac{I_\circ}{I}\right)(\mu E - (\gamma + b)I). \end{aligned} \tag{3.17}$$

But since

$$b = I_\circ H(I_\circ, S_\circ) + pbE_\circ + qbI_\circ + bS_\circ, \tag{3.18}$$

then

$$\begin{aligned} V^{\cdot} &= k_1(S - S_\circ)(I_\circ H(I_\circ, S_\circ) + pbE_\circ + qbI_\circ + bS_\circ) \\ &\quad - k_1(S - S_\circ)(IH(I,S) + pbE + qbI + bS) \\ &\quad + k_2(E - E_\circ)(IH(I,S) + pbE + qbI) - k_2(E - E_\circ)(\mu + b)E \\ &\quad + \frac{(I - I_\circ)}{I}(\mu E - (\gamma + b)I). \end{aligned} \tag{3.19}$$

Now, putting

$$\begin{gathered} a_{11} = k_1 \frac{(I_\circ H(I_\circ, S_\circ) + pbE_\circ + qbI_\circ + bS_\circ)}{(S - S_\circ)}, \\ a_{12} = k_2 \frac{(IH(I,S) + pbE + qbI)}{(E - E_\circ)} - k_1 \frac{(IH(I,S) + pbE + qbI + bS)}{(S - S_\circ)}, \\ a_{13} = a_{23} = 0, \\ a_{22} = -k_2 \frac{(\mu + b)E}{(E - E_\circ)}, \qquad a_{33} = \frac{(\mu E - (\gamma + b)I)}{I(I - I_\circ)}, \end{gathered} \tag{3.20}$$

let

$$X = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{3.21}$$

such that

$$\begin{aligned} v_1 &= (S-S_\circ), \\ v_2 &= (E-E_\circ), \\ v_3 &= (I-I_\circ). \end{aligned} \tag{3.22}$$

Thus we can write the derivative $V^\cdot$ as

$$\begin{aligned} V^\cdot &= a_{11}v_1^2 + a_{12}v_1v_2 + a_{22}v_2^2 + a_{33}v_3^2 \\ &= a_{11}v_1^2 + \frac{1}{2}a_{12}v_1v_2 + \frac{1}{2}a_{13}v_1v_3 + \frac{1}{2}a_{12}v_1v_2 + \frac{1}{2}a_{23}v_2v_3 \\ &\quad + a_{22}v_2^2 + \frac{1}{2}a_{23}v_2v_3 + \frac{1}{2}a_{13}v_1v_3 + a_{33}v_3^2, \end{aligned} \tag{3.23}$$

where $a_{ij} = a_{ji}$ with $a_{13} = a_{23} = 0$, $i, j = 1, 2, 3$.

But $V^\cdot = X^T AX = XAX^T = \langle AX, X\rangle$ (quadratic form), where A is a 3×3 real symmetric matrix, such that $A = (1/2)(A + A^T)$, and is given by

$$A = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{pmatrix}. \tag{3.24}$$

Let a_{ij}, $i, j = 1, 2, 3$, be such that the following hold:

(i) $a_{ij} \in C^1(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, R)$,

(ii) $\lim_{x\to x_\circ} a_{ij}$ exists as a finite number, where $x_\circ$ is the equilibrium point,

(iii) a_{ij} are bounded for all $i, j = 1, 2, 3$.

The characteristic roots of the matrix A are given by

$$\begin{aligned} \rho(\lambda, A) &= \det(A - \lambda I_{3\times 3}) \\ &= \lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 = 0, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} m_1 &= -\operatorname{trace} A = -(a_{11} + a_{22} + a_{33}), \\ m_2 &= \det\begin{vmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{vmatrix} + \det\begin{vmatrix} a_{11} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{13} & a_{33} \end{vmatrix} + \det\begin{vmatrix} a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{23} & a_{33} \end{vmatrix}, \\ m_3 &= -\det A. \end{aligned} \tag{3.26}$$

But since we have $a_{13} = a_{23} = 0$,

$$\begin{aligned} m_1 &= -(a_{11} + a_{22} + a_{33}), \\ m_2 &= a_{11}(a_{22} + a_{33}) - \frac{1}{4}a_{12}^2, \\ m_3 &= a_{33}\left(\frac{1}{4}a_{12}^2 - a_{11}a_{22}\right). \end{aligned} \tag{3.27}$$

Hence, by the Routh-Hurwitz criterion and [15, Lemma 6.1, page 177], it follows that A is negative definite if

$$m_1 < 0, \qquad m_3 < 0, \qquad m_1 m_2 > m_3, \tag{3.28}$$

and we have the following result.

Theorem 3.5. *If the following two conditions hold, then the equilibrium point $P = (S_\circ, E_\circ, I_\circ) \in \mathbb{R}^+_{\mathrm{SEI}}$ is globally asymptotically stable with respect to solution trajectories initiating from* $\operatorname{int}\mathbb{R}^+_{\mathrm{SEI}}$:

(i) $a_{ii} < 0$, $i = 1,2,3$,

(ii) $a_{11}a_{22} - (1/4)a_{12}^2 < 0$.

Proof. The proof follows from [15, Lemma 6.1] and Frobenius' theorem (1876). □

We will need the following lemma due to Butler-McGehee to obtain our results.

Lemma 3.6 [5, page 227]**.** *Let P be an isolated hyperbolic equilibrium in the omega limit set $\Omega(X)$ of an orbit $\vartheta(X)$. Then either $\Omega(X) = P$ or there exist points Q^+, Q^- in $\Omega(X)$ with $Q^+ \in M^+(P)$ and $Q^- \in M^-(P)$.*

Now, we present results on persistence, uniform persistence, and then we give sufficient conditions for the existence of a positive interior equilibrium point $P = (S^*, E^*, I^*, R^*)$.

Theorem 3.7. *Assume that*

(i) *$P_\circ = (1,0,0,0)$ is a hyperbolic saddle point and is repelling in both I- and R-directions (see Theorem 3.4);*

(ii) *system (1.1) is dissipative and the solutions initiating in* $\operatorname{int}\mathbb{R}^+_{\mathrm{SEIR}}$ *are eventually uniformly bounded;*

(iii) *the equilibrium points $\bar{P}_\circ = (1,0,0)$ and $P = (S_\circ, E_\circ, I_\circ)$ are globally asymptotically stable.*

Then system (1.1) is uniformly persistent.

Proof. The proof will depend on Lemma 3.6. Let $\Gamma = \{(S,E,I,R) \in \mathbb{R}^4_{\mathrm{SEIR}} : S + E + I + R = 1\} \subset \mathbb{R}^4_+$. We have shown in Theorem 3.2 that Γ is positively invariant and any solution of system (1.1) initiating at a point in $\Gamma \in \mathbb{R}^4$ is eventually uniformly bounded. However $\bar{P}_\circ = (1,0,0)$ are the only compact invariant sets on $\partial\mathbb{R}^4_+$. Let $M = P = (S^*, E^*, I^*, R^*)$ be such that $M \in \operatorname{int}\partial\mathbb{R}^4_+$. The proof will be completed by showing that no points $Q_i \in \partial\mathbb{R}^4_+$ belong to $\Omega(M)$. Suppose, on the contrary, that $P_\circ \notin \Omega(M)$. Suppose $P_\circ \in \Omega(M)$. Since $P_\circ$ is hyperbolic, $P_\circ \notin \Omega(M)$. By Lemma 3.6, there exists a point $Q_0^+ \in W^+((P_\circ)\backslash\{P_\circ\})$ such that $Q_0^+ \in \Omega(M)$. But since $W^+(P_\circ) \cap (\mathbb{R}^4_+ \backslash \{P_\circ\}) = \phi$, this contradicts the positive invariance property of Γ. Thus $P_\circ \notin \Omega(M)$. We also show that $P_1 = (S_\circ, E_\circ, I_\circ, 0) \notin \Omega(M)$. If $P_1 = (S_\circ, E_\circ, I_\circ, 0) \in \Omega(M)$, then there exists a point $Q_1^+ \in W^+((P_1)\backslash\{P_1\})$ such that $Q_1^+ \in \Omega(M)$. But $W^+(P_1) \cap (\mathbb{R}^4_+) = \phi$ and $P_1 = (S_\circ, E_\circ, I_\circ, 0)$ is globally asymptotically stable with respect to $\mathbb{R}^+_{\mathrm{SEI}}$. This implies that the closure of the orbit $\vartheta(Q_1^+)$ through Q_1^+ either contains $P_\circ$ or is unbounded. This is a contradiction.

Hence $P_1 = (S_\circ, E_\circ, I_\circ, 0) \notin \Omega(M)$. Thus we see that if $P_\circ$ is unstable, then

$$W^+(P_\circ) \cap (\mathbb{R}_+^4 \setminus \{P_\circ\}) = \phi. \tag{3.29}$$

Also, we deduce that if P_1 is unstable, then

$$\begin{aligned} W^+(P_1) \cap (\operatorname{int}\mathbb{R}_+^4) &= \phi, \\ W^-(P_1) \cap (\mathbb{R}^4 \setminus \mathbb{R}_+^4) &= \phi. \end{aligned} \tag{3.30}$$

Now, we show that $\partial\mathbb{R}_+^4 \cap \Omega(M) = \phi$. Let $Q \in \partial\mathbb{R}_+^4$ and $Q \in \Omega(M)$. Then the closure of the orbit through Q, $\vartheta\overline{(Q)}$, must either contain $P_\circ$ and P_1 or be unbounded, and the uniform persistence result follows since $\Omega(M)$ must be in $\operatorname{int}\mathbb{R}_+^4$. This completes the proof. □

Now, we discuss the Hopf-Andronov-Poincaré bifurcation for system (1.1) with bifurcation parameter η. System (1.1) can be recast into the form

$$X' = F(X, \eta), \tag{3.31}$$

where

$$X \in \mathbb{R}^4 = \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix} \tag{3.32}$$

and η is the bifurcation parameter. $F(X,\eta)$ is a C^r $(r \leq 5)$ function on an open set in $\mathbb{R}^4 \times \mathbb{R}^1$.

Let $B_\eta = \{P_\circ = (1,0,0,0),\ P = (S^*, E^*, I*, R^*)\}$ be the set of equilibrium points of (1.1) such that $F(B_\eta) = 0$, for some $\eta \in \mathbb{R}^1$, on a sufficiently large open set G containing each member of B_η. The linearized problem corresponding to (1.1) about any η is given by

$$y^{\cdot} = J_\eta(F(B_\eta))y, \quad y \in \mathbb{R}^4. \tag{3.33}$$

Here, we are interested in studying how the orbit structure near B_η changes as η varies.

Theorem 3.8. *Let condition (ii) of Theorem 3.3 be satisfied. Then the Hopf bifurcation cannot occur at* $P_\circ = (1,0,0,0)$.

Proof. By the same manner as in [15, page 185], the proof follows from the fact that when Theorem 3.3 holds, then $P_\circ = (1,0,0,0)$ is a hyperbolic saddle point and its stable manifold lies on an axis. □

Now, we consider the equilibrium point $P = (S^*, E^*, I*, R^*)$. The Jacobian matrix corresponding to $B_\eta = P = (S^*, E^*, I*, R^*)$ is given by

$$J_\eta(B_\eta) = \begin{pmatrix} a_{11}^* & a_{12}^* & a_{13}^* & 0 \\ a_{21}^* & a_{22}^* & a_{23}^* & 0 \\ 0 & a_{32}^* & a_{33}^* & 0 \\ 0 & 0 & a_{43}^* & a_{44}^* \end{pmatrix}, \tag{3.34}$$

where $a_{11}^* = -I(\partial H/\partial S) - b$, $a_{12}^* = -pb$, $a_{13}^* = -H - I(\partial H/\partial I) - qb$, $a_{21}^* = I(\partial H/\partial S)$, $a_{22}^* = pb - (\mu + b)$, $a_{23}^* = H + I(\partial H/\partial I) + qb$, $a_{32}^* = \mu$, $a_{33}^* = -(\gamma + b)$, $a_{43}^* = \gamma$, $a_{44}^* = -b$, and $a_{14}^* = a_{24}^* = a_{31}^* = a_{34}^* = a_{41}^* = a_{42}^* = 0$.

The characteristic equation corresponding to $J_\eta(B_\eta)$ is

$$\lambda^4 + \sigma_1\lambda^3 + \sigma_2\lambda^2 + \sigma_3\lambda + \sigma_4 = 0, \tag{3.35}$$

where

$$\begin{aligned}
\sigma_1 &= -\operatorname{trace} J_\eta(B_\eta) = -(a_{11}^* + a_{22}^* + a_{33}^* + a_{44}^*),\\
\sigma_2 &= (a_{11}^*a_{22}^* + a_{22}^*a_{33}^* + a_{22}^*a_{44}^* + a_{11}^*a_{33}^* + a_{11}^*a_{44}^* - a_{32}^*a_{23}^* - a_{12}^*a_{21}^*),\\
\sigma_3 &= (-a_{11}^*a_{22}^*a_{33}^* - a_{11}^*a_{22}^*a_{44}^* + a_{11}^*a_{32}^*a_{23}^* - a_{22}^*a_{33}^*a_{44}^* - a_{11}^*a_{33}^*a_{44}^*\\
&\qquad + a_{21}^*a_{12}^*a_{33}^* + a_{21}^*a_{12}^*a_{44}^* + a_{32}^*a_{23}^*a_{44}^* - a_{21}^*a_{32}^*a_{13}^*),\\
\sigma_4 &= \det J_\eta(B_\eta) = (a_{11}^*a_{22}^*a_{33}^*a_{44}^* + a_{21}^*a_{32}^*a_{13}^*a_{44}^* - a_{11}^*a_{32}^*a_{23}^*a_{44}^* - a_{21}^*a_{12}^*a_{33}^*a_{44}^*).
\end{aligned} \tag{3.36}$$

Going through the Routh-Hurwitz criterion as in [15], and sufficient conditions for the roots of (3.35) to have negative real parts are

$$(l_1)\sigma_1 > 0, \qquad \sigma_3 > 0, \qquad \sigma_4 > 0, \qquad (l_2)\sigma_1\sigma_2\sigma_3 > (\sigma_3)^2 + (\sigma_1)^2\sigma_4. \tag{3.37}$$

Now, in order to have Hopf bifurcation, we must violate either (l_1) or (l_2). Suppose each $\sigma_i > 0$, $i = 1,2,3,4$, such that

(i) $\sigma_2\sigma_3 - \sigma_4 > 0$,

(ii) (l_2) is violated such that

$$\sigma_1\sigma_2\sigma_3 = (\sigma_3)^2 + (\sigma_1)^2\sigma_4 \tag{3.38}$$

which implies that

$$\sigma_2\sigma_3 = \frac{(\sigma_3)^2}{\sigma_1} + \sigma_1\sigma_4 > \sigma_4. \tag{3.39}$$

Lemma 3.9. *Assume that the following conditions are satisfied:*

(i) $\sigma_i > 0$, $i = 1,2,3,4$,

(ii) $\sigma_2\sigma_3 - \sigma_4 > 0$,

(iii) $\sigma_1\sigma_2\sigma_3 \le (\sigma_3)^2 + (\sigma_1)^2\sigma_4$.

Then the characteristic equation (3.35) can be factorized into the form

$$(\lambda^2 + n_1)(\lambda + n_2)(\lambda + n_3) = 0, \quad n_i > 0,\ i = 1,2,3, \tag{3.40}$$

where $\sigma_1 = n_2 + n_3$, $\sigma_2 = n_2n_3 + n_1$, $\sigma_3 = n_1(n_2 + n_3)$, *and* $\sigma_4 = n_1n_2n_3$, *which implies that* $n_1 = \sigma_3/\sigma_1$ *and* n_2, n_3 *are satisfied by the quadratic equation*

$$x^2 - \sigma_1 x + \left(\sigma_1 - \frac{\sigma_3}{\sigma_1}\right) = 0. \tag{3.41}$$

In particular, the eigenvalues of (3.35) are given by $\{i\sqrt{n_1}, -i\sqrt{n_1}, -n_2, -n_3\}$. Thus, under the conditions of the above Lemma, the eigenvalues of the Jacobian matrix $J_\eta(B_\eta)$ have two pure imaginary roots for some value of η, say, $\eta = \eta^*$. For $\eta \in (\eta^* - \varepsilon, \eta^* + \varepsilon)$, the characteristic equation (3.35) cannot have real positive roots. But for $\eta \in (\eta^* - \varepsilon, \eta^* + \varepsilon)$, the roots are in the general form

$$\begin{aligned}
\lambda_1(\eta) &= \alpha(\eta) + i\beta(\eta),\\
\lambda_2(\eta) &= \alpha(\eta) - i\beta(\eta),\\
\lambda_3(\eta) &= -n_2 \neq 0,\\
\lambda_4(\eta) &= -n_3 \neq 0.
\end{aligned} \tag{3.42}$$

We now apply Hopf's transversality criterion to (3.35) in order to obtain the required condition for Hopf bifurcation to occur for this system. Hopf's transversality criterion is given by

$$\operatorname{Re}\left[\frac{d\lambda_j}{d\eta}\right]_{\eta=\eta^*} \neq 0, \quad j = 1,2. \tag{3.43}$$

Substituting $\lambda_j(\eta) = \alpha(\eta) + i\beta(\eta)$ into (3.35), we obtain

$$\begin{aligned}
4(\alpha+i\beta)^3(\alpha'+i\beta') + \sigma_1'(\alpha+i\beta)^3 + 3\sigma_1(\alpha+i\beta)^2(\alpha'+i\beta') + \sigma_2'(\alpha+i\beta)^2&\\
+ 2\sigma_2(\alpha+i\beta)(\alpha'+i\beta') + \sigma_3'(\alpha+i\beta) + \sigma_3'(\alpha'+i\beta') + \sigma_4' = 0.&
\end{aligned} \tag{3.44}$$

Comparing the real and imaginary parts in both sides of the above equation, we get

$$\begin{aligned}
A(\eta)\alpha'(\eta) - B(\eta)\beta'(\eta) + C(\eta) &= 0,\\
B(\eta)\alpha'(\eta) + A(\eta)\beta'(\eta) + D(\eta) &= 0,
\end{aligned} \tag{3.45}$$

where

$$\begin{aligned}
A(\eta) &= (4\alpha^3 - 12\alpha\beta^2) + 3\sigma_1(\alpha^2-\beta^2) + 2\sigma_2\alpha + \sigma_3,\\
B(\eta) &= (12\alpha^2\beta - 4\beta^3) + 6\sigma_1\beta + 2\sigma_2\beta,\\
C(\eta) &= \sigma_1'(\alpha^3 - 3\alpha\beta^2) + \sigma_2'(\alpha^2-\beta^2) + \sigma_2'\alpha + \sigma_4',\\
D(\eta) &= \sigma_1'(3\alpha\beta - \beta^3) + 2\sigma_2'\alpha\beta + \sigma_3'\beta.
\end{aligned} \tag{3.46}$$

Thus, from (3.45), we have

$$\begin{aligned}
\operatorname{Re}\left[\frac{d\lambda_j}{d\eta}\right]_{\eta=\eta^*} &= \frac{\det\begin{vmatrix} -C(\eta) & -B(\eta)\\ -D(\eta) & A(\eta)\end{vmatrix}}{\det\begin{vmatrix} A(\eta) & -B(\eta)\\ B(\eta) & A(\eta)\end{vmatrix}}\\
&= \frac{-(AC+BD)}{A^2+B^2}.
\end{aligned} \tag{3.47}$$

Since $(AC+BD) \neq 0$, then $\operatorname{Re}[d\lambda_j/d\eta]_{\eta=\eta^*} \neq 0$.

The above discussion proves the following result.

Theorem 3.10. *Suppose the equilibrium point $P = (S^*, E^*, I*, R^*)$ exists, $\sigma_i > 0$, $i = 1, 2, 3, 4$, and $\sigma_1\sigma_2\sigma_3 \le (\sigma_3)^2 + (\sigma_1)^2\sigma_4$; then system (1.1) exhibits a Hopf-Andronov-Poincaré bifurcation in the first orthant, leading to a family of periodic solutions that bifurcate from P for suitable values of η in the neighborhood of $\eta = \eta^*$.*

4. Discussion. In this paper, we studied a general SEIR model with vertical transmission for the dynamics of an infectious disease. We assumed that a fraction p and a fraction q of the offspring from the exposed and the infectious classes, respectively, are assumed to be infected at birth. The incidence term $H(I, S)$ is of nonlinear form and the immunity is assumed to be permanent. We established the local asymptotic stability of the disease-free equilibrium points $\bar{P}_\circ = (1, 0, 0)$ and $P_\circ = (1, 0, 0, 0)$ for systems (2.1) and (1.1), respectively. Our results are consistent with those obtained by Li et al. [12], where our condition (A1) of Theorem 2.1 are equivalent to the condition $R_\circ(p, q) \le 1$ in terms of the notation of Li et al. [12]. We have shown that if condition (A1) of Theorem 2.1 is satisfied, then the disease-free equilibrium point $\bar{P}_\circ = (1, 0, 0)$ is locally asymptotically stable in the interior of the feasible region and the disease always dies out. Also we have shown that if the two conditions (A2) and (A3) of Theorem 2.3 hold, then a unique endemic equilibrium point $P = (S_\circ, E_\circ, I_\circ)$ exists and is locally asymptotically stable in the interior of the feasible region; moreover, once the disease appears, it eventually persists at the unique endemic equilibrium level. The local stability of $\bar{P}_\circ = (1, 0, 0)$, $P = (1, 0, 0, 0)$, and $P = (S_\circ, E_\circ, I_\circ)$ was proved using the Routh-Hurwitz criterion that have been widely used in the Literature, see [2, 15]. The global stability of $\bar{P}_\circ = (1, 0, 0)$ and $P = (S_\circ, E_\circ, I_\circ)$ in Theorems 3.4 and 3.5 was established using Lyapunov functions similar to those discussed by Li and Wang [13] and Nani and Freedman [15], respectively. We applied the Hopf bifurcation and center manifold theories for system (2.1) using an approach similar to those due to [3]. We employed the mathematical tools of differential analysis, persistence theory, Hopf-Andronov-Poincaré bifurcation, and linear system theory to deduce the existence of a family of periodic solutions that bifurcate from $P = (S^*, E^*, I^*, R^*)$. We used a technique similar to that used by Nani and Freedman [15]. Our results obtained here improve and partially generalize those obtained in [3, 10, 12, 13].

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M. M. A. El-Sheikh: Department of Mathematics, Faculty of Science, Minoufiya University, Shebin El-Koom 230511, Egypt

E-mail address: msheikh_2000@yahoo.com

S. A. A. El-Marouf: Department of Mathematics, Faculty of Science, Minoufiya University, Shebin El-Koom 230511, Egypt

E-mail address: sobhy_2000_99@yahoo.com