

POLYNOMIAL SPLINE COLLOCATION METHODS FOR SECOND-ORDER VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

EDRIS RAWASHDEH, DAVID MCDOWELL, and LEELA RAKESH

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We have presented a method for the construction of an approximation to the initial-value second-order Volterra integrodifferential equation (VIDE). The polynomial spline collocation methods described here give a superconvergence to the solution of the equation.

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1. Introduction. In order to discuss the numerical solution of the second-order VIDEs we consider the following linear integrodifferential equation:

$$y^{(2)}(t) = q(t) + \sum_{i=0}^1 p_i(t)y^{(i)}(t) + \sum_{i=0}^1 \int_0^t k_i(t,s)y^{(i)}(s)ds, \quad t \in I := [0, T], \quad (1.1)$$

with

$$y(0) = y_0, \quad y^{(1)}(0) = y_1, \quad (1.2)$$

where $q : I \rightarrow R$, $p_i : I \rightarrow R$, and $k_i : D \rightarrow R$ ($i = 0, 1$) (with $D := \{(t, s) : 0 \leq s \leq t \leq T\}$) are given functions and are assumed to be (at least) continuous in the respective domains. For more details of these equations, many other interesting methods for the approximated solution and stability procedures are available in earlier literatures [1, 3, 4, 5, 6, 7, 8, 11]. The above equation is usually known as basis test equation and is suggested by Brunner and Lambert [5]. Since then it has been widely used for analyzing the solution and stability properties of various methods.

Second-order VIDEs of the above form (1.1) will be solved numerically using polynomial spline spaces. In order to describe these approximating polynomial spline spaces, let $\Pi_N : 0 = t_0 < t_1 < \dots < t_N = T$ be the mesh for the interval I , and set

$$\begin{aligned} \sigma_n &:= [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad n = 0, 1, \dots, N-1, \\ h &= \max \{h_n : 0 \leq n \leq N-1\} \quad (\text{mesh diameter}), \\ Z_N &:= \{t_n : n = 1, 2, \dots, N-1\}, \quad \bar{Z}_N = Z_N \cup \{T\}. \end{aligned} \quad (1.3)$$

Let π_{m+d} be the set of (real) polynomials of degree not exceeding $m+d$, where $m \geq 1$ and $d \geq -1$ are given integers. The solution y to the initial-value problem (1.1), (1.2)

will be approximated by an element u in the polynomial spline space,

$$S_{m+d}^{(d)}(Z_N) := \{u := u(t) |_{t \in \sigma_n} := u_n(t) \in \pi_{m+d}, n = 0, 1, \dots, N-1, \\ u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } j = 0, 1, \dots, d, t_n \in Z_N\}, \tag{1.4}$$

that is, by a polynomial spline function of degree $m + d$ which possesses the knots Z_N and is d times continuously differentiable on I . If $d = -1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuities at the knots Z_N . Initial value problems, such as (1.1) and (1.2), have often been solved by collocation method in polynomial spines spaces $S_m^{(0)}(Z_N)$ and $S_m^{(1)}(Z_N)$.

According to M. Miculá and G. Micula [10], an element $u \in S_{m+d}^{(d)}(Z_N)$ has the following form: for all $n = 0, 1, \dots, N-1$ and $t \in \sigma_n$,

$$u(t) = u_n(t) = \sum_{r=0}^d \frac{u_{n-1}^{(r)}(t_n)}{r!} (t - t_n)^r + \sum_{r=1}^m a_{n,r} (t - t_n)^{d+r}, \tag{1.5}$$

where

$$u_{-1}^r(0) := \left[\frac{d^r}{dt^r} u(t) \right]_{t=0} = y^{(r)}(0), \quad r = 0, 1, \dots, d. \tag{1.6}$$

From (1.5), we see that the element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined provided the coefficients $\{a_{n,r}\}_{r=1,\dots,m}$ are known. In order to determine these coefficients, we consider a set of collocation parameters $\{c_j\}_{j=1,\dots,m}$, where $0 < c_1 < \dots < c_m \leq 1$, and define the set of collocation points as

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \quad \text{with } X_n := \{t_{n,j} := t_n + c_j h_n, j = 1, 2, \dots, m\}. \tag{1.7}$$

The approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined by imposing the condition that u satisfies the following initial-value problem on $X(N)$:

$$u^{(2)}(t) = q(t) + \sum_{i=0}^1 p_i(t) u^{(i)}(t) + \sum_{i=0}^1 \int_0^t k_i(t,s) u^{(i)}(s) ds, \quad \forall t \in X(N), \tag{1.8}$$

with

$$u(0) = \gamma_0, \quad u^{(1)}(0) = \gamma_1, \tag{1.9}$$

with a uniform the mesh sequence $\{\prod_N\}$, $h_n = h$, for all $n = 0, 1, \dots, N-1$, but for small h , (1.8) has a unique solution $\{a_{n,j}\}_{j=1,\dots,m}$, for all $n = 0, 1, \dots, N-1$.

Therefore, the modified collocation equation (1.8) can be rewritten as

$$\begin{aligned}
 u_n''(t_{n,j}) &= q(t_{n,j}) + p_0(t_{n,j})u_n(t_{n,j}) + p_1(t_{n,j})u_n'(t_{n,j}) \\
 &+ h_n \phi_{n,n}^j[u_n] + \sum_{i=0}^{n-1} h_i \phi_{n,i}^j[u_i] \\
 &+ \psi_{n,n}^j[u_n'] + \sum_{i=0}^{n-1} \psi_{n,i}^j[u_i'], \quad j = 1, \dots, m, \quad n = 0, 1, \dots, N-1,
 \end{aligned}
 \tag{1.10}$$

where

$$\begin{aligned}
 \phi_{n,i}^j[u_i] &= \begin{cases} \int_0^{c_j} k_0(t_{n,j}, t_n + v h_n) u_n(t_n + v h_n) dv, & \text{if } i = n, \\ \int_0^1 k_0(t_{n,j}, t_i + v h_i) u_i(t_i + v h_i) dv, & \text{if } i = 0, 1, \dots, n-1, \end{cases} \\
 \psi_{n,i}^j[u_i] &= \begin{cases} \int_0^{c_j} k_1(t_{n,j}, t_n + v h_n) u_n'(t_n + v h_n) dv, & \text{if } i = n, \\ \int_0^1 k_1(t_{n,j}, t_i + v h_i) u_i'(t_i + v h_i) dv, & \text{if } i = 0, 1, \dots, n-1. \end{cases}
 \end{aligned}
 \tag{1.11}$$

In most applications the integrals appearing in (1.11) cannot be evaluated analytically, so we seek suitable quadrature formulas as follows:

$$\hat{\phi}_{n,i}^j[u_i] = \begin{cases} \sum_{l=1}^{\mu_1} w_{j,l} k_0(t_{n,j}, t_n + d_{j,l} h_n) u_n(t_n + d_{j,l} h_n), & \text{if } i = n, \\ \sum_{l=1}^{\mu_0} w_l k_0(t_{n,j}, t_i + d_l h_i) u_i(t_i + d_l h_i), & \text{if } i = 0, 1, \dots, n-1, \end{cases}
 \tag{1.12}$$

$$\hat{\psi}_{n,i}^j[u_i] = \begin{cases} \sum_{l=1}^{\mu_1} w_{j,l} k_1(t_{n,j}, t_n + d_{j,l} h_n) u_n'(t_n + d_{j,l} h_n), & \text{if } i = n, \\ \sum_{l=1}^{\mu_0} w_l k_1(t_{n,j}, t_i + d_l h_i) u_i'(t_i + d_l h_i), & \text{if } i = 0, 1, \dots, n-1, \end{cases}
 \tag{1.13}$$

where μ_0 and μ_1 are two given positive integers, $\{d_l\}$ and $\{d_{j,l}\}$ are two sets of parameters satisfying the following conditions:

$$0 \leq d_1 < \dots < d_{\mu_0} \leq 1, \quad 0 \leq d_{j,1} < \dots < d_{j,\mu_1} \leq c_j, \quad (j = 1, \dots, m),
 \tag{1.14}$$

and $w_l, w_{j,l}$ denote the quadrature weights.

The corresponding quadrature error terms are defined as

$$\begin{aligned}
 E_{n,i}^j[u_i] &= \phi_{n,i}^j[u_i] - \hat{\phi}_{n,i}^j[u_i], \\
 \dot{E}_{n,i}^j[u_i'] &= \psi_{n,i}^j[u_i'] - \hat{\psi}_{n,i}^j[u_i'], \\
 j &= 1, 2, \dots, m, \quad i = 0, 1, \dots, n \quad (n = 0, 1, \dots, N-1).
 \end{aligned}
 \tag{1.15}$$

By the above quadrature process (1.10) is a fully discretized collocation method which defines an element $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ (which, in general, will be different from the approximate solution u determined by the exact collocation equation (1.10)). The full discretized version of the collocation equation (1.10) is

$$\begin{aligned} \hat{u}_n''(t_{n,j}) &= q(t_{n,j}) + p_0(t_{n,j})\hat{u}_n(t_{n,j}) + p_1(t_{n,j})\hat{u}_n'(t_{n,j}) \\ &+ h_n \hat{\phi}_{n,n}^j[\hat{u}_n] + \sum_{i=0}^{n-1} h_i \hat{\phi}_{n,i}^j[\hat{u}_i] \\ &+ \hat{\psi}_{n,n}^j[\hat{u}_n'] + \sum_{i=0}^{n-1} \hat{\psi}_{n,i}^j[\hat{u}_i'], \quad j = 1, \dots, m, \quad n = 0, 1, \dots, N-1. \end{aligned} \tag{1.16}$$

Similarly, the approximate discretized solution $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ has the following form:

$$\hat{u}(t) = \hat{u}_n(t) = \sum_{r=0}^d \frac{\hat{u}_{n-1}^{(r)}(t_n)}{r!} (t - t_n)^r + \sum_{r=1}^m \hat{a}_{n,r} (t - t_n)^{d+r}, \tag{1.17}$$

with the assumption that

$$\hat{u}_{-1}^{(r)}(0) := y^{(r)}(0), \quad r = 0, 1, \dots, d. \tag{1.18}$$

The collocation equations (1.10) and (1.16) represent recursive systems, for each $n = 0, 1, \dots, N-1$, which yield the coefficients $\{a_{n,r}\}_{r=1, \dots, m}$ and $\{\hat{a}_{n,r}\}_{r=1, \dots, m}$, respectively. Once the coefficients are known, the values of u, \hat{u} together with their derivatives are determined on σ_n by (1.5) and (1.17), respectively.

On each of the N subintervals of I we have to solve an $m \times m$ system of linear equations. On the first subinterval σ_0 , $d + 1$ additional equations are furnished by the $d + 1$ initial conditions (1.6) and (1.18).

2. Global convergence criteria. If the given functions q, p_i , and k_i ($i = 0, 1$) are of class $m + d$ on their domain of definition, then the VIDE (1.1) has a unique solution y , which is of class $m + d + 2$. Let φ_n be the restriction of the function φ on I in the subinterval σ_n , for $n = 0, 1, \dots, N-1$, and the norm is defined by

$$\|\varphi\|_\infty = \sup \{ |\varphi_n(t)| : t \in \sigma_n, \quad n = 0, 1, \dots, N-1 \}. \tag{2.1}$$

The following theorem describes the order of global convergence of $u \in S_{m+d}^{(d)}(Z_N)$, $d \in \{0, 1, 2\}$.

THEOREM 2.1. *Let q, p_i , and k_i be $m + d$ times continuously differentiable in the respective domains I and S . For $d \in \{0, 1, 2\}$, every choice of collocation parameters $\{c_j\}_{j=1, \dots, m}$, with $0 < c_1 < \dots < c_m \leq 1$, and all quasiuniform mesh sequences $\{[\]_N\}$ with sufficiently small $h > 0$, the following hold:*

- (i) the exact collocation equation (1.10) defines a unique approximation $u \in S_{m+d}^{(d)}(Z_N)$ and the resulting error function $e := y - u$ satisfies

$$\|e^{(k)}\|_\infty \leq C_k h^{m+d+1-k}, \quad \forall k = 0, 1, \dots, m+d, \tag{2.2}$$

where C_k 's are finite constants independent of h ;

- (ii) if the quadrature formulas (1.12) and (1.13) satisfy that for $i = 0, 1, \dots, n-1$,

$$\int_0^1 \phi(t_i + \tau h_i) d\tau - \sum_{l=1}^{\mu_1} w_l \phi(t_i + d_l h_i) = O(h_i^{r_1}), \tag{2.3}$$

$$\int_0^1 \psi(t_i + \tau h_i) d\tau - \sum_{l=1}^{\mu_1} w_l \psi(t_i + d_l h_i) = O(h_i^{r_1}), \tag{2.4}$$

and for $j = 1, \dots, m$,

$$\int_0^{c_j} \phi(t_n + \tau h_n) d\tau - \sum_{l=1}^{\mu_0} w_{j,l} \phi(t_n + d_{j,l} h_n) = O(h_n^{r_0}), \tag{2.5}$$

$$\int_0^{c_j} \psi(t_n + \tau h_n) d\tau - \sum_{l=1}^{\mu_0} w_{j,l} \psi(t_n + d_{j,l} h_n) = O(h_n^{r_0}), \tag{2.6}$$

whenever the integrand is sufficiently smooth, then for all the discretized approximation solution $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ defined by (1.17), the following relationships hold:

$$\|\varepsilon^{(k)}\|_\infty := \|u^{(k)} - \hat{u}^{(k)}\|_\infty \leq Q_k h^{s_0-k}, \quad \text{for } k = 0, 1, \dots, s_0, \tag{2.7}$$

$$\|\hat{e}^{(k)}\|_\infty := \|y^{(k)} - \hat{u}^{(k)}\|_\infty \leq \hat{C}_k h^{s_1-k}, \quad \text{for } k = 0, 1, \dots, s_1, \tag{2.8}$$

where $s_0 = \min\{r_0 + 1, r_1\} + 1$, $s_1 = \min\{s_0, m + d + 1\}$, and Q_k, \hat{C}_k are finite constants independent of h .

PROOF. We will prove the theorem by induction using the same technique as in [2, 3, 9].

(i) For $n = 0, 1, \dots, N-1$ and for all $t = t_n + \tau h_n \in \sigma_n$ ($\tau \in (0, 1]$), the exact solution y can be obtained by Taylor series expansion:

$$y(t_n + \tau h_n) = \sum_{r=0}^{m+d} \frac{y^{(r)}(t_n) \tau^r}{r!} h_n^r + h_n^{m+d+1} R_n(\tau), \tag{2.9}$$

where

$$R_n(\tau) = \frac{1}{(m+d)!} \int_0^\tau y^{(m+d+1)}(t_n + \eta h_n) (\tau - \eta)^{m+d} d\eta. \tag{2.10}$$

If the restriction of $u \in S_{m+d}^{(d)}(Z_N)$ to the subinterval σ_n is given by (1.5), then by using (2.9), the error function on this subinterval has the form

$$e_n(t_n + \tau h_n) = \sum_{r=0}^d \frac{e_{n-1}^{(r)}(t_n)}{r!} \tau^r h_n^r + h_n^p \sum_{r=1}^m \beta_{n,r} \tau^{d+r} + h_n^{m+d+1} R_n(\tau), \tag{2.11}$$

where

$$h_n^p \beta_{n,r} := \left(\frac{y^{(d+r)}(t_n) - a_{n,r}}{(d+r)!} \right) h_n^{d+r}. \tag{2.12}$$

Subtracting the collocation equation (1.8) from the integrodifferential equation (1.1), replacing t by $t_{n,j}$, using (1.6) for $n = 0$, and utilizing the expression (2.11) for e_n yields

$$h_0^{p-2} D_0 \beta_0 = h_0^{m+d-1} r_0, \quad \text{for } n = 0, \tag{2.13}$$

$$h_n^{p-2} D_n \beta_n = F_n E_n + h_n^{m+d-1} r_n + \sum_{i=1}^{n-1} (h_i q_{n,i} + p_{n,i}), \quad \text{for } n > 0, \tag{2.14}$$

where D_n is an $m \times m$ matrix, F_n is an $m \times (d+1)$ matrix, E_n is a $(d+1)$ vector, and $\beta_n, r_n, q_{n,i}, p_{n,i}$ are m vectors whose elements are given by

$$\begin{aligned} (D_n)_{j,r} &= c_j^{d+r-2} [(d+r)(d+r-1) - h_n^2 p_0(t_n + d_j h_n) c_j^2 - (d+r) c_j h_n p_1(t_n + c_j h_n)] \\ &\quad - h_n^3 \int_0^{c_j} k_0(t_{n,j}, t_n + \tau h_n) \tau^{d+r} d\tau \\ &\quad - (d+r) h_n^2 \int_0^{c_j} k_1(t_{n,j}, t_n + \tau h_n) \tau^{d+r-1} d\tau, \\ (E_n)_{j,r} &= \begin{cases} p_0(t_{n,j}) + h_n \int_0^{c_j} k_0(t_{n,j}, t_n + \tau h_n) d\tau, & \text{if } r = 0, \\ h_n c_j p_0(t_{n,j}) h_n + p_1(t_{n,j}) + h_n^2 \int_0^{c_j} k_0(t_{n,j}, t_n + \tau h_n) \tau d\tau \\ \quad + h_n \int_0^{c_j} k_1(t_{n,j}, t_n + \tau h_n) d\tau, & \text{if } r = 1, \\ \frac{h_n^{r-2}}{r!} [-r(r-1) c_j^{r-2} + h_n^2 c_j^r p_0(t_{n,j}) + r h_n c_j^{r-1} p_1(t_{n,j})] \\ \quad + h_n^3 \int_0^{c_j} k_0(t_{n,j}, t_n + \tau h_n) \tau^r d\tau \\ \quad + r h_n^2 \int_0^{c_j} k_1(t_{n,j}, t_n + \tau h_n) \tau^{r-1} d\tau, & \text{if } r \geq 2, \end{cases} \end{aligned}$$

$$(E_n)_r = e_{n-1}^{(r)}(t_n),$$

$$(\beta_n)_r = \beta_{n,r} \quad \text{as defined in (2.12),}$$

$$\begin{aligned} (r_n)_j &= -R_n''(c_j) + h_n^2 p_0(t_{n,j}) R_n(c_j) + h_n p_1(t_{n,j}) R_n'(c_j) \\ &\quad + h_n^3 \int_0^{c_j} k_0(t_{n,j}, t_n + \tau h_n) R_n(\tau) d\tau \\ &\quad + h_n^2 \int_0^{c_j} k_1(t_{n,j}, t_n + \tau h_n) R_n'(\tau) d\tau, \end{aligned}$$

$$(q_{n,i})_j = \int_0^{c_j} k_0(t_{n,j}, t_i + \tau h_i) e_i(t_i + \tau h_i) d\tau,$$

$$(p_{n,i})_j = \int_0^{c_j} k_1(t_{n,j}, t_i + \tau h_i) e_i'(t_i + \tau h_i) d\tau.$$

(2.15)

Note that the matrix D_n defined by the coefficient of $\{\beta_{n,j}\}$ on the left-hand side of (2.13) and (2.14) is invertible whenever h_n is sufficiently small. This follows from the assumptions of the theorem and the fact that for $h_n = 0$ the determinant of this matrix is essentially a Vandermonde determinant. Therefore, for sufficiently small $h_n > 0$, the matrix D_n possesses a uniformly bounded inverse.

By the hypothesis on p_i and k_i , the ℓ_1 -norm of the vector r_0 is uniformly bounded. Hence, for $p = m + d + 1$, (2.13) leads to

$$\|\beta_0\|_1 := \sum_{l=1}^m |\beta_{0,l}| \leq \|D_0^{-1}\| \|r_0\| =: M_0. \tag{2.16}$$

From (2.11) we have the following result:

$$\begin{aligned} |e_0(t_0 + \tau h_0)| &\leq h^{m+d+1} (M_0 + |R_0(\tau)|) \\ &\leq C_0 h^{m+d+1}, \quad \text{for } \tau \in (0, 1]. \end{aligned} \tag{2.17}$$

By differentiating (2.11) k times ($k = 1, 2, \dots, m + d$) and using (2.16), we obtain

$$|e_0^k(t_0 + \tau h_0)| \leq C_0^k h^{m+d+1-k}, \quad \text{for } \tau \in (0, 1]. \tag{2.18}$$

Suppose, for $j = 0, 1, \dots, n - 1$,

$$|e_j^k(t_j + \tau h_j)| \leq C_j^k h^{m+d+1-k}, \quad \text{for } \tau \in (0, 1], k = 0, 1, \dots, m + d; \tag{2.19}$$

we prove that (2.19) holds for $j = n$. By the assumption of the theorem on q , p_i , and k_i and (2.19), it follows that for sufficiently small h , $\|E_n\|_1 = O(h^{m+1})$, $\|q_{n,i}\|_1 = O(h^{m+d+1})$, $\|p_{n,i}\|_1 = O(h^{m+d})$ ($i = 0, 1, \dots, n - 1$), and $\|r_n\|_1$ is bounded. Thus, for $p = m + d + 1$ and $d \in \{0, 1, 2\}$, (2.14) leads to

$$\|\beta_n\|_1 := \sum_{l=1}^m |\beta_{n,l}| \leq M_n + M_n' h. \tag{2.20}$$

Then, from (2.20) and (2.11), it follows that

$$|e_n^k(t_n + \tau h_n)| \leq C_n^k h^{m+d+1-k}, \quad \text{for } \tau \in (0, 1], k = 0, 1, \dots, m + d. \tag{2.21}$$

This completes the first assertion of the theorem.

(ii) By (1.5) and (1.17) for every $n = 0, 1, \dots, N - 1$, the error function $\varepsilon := u - \hat{u}$ can be written as

$$\varepsilon_n(t_n + \tau h_n) = \sum_{r=0}^d \frac{\varepsilon_{n-1}^{(r)}(t_n)}{r!} \tau^r h_n^r + h_n^{s_0} \sum_{r=1}^m \eta_{n,r} \tau^{d+r}, \tag{2.22}$$

where

$$h_n^{s_0} \eta_{n,r} := (a_{n,r} - \hat{a}_{n,r}) h_n^{d+r}. \tag{2.23}$$

By subtracting the discretized equation (1.16) from the exact collocation equation (1.10) and using (1.15) and (2.22), we have the following systems:

$$h_0^{s_0-2} \hat{D}_0 \eta_0 = h_0 r_{0,0} + \bar{r}_{0,0}, \quad \text{for } n = 0, \tag{2.24}$$

$$h_n^{s_0-2} \hat{D}_n \eta_n = \hat{F}_n \xi_n + h_n r_{n,n} + \sum_{i=1}^{n-1} h_i r_{n,i} + \bar{r}_{n,n} + \sum_{i=1}^{n-1} \bar{r}_{n,i}, \quad \text{for } n > 0, \tag{2.25}$$

where $r_{n,i} := (E_{n,i}^1[u_i], \dots, E_{n,i}^m[u_i])^T$ and $\bar{r}_{n,i} := (\hat{E}_{n,i}^1[u'_i], \dots, \hat{E}_{n,i}^m[u'_i])^T$. The matrices $\hat{D}_n, \hat{F}_n, \xi_n$ have the same orders as D_n, F_n, E_n from (2.13) and (2.14). The difference between them is that the integrals are replaced with quadrature formulas (1.12) and (1.13).

The expressions (2.24) and (2.25) have the same structure as (2.13) and (2.14), respectively. The smoothness hypothesis and the assumption on the order of the quadrature formulas (2.3), (2.4), (2.5), and (2.6) imply $\|r_{n,n}\|_1 = \|\bar{r}_{n,n}\|_1 = O(h_n^{r_0})$ and $\|r_{n,i}\|_1 = \|\bar{r}_{n,i}\|_1 = O(h_n^{r_1})$. Therefore, by the same reasoning in the proof of assertion (i), inequality (2.7) is true.

From (2.2) and (2.7),

$$\|\hat{e}^{(k)}\|_\infty := \|\mathcal{Y}^{(k)} - \hat{u}^{(k)}\|_\infty \leq \|e^{(k)}\|_\infty + \|\varepsilon^{(k)}\|_\infty \leq \hat{C}_k h^{s_1-k}, \tag{2.26}$$

for all $k = 0, 1, \dots, s_1$, with $s_1 = \min\{s_0, m + d + 1\}$. □

COROLLARY 2.2. *Let the assumptions of Theorem 2.1 hold. If the quadrature formulas (1.12) and (1.13) are of interpolatory type, with $\mu_0 = \mu_1 = m + d$, then the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_n)$ defined by the discretized collocation equation (1.16) leads to an error $\hat{e}(t)$ satisfying*

$$\|\hat{e}^{(k)}\|_\infty := O(h^{m+d+1-k}), \tag{2.27}$$

for $k = 0, \dots, m + d$, every choice of collocation parameters $\{c_j\}_{j=1, \dots, m}$, with $0 < c_1 < \dots < c_m \leq 1$, and all quasiuniform mesh sequences $\{\prod_N\}$ with sufficiently small $h > 0$, and $d \in \{0, 1, 2\}$.

If we use $\mu_0 = \mu_1 = m$, $d_j = c_j$, and $d_{j,l} = c_j c_l$ ($j, l = 1, 2, \dots, m$) in the quadrature formulas, then our method leads to some simplifications. These simplifications are useful when they do not affect the convergence order given by Theorem 2.1, namely, $s_1 = m + d + 1$.

COROLLARY 2.3. *If in the VIDE (1.1), $q \in C^{m+d}(I)$, $p_i \in C^{m+d}(I)$, $k_i \in C^{m+d}(S)$, and $m \geq d$, then there exists the set of collocation parameters $\{c_j\}_{j=1, \dots, m}$ such that for the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_n)$ defined by the discretized collocation equation (1.16), where $\mu_0 = \mu_1 = m$, $d_l = c_l$, and $d_{j,l} = c_j c_l$,*

$$\|\hat{e}^{(k)}\|_\infty := O(h^{m+d+1-k}), \tag{2.28}$$

for $k = 0, \dots, m + d$, $d \in \{0, 1, 2\}$.

PROOF. If $\mu_0 = \mu_1 = m$, $m \geq d$, then we can choose the collocation parameters to be the zeros of the shifted Legendre polynomial $P_m(2s - 1)$ (i.e., the Gauss points for $(0, 1)$), as the collocation parameters $\{c_j\}_{j=1,\dots,m}$. Then in the quadrature formulas (2.3), (2.4), (2.5), and (2.6), $r_0 = r_1 = 2m$. Therefore, $s_0 = 2m + 1$ and $s_1 = m + d + 1$ in (2.7) and (2.8), respectively, which proves the corollary. \square

3. Local superconvergence on \bar{Z}_N . In many applications one is especially interested in obtaining higher-order approximations at the mesh points \bar{Z}_N . There arises the question as to whether there exist particular sets of collocation parameters leading to a discrete convergence order (on \bar{Z}_N) that is higher than the global order.

In the subsequent analysis, the integrals

$$J_k := \int_0^1 s^k \prod_{j=1}^m (s - c_j) ds, \quad k \in \mathbb{N}, \tag{3.1}$$

play a critical role.

THEOREM 3.1. *Suppose that the given functions q , p_i , and k_i ($i = 0, 1$) in the VIDE (1.1) are $m + p$ times continuously differentiable on their respective domains I and S ($d + 1 < p \leq m$ and $d \in \{0, 1, 2\}$), and assume that the collocation parameters $\{c_j\}_{j=1,\dots,m}$ are chosen so that $J_k = 0$ for $k = 0, 1, \dots, p - 1$, and $J_p \neq 0$. Then, for all quasiuniform mesh sequences $\{\Pi_N\}$ with sufficiently small $h > 0$, the following hold:*

- (i) *if $u \in S_{m+d}^{(d)}(Z_N)$ is the approximate solution defined by the exact collocation equation (1.8) and y is the exact solution of (1.1), then*

$$\max_{t_n \in \bar{Z}_N} |y(t_n) - u(t_n)| = O(h^{m+p}), \quad \text{as } h \rightarrow 0, Nh \leq \gamma T, \tag{3.2}$$

- (ii) *if the quadrature formulas (1.12) satisfy (2.3) and (2.4), the quadrature formulas (1.13) satisfy (2.5) and (2.6), and $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ is the approximate solution defined by the discretized collocation equation (1.16), then*

$$\max_{t_n \in \bar{Z}_N} |y(t_n) - \hat{u}(t_n)| = O(h^\alpha), \quad \text{as } h \rightarrow 0, Nh \leq \gamma T, \tag{3.3}$$

where $\alpha = \min\{m + p, s_0\}$ and $s_0 = \min\{r_0 + 1, r_1\} + 1$.

PROOF. (i) The exact collocation equation (1.8) can be written in the form

$$\begin{aligned} u''(t) &= q(t) + p_0(t)u(t) + p_1(t)u'(t) - \delta(t) \\ &\quad + \int_0^t k_0(t,s)u(s)ds + \int_0^t k_1(t,s)u'(s)ds, \end{aligned} \tag{3.4}$$

where $\delta(t)$ (the residual) is (at least) continuous on each of the subintervals σ_n and vanishes at the collocation points $X(N)$, $t \in I$. Then, by (3.4) and (1.1), the error function

satisfies

$$e''(t) = p_0(t)e(t) + p_1(t)e'(t) + \delta(t) + \int_0^t k_0(t,s)e(s)ds + \int_0^t k_1(t,s)e'(s)ds, \tag{3.5}$$

with initial conditions $e(0) = 0, e'(0) = 0$.

The solution of (3.5) for $t = t_n \in \bar{Z}_N$ can be expressed in the form (see [4, pages 130-131])

$$e(t_n) = h \sum_{i=0}^{n-1} E^{n,i}, \tag{3.6}$$

where $E^{n,i}$ is the quadrature error which, by (3.1), has an order $m + p$.

Hence,

$$e(t_n) = |\gamma(t_n) - u(t_n)| = O(h^{m+p}), \tag{3.7}$$

which proves the first assertion of the theorem.

(ii) The second assertion of [Theorem 3.1](#) now follows from (2.7) and (3.2). □

It is well known that the orthogonality conditions $J_k = 0$ ($k = 0, \dots, p - 1$) imply that the degree of precision of the m -point interpolatory quadrature formula on $[0, 1]$ based on the abscissas $\{c_j\}_{j=1, \dots, m}$ is $m + p - 1$. Since this degree of precision cannot exceed the value $2m - 1$, we always have $p \leq m$. Moreover, since the local order is required to be greater than or equal to the global order ($m + d + 1$), it is necessary that $d + 2 \leq p$. The following corollary deals with some important special cases and its proof relies on the above theorem.

COROLLARY 3.2. *Let the assumptions of [Theorem 3.1](#) hold, then*

(i) *if the collocation parameters $\{c_j\}_{j=1, \dots, m}$ are the Gauss(-Legendre) points for $(0, 1)$, that is, the zeros of the shifted Legendre polynomial $P_m(2s - 1)$, then $J_k = 0$ for $k = 0, \dots, m - 1$, with $J_m \neq 0$, and hence*

$$\max_{t_n \in \bar{Z}_N} |\gamma(t_n) - u(t_n)| = O(h^{2m}), \quad \text{as } h \rightarrow 0, Nh \leq \gamma T, \tag{3.8}$$

(ii) *if the collocation parameters $\{c_j\}_{j=1, \dots, m}$ are the Radau II points for $(0, 1)$, that is, the zeros of the polynomial $P_{m-1}(2s - 1) - P_m(2s - 1)$, then $J_k = 0$ for $k = 0, \dots, m - 2$, with $J_{m-1} \neq 0$, and hence*

$$\max_{t_n \in \bar{Z}_N} |\gamma(t_n) - u(t_n)| = O(h^{2m-1}), \quad \text{as } h \rightarrow 0, Nh \leq \gamma T. \tag{3.9}$$

4. Numerical example. The convergence results derived in the preceding sections will be illustrated by applying various collocation methods to the following problem.

EXAMPLE 4.1. Consider the following integrodifferential equation of second order:

$$\gamma''(t) = q(t) + \gamma(t) + \int_0^t ts\gamma(s)ds, \quad \gamma(0) = 1, \gamma'(0) = 1, t \in [0, 1], \tag{4.1}$$

TABLE 4.1. Approximate error when $m = 2$ and $(\{c_1 = 1/2, c_2 = 1\})$ (uniformly distributed collocation parameters).

d	e_1	$e_{N/2}$	e_N
0	1.68×10^{-7}	0.593657978	1.089198840
1	0	0.000304350	0.012944677
2	0	0.000303703	0.012941467
3	0	0.000303661	0.012940922

TABLE 4.2. Approximate error when $m = 3$ and $(\{c_1 = 1/3, c_2 = 2/3, c_3 = 1\})$ (uniformly distributed collocation parameters).

d	e_1	$e_{N/2}$	e_N
0	2.45×10^{-5}	0.015164638	0.172113704
1	0	0.000303699	0.012941460
2	0	0.000303701	0.012941459
3	0	0.000303688	0.012941212

TABLE 4.3. Approximate error when $m = 2$ and (Radau II parameters) $(\{c_1 = 1/3, c_2 = 1\})$.

d	e_1	$e_{N/2}$	e_N
0	1.68×10^{-7}	0.593423722	1.089594757
1	0	0.000304350	0.012944677
2	0	0.000303704	0.012941470
3	0	17489.97760	1.97×10^{19}

TABLE 4.4. Approximate error when $m = 3$ and (Radau II parameters) $(\{c_1 = (4 - \sqrt{6})/10, c_2 = (4 + \sqrt{6})/10, c_3 = 1\})$.

d	e_1	$e_{N/2}$	e_N
0	0.000027239	0.57170808	0.126648551
1	0	0.0003033700	0.012941460
2	0	0.000303703	0.012941464
3	0	1.89×10^{10}	1.36×10^{34}

with $q(t)$ chosen in such a way that it has the following exact solution:

$$y(t) = t^2 + e^t. \tag{4.2}$$

Following Tables 4.1, 4.2, 4.3, and 4.4 illustrate the error approximations as

$$\begin{aligned} e_1 &:= |y(0.01) - u(0.01)|, \\ e_{N/2} &:= |y(0.5) - u(0.5)|, \\ e_N &:= |y(1) - u(1)|, \end{aligned} \tag{4.3}$$

where $u \in S_{m+d}^d(Z_N)$ is the approximate solution if $t_i := ih \in Z_N$ with $h = 0.01$ when $m \in \{2, 3\}$ and $d \in \{0, 1, 2, 3\}$.

As we can see from the above tables, the collocation spline method yields very good approximations when $d = 1, 2$. However, for $d = 0$, the method performs poorly. But for the case $d = 3$, the method converges if the collocation parameters are uniformly distributed and diverges if the collocation parameters are Radau II points, which can be seen from Tables 4.1, 4.2, 4.3, and 4.4.

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Edris Rawashdeh: Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, USA

E-mail address: edirash@yahoo.com

David McDowell: Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, USA

E-mail address: mcdow1d@cmich.edu

Leela Rakesh: Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, USA

E-mail address: lrakesh@aol.com